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REMARKS ON THE KKM THEORY OF ABSTRACT CONVEX MINIMAL SPACES

Sehie Park

The National Academy of Sciences, Seoul 137–044, Republic of Korea
and
Department of Mathematical Sciences, Seoul National University
Seoul 151–747, Korea
e-mail: park35@snu.ac.kr; sehiepark@gmail.com

Abstract. Recently, Allmohammady et al. [1-3], Darzi et al. [5], and Delavar et al. [6] dealt with some results in the KKM theory on generalized convex minimal spaces. By establishing a kind of the KKM principle in these spaces, they obtained some results on coincidence or fixed point theorems and others. Our aim in the present paper is to show that their results are consequences of corresponding ones for abstract convex minimal spaces in our previous paper [12] and hence, can be extended to more general setting.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptiness of the intersection of certain family of subsets of an underlying set. One of the remarkable results on the nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in 1929 [7], which was concerned with certain types of multimaps called the KKM maps later.

The KKM theory, first named by the author in 1992, is nowadays the study of applications of various equivalent formulations of the KKM principle and their generalizations. In the last two decades, the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author and his followers. Moreover, in 2006 [10], we introduced a new concept of

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abstract convex spaces which is adequate to establish the KKM theory. With this new concept, we generalized and simplified known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. For details, see [13-16] and the references therein.

Apparently motivated by the present author's works on G-convex spaces, in 2008, Alimohammady et al. [1] introduced the notion of G-convex minimal spaces and obtained the open and closed versions of the KKM principle in this new setting. Their method is just replacing the topological structure in the relevant results by the more general minimal structure as in [2].

In the same year [12], we introduced a new concept of abstract convex minimal spaces and established typical results in the KKM theory of such spaces. Since any minimal space can be made into a topological space, results on abstract convex minimal spaces can be deduced from the theory on abstract convex spaces. In this way, the KKM type theorems were used there to obtain coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces.

Recently, Allmohammady et al. [3], Darzi et al. [5], and Delavar et al. [6] dealt with some results in the KKM theory on generalized convex minimal spaces. By establishing a kind of the KKM principle in these spaces, they obtained some results on coincidence or fixed point theorems and others. Our aim in the present paper is to show that their results are consequences of corresponding ones for abstract convex minimal spaces in our previous work [12] and hence, can be extended to more general setting.

The authors of [5] claimed that generalized versions of Ky Fan's lemma, Fan-Browder fixed point theorem, Nash equilibrium theorem and some Urai type fixed point theorems in G-convex minimal spaces were given. In fact, most of their results are formal extensions of corresponding ones for G-convex spaces due to the present author. These are already extended to more general abstract convex minimal spaces in [12].

Our aim in the present paper is to show that all of the results in [5, 6] are consequences of corresponding ones in our previous paper [12] and hence, can be extended to abstract convex minimal spaces. After two preliminary sections, the main results extending Theorems 2-8 of [5] will be given. We indicate that other consequences of them in [3, 5, 6] can be also extended and improved.

2. Abstract convex spaces

In this section, we recall definitions and some basic results on abstract convex spaces given in [10, 12-15] and some of their new consequences.

A *multimap* or simply a *map* $F : X \multimap Y$ is a function from a set X into the power set $\mathcal{P}(Y)$ of a set Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^{-}(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \,|\, x \in A\}$. For any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by

$$F^{-}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a map $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a Γ-*convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

An abstract convex space with a topology on E is sometimes called *an abstract convex topological space.*

Example 2.1. Examples of abstract convex spaces were given in [10, 13-15].

For abstract convex spaces, we can define KKM maps as in [10]:

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D\rangle,$$

then G is called a KKM *map* with respect to F. A KKM *map* $G : D \multimap E$ is a KKM map with respect to the identity function 1_E.

A map $F : E \multimap Z$ is said to have the KKM *property* and called a $\mathfrak{K}$-*map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E,Z) \subset \mathfrak{KC}(E,Z) \cap \mathfrak{KO}(E,Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical.

Example 2.2. (1) Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E,Z)$ for any nonempty set Z and for any class of KKM maps $G : D \multimap Z$ with respect to F. In fact, for each $x \in E$, choose $F(x) := Z$ or let $F(x)$ contain some $z_0 \in Z$.
(2) Further examples were given in Section 5 of [10].

Definition 2.3. The *partial* KKM *principle* for an abstract convex topological space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E,E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The KKM *principle* is the statement $1_E \in \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) KKM *space* if it satisfies the (partial) KKM principle, respectively.

Example 2.3. The following are typical examples of KKM spaces. Others can be seen in [13] and the references therein.

(1) A *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a topological space E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection.

(2) A *convex space* $(X, D) = (X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\operatorname{co} D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the convex space due to Lassonde for $X = D$.

(3) An abstract convex topological space $(X, D; \Gamma)$ is called an H-*space* if $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$. If $D = X$, $(X; \Gamma)$ is called a *c*-space by Horvath.

(4) A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex topological space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$.

(5) A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Every ϕ_A-space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with $|A| = n+1$ is a KKM space which is not G-convex; see [16].

Note that each of the above examples has a large number of concrete examples. Note also that [13] contains some incorrectly stated statements such as (VI), Theorem 4, (XVI), and (XVII). These can be corrected easily.

Recently, Kulpa and Szymanski [8] found some partial KKM spaces which are not KKM spaces.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space
$\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

3. Abstract convex minimal spaces

Definition 3.1. ([1,2]) A family $\mathcal{M} \subset \mathcal{P}(X)$ is called a *minimal structure* on a set X if $\emptyset, X \in \mathcal{M}$. In this case, $(X, \mathcal{M})$ is called a *minimal space*. Any element of $\mathcal{M}$ is called an *m-open set* of X and a complement of an m-open set is called an *m-closed set* of X. For minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a function $f : X \to Y$ is said to be *continuous* (more precisely, *m-continuous* or $(\mathcal{M}, \mathcal{N})$-*continuous*) if $f^{-1}(V) \in \mathcal{M}$ for each $V \in \mathcal{N}$.

From now on, an abstract convex space $(E, D; \Gamma)$ with a minimal structure on E will be called *an abstract convex minimal space*.

Example 3.1. (1) Any topological space is a minimal space and not conversely. However, any minimal space can be made into a topological space; see Proposition 1 below.

(2) Any t.v.s. is a minimal vector space. There is some linear minimal space which is not a t.v.s. [1].

(3) A *generalized convex minimal space* or a G-*convex minimal space* $(X, D; \Gamma)$ consists of a minimal space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. See [1].

(4) A G-convex space is a G-convex minimal space, and the converse does not hold; for an example, see [1].

(5) A minimal ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consisting of a minimal space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D\rangle$ with $|A| = n+1$ is an abstract convex minimal space by putting $\Gamma_A := \phi_A(\Delta_n)$; see [11].

It is obvious that basic facts on G-convex spaces (e.g. in [9]) can be extended to corresponding ones on G-convex minimal spaces. However, we have the following:

Proposition 3.1. ([12]) (i) *A minimal space* $(X, \mathcal{M})$ *can be made into a topological space* $(X, \mathcal{T})$.
(ii) *A continuous map* $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ *between minimal spaces can be regarded as a continuous map between the corresponding topological spaces.*

Proposition 3.2. ([12]) *A minimal* ϕ_A*-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ *with a minimal space* $(X, \mathcal{M})$ *can be regarded as the one with a topological space* $(X, \mathcal{T})$.

Consequently, a G-convex minimal space can be made into a G-convex space. This implies that the study of G-convex minimal spaces is not essential.

Definition 3.2. Let $(E, D; \Gamma)$ be an abstract convex space and $(Z, \mathcal{M})$ a minimal space. Then an $m\mathfrak{KC}$-map is defined for m-closed-valued maps G, and an $m\mathfrak{KO}$-map for m-open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset m\mathfrak{KC}(E, Z) \cap m\mathfrak{KO}(E, Z).$$

For a KKM map on a G-convex minimal space, the following is known:

Proposition 3.3. ([1, Theorems 3.2 and 3.5]) *Let* $(E, D; \Gamma)$ *be a G-convex minimal space and* $F : D \multimap E$ *a* KKM *map with m-closed* [*resp., m-open*] *values. Then* $\{F(z)\}_{z\in D}$ *has the finite intersection property.*

Usually, a KKM type theorem is a claim $1_E \in \mathfrak{K}(E, E)$ for an abstract convex space $(E, D; \Gamma)$. There are a large number of works on various forms of the KKM type theorems for convex spaces, H-spaces, or G-convex spaces and their applications. See Section 5 of [10] and the references at the end.

Definition 3.3. For an abstract convex minimal space $(E, D; \Gamma)$, the mKKM *principle* is the statement $1_E \in m\mathfrak{KC}(E, E) \cap m\mathfrak{KO}(E, E)$, and the *partial* mKKM *principle* the statement $1_E \in m\mathfrak{KC}(E, E)$.

A *minimal* KKM *space* (or simply, mKKM *space*) is an abstract convex minimal space satisfying the mKKM principle. A *minimal partial KKM space* (or simply, *partial* mKKM *space*) is an abstract convex minimal space satisfying the partial mKKM principle.

Example 3.2. In view of Proposition 3.3, a G-convex minimal space is an mKKM-space. The converse does not hold; see [11].

Proposition 3.3 can be extended as follows:

Proposition 3.4. *A minimal ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ is an mKKM space.*

Proof. By Proposition 3.2, any ϕ_A-space with a minimal space $(X, \mathcal{M})$ can be regarded as the one with a topological space $(X, \mathcal{T})$. Let $\Gamma : \langle D \rangle \multimap X$ be defined by $\Gamma_A := \phi_A(\Delta_{|A|-1})$ for each $A \in \langle D \rangle$. Then $(X, D; \Gamma)$ is a KKM space as in [16]. Hence the proposition holds. □

4. The KKM type theorems in abstract convex spaces

We begin with the following in [12, Theorem 1]:

Theorem 4.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ a map. Then $F \in \mathfrak{K}(E, Z)$ iff for any map $G : D \multimap Z$ satisfying*

$$F(\Gamma_N) \subset G(N) \quad \textit{for any } N \in \langle D \rangle,$$

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Remark 4.1. If Z has any minimal structure and if $F \in m\mathfrak{KO}(E, Z)$ [resp., $F \in m\mathfrak{KC}(E, Z)$], then we have to assume G is m-open-valued [resp., m-closed-valued].

Definition 4.1. ([1]) A subset K of a minimal space $(Z, \mathcal{M})$ is said to be *m-compact* if any family $\{A_\alpha\}$ of m-open sets such that $K \subset \bigcup_\alpha A_\alpha$ has a finite subfamily $\{A_{\alpha_i}\}$ such that $K \subset \bigcup_i A_{\alpha_i}$.

For a subset A of a minimal space $(Z, \mathcal{M})$, let $\operatorname{Int} A = m\text{-}\operatorname{Int} A := \bigcup\{U \in \mathcal{M} \mid U \subset A\}$ and $\overline{A} = m\text{-}\operatorname{Cl} A := \bigcap\{V \mid A \subset V,\ V^c \in \mathcal{M}\}$. Note that $\overline{A}$ is m-closed if and only if arbitrary union of m-open sets is m-open [17].

Under an additional requirement, we have the whole intersection property for the map-values of a KKM map as in [12, Corollary 1.2]:

Theorem 4.2. *Let $(E, D; \Gamma)$ be a partial mKKM space and $G : D \multimap E$ a map satisfying*

(1) G has m-closed values, and

(2) G is a KKM map.

Then $\{G(y)\}_{y \in D}$ has the finite intersection property.

Further if

(3) $\bigcap_{z\in M} G(z)$ *is m-compact for some* $M \in \langle D\rangle$,
then we have

$$\bigcap_{y\in D} G(y) \neq \emptyset.$$

Note that, if $(E, D; \Gamma)$ is a G-convex minimal space, then Theorem 4.2 reduces to [5, Theorem 2].

Modifying the definition in [3], for an abstract convex minimal space $(E, D; \Gamma)$, we define as follows:

Definition 4.2. A map $G : D \multimap E$ has *intersectionally m-closed values* if

$$\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$$

and *transfer m-closed values* if

$$\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y).$$

Corollary 4.1. *Let* $(E, D; \Gamma)$ *be a partial mKKM space, and* $G : D \multimap E$ *a map satisfying*
(1) G *has intersectionally m-closed values,*
(2) $\overline{G}$ *has m-closed values,*
(3) $\overline{G}$ *is a KKM map, and*
(4) $\bigcap_{z\in M} \overline{G(z)}$ *is m-compact for some* $M \in \langle D\rangle$.
Then we have

$$\bigcap_{y\in D} G(y) \neq \emptyset.$$

Proof. Since $\overline{G}$ satisfies requirements of G in Theorem 4.2, we have $\emptyset \neq \bigcap_{y\in D} \overline{G}(y)$. This implies $\overline{\bigcap_{y\in D} G(y)} \neq \emptyset$ by (1). Hence the conclusion holds. □

Corollary 4.1 reduces to [12, Corollary 1.3] whenever G is a transfer m-closed map. Consequently, it contains results in [9] for G-convex spaces, [10, Proposition 5] for abstract convex spaces, and [1, Theorem 3.3] for G-convex minimal spaces.

The following is [12, Corollary 1.4]:

Theorem 4.3. *Let* $(E, D; \Gamma)$ *be an abstract convex minimal space with the identity map* $1_E \in m\mathfrak{KO}(E, E)$, *and* $G : D \multimap E$ *a map satisfying*
(1) G *has m-open values, and*
(2) G *is a KKM map.*

Then $\{G(y)\}_{y\in D}$ *has the finite intersection property.*

Further if

(3) $\bigcap_{z\in M}\overline{G(z)}$ *is m-compact for some* $M\in\langle D\rangle$, *and*

(4) $\overline{G}$ *has m-closed values,*

then we have

$$\bigcap_{y\in D}\overline{G(y)}\neq\emptyset.$$

Note that Theorems 4.2 and 4.3 hold for an mKKM space $(E,D;\Gamma)$ and hence for any G-convex minimal space. Therefore, Theorem 4.3 reduces to [5, Theorem 3]. Moreover, note that [5, Corollary 1] simply tells that any G-convex minimal space is an mKKM space.

5. Coincidence and fixed point theorems

In the KKM theory, there exist some basic results from which we can deduce several equivalent formulations that can be used to applications; see [9, 13-15]. In this section, we introduce some of such basic results.

For abstract convex spaces, we have the following coincidence theorem [12, Theorem 3]:

Lemma 5.1. *Let* $(E,D;\Gamma)$ *be an abstract convex space,* Z *a set,* $S:D\multimap Z$, $T:E\multimap Z$ *maps, and* $F\in\mathfrak{K}(E,Z)$. *Suppose that*

(1) *for each* $z\in F(E)$, $\mathrm{co}_\Gamma S^-(z)\subset T^-(z)$ *[that is,* $T^-(z)$ *is* Γ*-convex relative to* $S^-(z)$*]; and*

(2) $F(E)\subset S(N)$ *for some* $N\in\langle D\rangle$.

Then there exists an $\bar{x}\in E$ *such that* $F(\bar{x})\cap T(\bar{x})\neq\emptyset$.

Remark 5.1. If Z has a minimal structure and S has m-open [resp., m-closed] values, then S has relatively m-open [resp., m-closed] values in $F(E)$. Then we can assume $F\in m\mathfrak{KC}(E,Z)$ [resp., $F\in m\mathfrak{KO}(E,Z)$].

The following is an immediate consequence of Lemma 5.1:

Theorem 5.1. *Suppose that* $(E,D;\Gamma)$ *is an abstract convex minimal space,* $(Z,\mathcal{M})$ *is a minimal space,* $S:D\multimap Z$, $T:E\multimap Z$ *and* $F\in m\mathfrak{KC}(E,Z)$ *are multimaps satisfying*

(a) S *has m-open values;*

(b) *for each* $z\in F(E)$, $\mathrm{co}_\Gamma S^-(z)\subset T^-(z)$;

(c) $Z=S(N)$ *for some* $N\in\langle D\rangle$.

Then F *and* T *have a coincidence point, that is, the conclusion of Lemma* 5.1 *holds.*

Note that, if $(E, D; \Gamma)$ is a G-convex minimal space, then Theorem 5.1 reduces to [5, Theorem 4].

From Lemma 5.1, we also have the following:

Theorem 5.2. *Suppose that $(E, D; \Gamma)$ is an abstract convex minimal space, $(Z, \mathcal{M})$ is a minimal space, $S : D \multimap Z$, $T : E \multimap Z$ and $F \in m\mathfrak{KO}(E, Z)$ are multimaps satisfying*
(a) *S has m-closed value;*
(b) *for each $z \in F(E)$, $\mathrm{co}_\Gamma S^-(z) \subset T^-(z)$;*
(c) *$Z = S(N)$ for some $N \in \langle D \rangle$.*
Then F and T have a coincidence point.

Note that, if $(E, D; \Gamma)$ is a G-convex minimal space, then Theorem 5.2 reduces to [5, Theorem 5]. Moreover, [5, Corollary 6] can be improved as for Theorem 5.2.

From Theorems 5.1 and 5.2 with $E = Z$ and $F = 1_E$, we have the following prototype [12, Corollary 3.1] of the Fan-Browder fixed point theorem [4]:

Corollary 5.1. *Let $(E, D; \Gamma)$ be an mKKM space, and $G : E \multimap D$, $H : E \multimap E$ maps satisfying*
(1) *for each $x \in E$, $\mathrm{co}_\Gamma G(x) \subset H(x)$; and*
(2) *$E = G^-(N)$ for some $N \in \langle D \rangle$.*
(3) *G^- has m-open [resp., m-closed] values.*
Then H has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in H(\bar{x})$.

All of [5, Corollaries 2-5], which were claimed in [5] to be generalizations of our previous results, are simple consequences of Corollary 5.1 whenever $(E, D; \Gamma)$ is a G-convex minimal space.

From Lemma 5.1 with $Z = E$, $F = 1_E$, we have the following:

Theorem 5.3. *Suppose that $(E, D; \Gamma)$ is an mKKM space and $S : D \multimap E$ and $T : E \multimap E$ are two multimaps satisfying*
(a) *S has m-open [resp., m-closed] values;*
(b) *for each $y \in E$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$;*
(c) *$T(E) \subset S(N)$ for $N \in \langle D \rangle$;*
(d) *T^- has nonempty values.*
Then T has a fixed point.

Proof. For each $x \in E$, by (d), there exists $y \in T^-(x)$ or $x \in T(y)$. Therefore $E = T(E)$ and hence $E \subset S(N)$ by (c). This shows that condition (2) of Lemma 5.1 holds. Since other requirements of Lemma 5.1 holds, the conclusion follows. □

Note that [5, Theorem 6] is a particular form of Theorem 5.3 for a G-convex minimal space $(E, D; \Gamma)$.

From Corollary 5.1, we deduced some new forms of the Fan-Browder type fixed point theorems as follows [12, Corollary 3.3]:

Theorem 5.4. *Let $(E, D; \Gamma)$ be an mKKM space and $S : E \multimap D$, $T : E \multimap E$ maps such that*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and*

(2) *there exist $D' := \{y_1, y_2, \ldots, y_n\} \in \langle D \rangle$ and m-open [resp., m-closed] subsets $\{G_i\}_{i=1}^n$ of E such that*

$$E = \bigcup_{i=1}^{n} G_i \quad \text{and} \quad G_i \subset S^-(y_i) \quad \text{for each } i.$$

Then T has a fixed point $x_* \in E$.

Note that [5, Theorem 7] is a particular form of Theorem 5.4 for a G-convex minimal space $(E, D; \Gamma)$ and $S = T$.

Modifying the definition in [3], for a minimal space X and a set Y, we define as follows:

Definition 5.1. A map $F : Y \multimap X$ has *unionly m-open values* if

$$\mathrm{Int} \bigcup_{y \in Y} F(y) = \bigcup_{y \in Y} \mathrm{Int}\, F(y);$$

and *transfer m-open values* if

$$\bigcup_{y \in Y} F(y) = \bigcup_{y \in Y} \mathrm{Int}\, F(y).$$

Corollary 5.2. *Suppose $(X, D; \Gamma)$ is an m-compact mKKM space, Y is a minimal space, $S : X \multimap D$, $F : X \multimap Y$ and $T : X \multimap X$ are multimaps such that:*

(a) *$x \in X$ and $M \in \langle S(x) \rangle$ imply that $\Gamma_M \subset T(x)$,*

(b) *$F^- : Y \multimap X$ has unionly m-open values and $F(x)$ is nonempty for each $x \in X$,*

(c) *for any $y \in Y$ there exists $z \in D$ such that $F^-(y) \subset S^-(z)$.*

Then T has a fixed point.

Proof. Note that (a) implies condition (1) of Theorem 5.4 with $E = X$. By (b), for each $x \in X$, there exists a $y \in Y$ such that $x \in F^-(y)$, and hence

$$X = \bigcup_{y \in Y} F^-(y) = \mathrm{Int} \bigcup_{y \in Y} F^-(y) = \bigcup_{y \in Y} \mathrm{Int}\, F^-(y).$$

Since X is m-compact, there exists $\{y_1, \ldots, y_n\} \subset Y$ such that $X = \bigcup_{i=1}^n \operatorname{Int} F^-(y_i)$. By (c), there exists $\{z_1, \ldots, z_n\}$ such that $F^-(y_i) \subset S^-(z_i)$. By putting $G_i := \operatorname{Int} F^-(y_i) \subset S^-(z_i)$ for each i, condition (2) of Theorem 5.4 holds. Now the conclusion follows from Theorem 5.4. □

For a G-convex minimal space and a transfer m-open valued F, Corollary 5.2 reduces [3, 6, Theorem 2.1] which plays the key role in [3, 6]. Hence these papers can be improved by applying Corollary 5.2. But we will not follow them.

Lemma 5.2. *Suppose that $(X, D; \Gamma)$ with $\Gamma_A = \phi_A(\Delta_n)$ for $A \in \langle D \rangle$ with $|A| = n + 1$ is a minimal ϕ_A-space and D' is a nonempty subset of D. Then its Γ-convex subspace $(\mathrm{co}_\Gamma D', D'; \Gamma|_{\langle D' \rangle})$ is an mKKM space.*

Proof. In the subspace, $\Gamma_A = \phi_A(\Delta_n)$ for $A \in \langle D' \rangle$ with $|A| = n + 1$. Hence, by Proposition 3.4, the subspace is an mKKM space. □

In particular, if $(X, D; \Gamma)$ is a G-convex minimal space, so is any Γ-convex subspace by [5, Lemma 1], and hence, any subspace is an mKKM space.

Theorem 5.5. *Suppose $(E, D; \Gamma)$ is an abstract convex minimal space, $S : E \multimap D$ is a multimap and also suppose that there exist $D' = \{y_1, \ldots, y_n\} \subset D$ and nonempty m-open [resp., m-closed] subsets $G_i \subset S^-(y_i)$ for each $i = 1, \ldots, n$. If $(\mathrm{co}_\Gamma D', D'; \Gamma|_{\langle D' \rangle})$ is an mKKM space and $\mathrm{co}_\Gamma D' \subset \bigcup_{i=1}^n G_i$, then the map $\mathrm{co}_\Gamma S$ has a fixed point.*

Proof. Let $(E, D; \Gamma) := (\mathrm{co}_\Gamma D', D'; \Gamma|_{\langle D' \rangle})$. Define $G : E \multimap D$ by $G^-(y_i) := G_i$ for each i and $H : E \multimap E$ by $H(x) := \mathrm{co}_\Gamma S(x)$ for each $x \in \mathrm{co}_\Gamma D'$. Then apply Corollary 5.1 of Theorems 5.1 and 5.2. □

When $(E, D; \Gamma)$ is a G-convex minimal space, then any subspace is an mKKM space, and hence Theorem 5.5 reduces to [5, Theorem 8]. In [5], its authors claimed that [5, Theorem 8] and its Corollaries [5, Corollaries 7-11] are all extended versions of our previous results. Now their results for G-convex minimal spaces can be extended to mKKM spaces in view of our new Theorem 5.5.

Moreover, in view of Proposition 3.4, all of Theorems 4.2, 4.3, 5.1–5.5 holds for minimal ϕ_A-spaces. Finally, note that any results on minimal spaces can be deduced from the corresponding ones on topological spaces. Hence such study on minimal spaces is not essential.

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