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ON INTEGRAL ESTIMATES FOR POLAR DERIVATIVE OF A POLYNOMIAL WITH RESTRICTED ZEROS

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Abstract. If $P(z)$ is a polynomial of degree n, having all the zeros in $|z| \leq k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, Liman [5] proved that, for every $\beta \in C$ with $|\beta| \leq 1$ and for each $q > 0$,

$$n\left\|P(z) - \frac{m\beta z^n}{k^n}\right\|_q \leq \left\|1 + zS(\mu,k)\right\|_q \left\|P'(z) - \frac{m\beta}{k^n} nz^{n-1}\right\|_q,$$

where

$$S_{(\mu,k)} = \left\{\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{u-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}\right\}.$$

In this paper, we improve and extend the above inequality and related result for polar derivatives of a polynomial. Our results generalizes certain well known polynomial inequalities.

1. Introduction and Preliminaries

Let P_n be a space of polynomials of degree at most n and $P \in P_n$. Let polynomial $P(z)$ has all its zeros in $|z| \leq 1$. Then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

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The result is sharp and equality holds for $P(z) = \alpha z^n + b$, $|a| = |b|$, which is an inequality of Turan [10].

As an extension of (1.1), Malik [6] proved that, if polynomial $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\frac{n}{1+k} \max_{|z|=1} |P(z)| \leq \max_{|z|=1} |P'(z)|. \tag{1.2}$$

Later on Govil [4] improved the result and proved that, if polynomial $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Let $D_\alpha P(z)$ denote polar differentiation of the polynomial $P(z)$ of degree n with respect to α. Then

$$D_\alpha P(z) := nP(\alpha) + (\alpha - z)P^{'}(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Recently Shah [10] extended (1.3) to the polar derivative of $P(z)$ and proved that, if all the zeros of $P(z)$ lie in $|z| \leq 1$, then

$$\frac{n}{2}(|\alpha| - 1) \max_{|z|=1} |P(z)| \leq \max_{|z|=1} |D_\alpha P(z)|. \tag{1.4}$$

The result is sharp and equality holds for $P(z) = (\frac{z-1}{z})^n$.

Inequality (1.4) was later generalized and proved that, if all the zeros of $P(z)$ lie in $|z| \leq k, k \geq 1$, then for any complex number α with $|\alpha| \geq k$

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left\{ \frac{|\alpha| - k}{1+k} \right\} \max_{|z|=1} |P(z)|. \tag{1.5}$$

The result is sharp and equality holds for $P(z) = (z-k)^n$.

Malik [7] extended (1) to L_p norm by proving the following more general result. If $P(z)$ has all its zeros in closed convex unit disc $|z| \leq 1$, then for each $q > 0$

$$n\|P\|_q \leq \|1+z\|_q \max_{|z|=1} |P'(z)|, \tag{1.6}$$

where

$$\|P\|_q = \left\{ \int_0^{2\pi} |P(e^{i\theta}|^q \right\}^{\frac{1}{q}}.$$

Further, Aziz [1] improved (1.6) and generalized (1.2) by proving if $P(z)$ has all its zeros in $|z| \leq k \leq 1$, then for each $q > 0$

$$n\|P\|_q \leq \|1 + kz\|_q \max_{|z|=1} |P'(z)|. \tag{1.7}$$

Letting $q \to \infty$ in (1.6) and in (1.7) and making use of facts of analysis (see [9, p.91]) that

$$n\|P\|_q \leq \max_{0 \leq \theta \leq 2\pi} |P(e^{i\theta})|,$$

as $q \to \infty$, we get inequality (1.1) and (1.2) respectively.

Dewan et all [3] generalizes (1.5) and (1.7) and proved the following result for polar derivative of a polynomial and proved that if $P(z)$ has all its zeros in $|z| \leq k \leq 1$, then for $|\alpha| \geq k$ and for each $q > 0$

$$n\||\alpha| - k\|_q \leq \|1 + kz\|_q \max_{|z|=1} |D_\alpha P(z)|. \tag{1.8}$$

Dividing both sides of (1.8) by $|\alpha|$ and letting $|\alpha| \to \infty$ we get (1.7). If we let $q \to \infty$ in (1.8) we get (1.5).
Liman [5] considered a class of polynomials

$$P_{(n,\mu)} = a_n z^n + \sum_{\mu}^{n} a_{n-j} z^{n-j}, 1 \leq \mu \leq k$$

and proved the following four results:

Theorem 1.1. *If $P \in P_{(n,\mu)}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for any real or complex number β with $|\beta| < 1$ and each $q > 0$,*

$$n \left\| \frac{P(e^{i\theta}) - \frac{m\beta z^n}{k^n}}{P'(e^{i\theta}) - \frac{mn\beta z^{n-1}}{k^n}} \right\|_q \leq \left\| 1 + S_{(\mu,k)} z \right\|_q, \tag{1.9}$$

where

$$S_{(\mu,k)} = \left\{ \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{u-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|} \right\}. \tag{1.10}$$

Using Holder's inequality, Liman [5] proved that:

Theorem 1.2. *If $P \in P_{(n,\mu)}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$. If $m = \min_{|z|=k} |P(z)|$, then for each $q > 0$, $s > 1$, $r > 1$ with $r^{-1} + s^{-1} = 1$ and for any β with $|\beta| < 1$*

$$n\|P(e^{i\theta}) - \frac{m\beta z^n}{k^n}\|_q \leq \|1 + S_{(1,k)}z\|_{qr}\|P'(e^{i\theta}) - \frac{mn\beta z^{n-1}}{k^n}\|_{qs}, \tag{1.11}$$

where $S_{(1,k)}$ is defined in (1.10) by choosing $\mu = 1$.

Theorem 1.3. *If $P \in P_{(n,\mu)}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$. If $m = \min_{|z|=k} |P(z)|$, then for each $q > 0$ and for any β with $|\beta| < 1$,*

$$n\left\|\frac{P(z) - \frac{m\beta}{k^n}z^n}{P'(z) - \frac{m\beta}{k^n}nz^{n-1}}\right\|_q \leq \left\|1 + k^\mu z\right\|_q. \tag{1.12}$$

Theorem 1.4. *If $P \in P_{(n,\mu)}$ and $P(z)$ has all its zeros in $|z| \leq k$, where $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every complex number β with $|\beta| < 1$ and each $q > 0$, $s > 1$, $r > 1$ with $r^{-1} + s^{-1} = 1$*

$$n\left\|P(z) - \frac{m\beta z^n}{k^n}\right\|_q \leq \left\|1 + k^\mu z\right\|_{qr}\left\|P'(z) - \frac{m\beta}{k^n}nz^{n-1}\right\|_{qs}. \tag{1.13}$$

2. Lemmas

For the proofs of our main results, we need the following lemmas. The first lemma is due to Aziz and Rather [2].

Lemma 2.1. *If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ is a polynomial of degree n having all the zeros in the closed disc $|z| \leq k \leq 1$, then for $|z| = 1$, $1 \leq \mu \leq n$.*

$$|Q'(z)| \leq k^\mu |P'(z)|. \tag{2.1}$$

Lemma 2.2. *If $P \in P_{n,m}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ then for $|z| = 1$*

$$|Q'(z)| \leq S(\mu, k)|P'(z)|,$$

where

$$S(\mu, k) = \left\{\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}}\right\}.$$

Lemma 2.3. *If* $P(z) = a_n z^n + \sum\limits_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ *is a polynomial of degree* n *having all the zeros in the colsed and convex disk* $|z| \leq k \leq 1$, *then for every real or complex number* α *with* $|\alpha| \geq S_\mu$ *and* $|z| = 1$

$$|D_\alpha P(z)| \geq (|\alpha| - S_\mu)|P'(z)|.$$

Proof. Let $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then

$$|Q^{'}(z)| = |nP(z) - zP^{'}(z)|$$

for $|z| = 1$. Thus for $|z| = 1$, we have

$$|D_\alpha P(z)| = |nP(z) + (\alpha - z)P'(z)|.$$

This implies

$$|D_\alpha P(z)| \geq |\alpha||P'(z)| - |nP(z) - zP'(z)|. \tag{2.2}$$

(2.1) in conjuction with (2.2) gives

$$|D_\alpha P(z)| \geq (|\alpha| - S_\mu)|P'(z)|.$$

□

Lemma 2.4. *If* $P(z) = a_n z^n + \sum\limits_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ *is a polynomial of degree* n *having all the zeros in the colsed and convex disk* $|z| \leq k \leq n$, *then for every real or complex number* α *with* $|\alpha| \geq k^\mu$ *and* $|z| = 1$,

$$|D_\alpha P(z)| \geq (|\alpha| - k^\mu)|P'(z)|.$$

Proof. If $P(z) = a_n z^n + \sum\limits_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ is a polynomial of degree n having all the zeros in $|z| \leq k \leq 1$ and $q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ then on $|z| = 1$

$$|q'(z)| \leq S(\mu, k)|P'(z)|.$$

But

$$S(\mu, k) \leq k^\mu,$$

this implies

$$|q'(z)| \leq k^\mu |P'(z)|.$$

Now, let

$$|q'(z)| = |nP(z) - zP'(z)| \quad \text{on} \quad |z| = 1.$$

Thus on $|z| = 1$,

$$|D_\alpha P(z)| = |nP(z) + (\alpha - z)P'(z)|,$$
$$|D_\alpha P(z)| \geq |\alpha||P'(z)| - |nP(z) - zP'(z)|.$$

Using (2.1), we get,

$$|D_\alpha P(z)| \geq |\alpha||P'(z)| - k^\mu |P'(z)|,$$

therefore,

$$|D_\alpha P(z)| \geq (|\alpha| - k^\mu)|P'(z)|.$$

□

3. Main Results

In this section, we prove some results which improve and extent the above results.

Theorem 3.1. *If $P \in P_{(n,\mu)}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for each $q > 0$,*

$$n\left\|(|\alpha| - S_{(\mu,k)})\frac{P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}}{D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n e^{i(n-1)\theta}}}\right\|_q \leq \left\|1 + S_{(\mu,k)} e^{i\theta}\right\|_q, \tag{3.1}$$

where

$$S_{(\mu,k)} = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}.$$

Proof. Let $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$ and $m = \min_{|z|=k} |P(z)|$. Suppose $P(z)$ has all the zeros in $|z| \leq k$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and there is nothing to prove. Now suppose that $P(z)$ vanishes in $|z| < k$, so that $m > 0$ and we have $m \leq |P(z)|$ for $|z| = k$. Therefore for every real or complex number β with $|\beta| < 1$, we have $|\frac{m\beta z^n}{k^n}| < |P(z)|$ for $|z| = k$. By Rouches theorem, it follows that all the zeros of

$$F(z) = P(z) - \frac{m\beta z^n}{k^n}$$

also lie in $|z| < k$. By Gauss Lucas theorem, the polynomial

$$F'(z) = P'(z) - \frac{mn\beta z^{n-1}}{k^n}$$

has all its zeros in $|z| < k$.

Again $F(z) = P(z) - \frac{m\beta z^n}{k^n}$, therefore

$$G(z) = z^n \overline{F(\frac{1}{\bar{z}})} = z^n \left\{ \overline{P(\frac{1}{\bar{z}})} - \frac{m\beta}{k^n z^n} \right\} = Q(z) - \frac{m\beta}{k^n}$$

and it can be easily verified that for $|z| = 1$,

$$|F'(z)| = |nG(z) - zG'(z)|. \tag{3.2}$$

Since $F(z)$ has all its zeros in $|z| \le k \le 1$, therefore using inequality (3.2) and Lemma 2.2, we get

$$|G'(z)| \le S(\mu,k)|nG(z) - zG'(z)|, \tag{3.3}$$

where

$$S(\mu,k) = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}$$

for $|z| = 1$, $1 \le \mu \le n$. From (3.3), we conclude that the function

$$W(z) = \frac{zG'(z)}{S(\mu,k)(nG(z) - zG'(z))}$$

is analytic in $|z| \le 1$, $|W(z)| \le 1$, $|z| = 1$ and $W(0) = 0$. Thus the function $1+S(\mu,k)W(z)$ is subordinate to $1+zS(\mu,k)$ for $|z| \le k$. Hence by well known property of subordinate, [4, P. 422] for each $q > 0$ and $|z| = 1$

$$\int_0^{2\pi} |1 + S(\mu,k)W(e^{i\theta})|^q d\theta \le \int_0^{2\pi} |1 + S(\mu,k)e^{i\theta}|^q d\theta. \tag{3.4}$$

Now

$$|1 + S(\mu,k)W(z)| = \frac{n|P(z) - \frac{m\beta z^n}{k^n}|}{|P'(z) - \frac{mn\beta z^{n-1}}{k^n}|}. \tag{3.5}$$

That is,

$$n|F(z)| = |1 + S(\mu,k)W(z)||F'(z)|. \tag{3.6}$$

Inequality (3.6) in conjuction with Lemma 2.3 gives

$$n|F(z)| \le |1 + S(\mu,k)W(z)| \frac{D_\alpha F(z)}{|\alpha| - S(\mu,k)}. \tag{3.7}$$

Therefore

$$n|F(z)|(|\alpha| - S(\mu,k)) \le |1 + S(\mu,k)W(z)||D_\alpha F(z)|$$

or

$$\frac{n|F(z)|(|\alpha| - S(\mu,k))}{|D_\alpha F(z)|} \le |1 + S(\mu,k)W(z)|.$$

Integrating both sides from 0 to 2π and using (3.4), we get

$$n(|\alpha| - S(\mu,k)) \int_0^{2\pi} \left| \frac{F(z)}{D_\alpha F(z)} \right|^q d\theta \le \int_0^{2\pi} |1 + S(\mu,k)e^{i\theta}|^q d\theta.$$

This implies

$$n(|\alpha| - S(\mu,k)) \int_0^{2\pi} \left| \frac{P(z) - \frac{m\beta z^n}{k^n}}{D_\alpha(P(z) - \frac{m\beta e^{in\theta}}{k^n})} \right|^q d\theta \le \int_0^{2\pi} |1 + S(\mu,k)(\mu,k)e^{i\theta}|^q d\theta.$$

Equivalently

$$\begin{aligned} &n(|\alpha| - S(\mu,k)) \int_0^{2\pi} \left| \frac{P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}}{D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n} D_\alpha(e^{in\theta}))} \right|^q d\theta \\ &\leq \int_0^{2\pi} |1 + S(\mu,k)e^{i\theta}|^q d\theta. \end{aligned} \tag{3.8}$$

□

Choosing $\beta = 0$ in Theorem 3.1, we get the following:

Corollary 3.2. *If $P \in P_{(n,\mu)}$ and $P(z)$ has all its zeros in $|z| \leq k$, where $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for each $q > 0$,*

$$n \left\| (|\alpha| - S_{(\mu,k)}) \frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})} \right\|_q \leq \left\| 1 + S_{(\mu,k)} e^{i\theta} \right\|_q,$$

where $S_{(\mu,k)}$ is already defined in (1.10).

Taking $\mu = 1$, (3.1) reduces to

Corollary 3.3. *If $P \in P_{(n,\mu)}$ and $P(z)$ has all its zeros in $|z| \leq k$, where $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for each $q > 0$,*

$$n \left\| (|\alpha| - S_{(1,k)}) \frac{P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}}{D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n e^{i(n-1)\theta}}} \right\|_q \leq \left\| 1 + S_{(1,k)} e^{i\theta} \right\|_q,$$

where

$$S_{(1,k)} = \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + \mu|a_{n-1}|}.$$

Remark 3.4. Dividing numerator and denominator on left hand side by $|\alpha|$ and letting $|\alpha| \to \infty$ in Theorem 3.1, we get Theorem 1.1.

Theorem 3.5. *If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every real or complex number β with $|\beta| \leq 1$ and $q > 0$, $r > 1$ and $s > 1$ with*

$\frac{1}{r} + \frac{1}{s} = 1$,

$$\begin{aligned} n(|\alpha| - S_{(\mu,k)})\left\|P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}\right\|_q \\ \leq \left\|1 + S_{(\mu,k)}e^{i\theta}\right\|_{qr}\left\|D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n}D_\alpha e^{in\theta}\right\|_{qs}. \end{aligned} \tag{3.2}$$

Proof. Proceeding similarly as in the proof of above theorem, we have from (3.5) for each $q > 0$

$$n^q(|\alpha| - S(\mu,k))^q \int_0^{2\pi} |P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}|^q d\theta$$
$$\leq \int_0^{2\pi} \left\{|1 + S(\mu,k)W(e^{i\theta})||D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n}D_\alpha(e^{in\theta})|\right\}^q d\theta.$$

This gives with the help of Holders inequality for $s > 1$, $r > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$

$$\begin{aligned} n^q(|\alpha| - S(\mu,k))^q \int_0^{2\pi} \left|P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}\right|^q d\theta \\ \leq \left\{\int_0^{2\pi} |1+S(\mu,k)W(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{r}}\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n}D_\alpha(e^{in\theta})|^{qs} d\theta\right\}^{\frac{1}{s}}. \end{aligned} \tag{3.10}$$

Using inequality (3.4) with q replaced by qr in (3.10), we obtain for each $q > 0$, $s > 1$, $r > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$,

$$n^q(|\alpha| - S(\mu,k))^q \int_0^{2\pi} |P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}|^q d\theta$$
$$\leq \left\{\int_0^{2\pi} |1 + zS(\mu,k)|^{qr} d\theta\right\}^{\frac{1}{r}} \left\{\int_0^{2\pi} \left|D_\alpha P(i\theta) - \frac{m\beta}{k^n}D_\alpha(e^{in\theta})\right|^{qs} d\theta\right\}^{\frac{1}{s}}.$$

Equivalently

$$n(|\alpha| - S(\mu,k))\left\|P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}\right\|_q$$
$$\leq \left\|1 + e^{i\theta}S(\mu,k)\right\|_{qr}\left\|D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n}D_\alpha(e^{in\theta})\right\|_{qs}.$$

□

Letting $s \to \infty$ (so that $r \to 1$) in Theorem 3.5, we get

Corollary 3.6. *If* $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k} |P(z)|$, *then for every real or complex number* β *with* $|\beta| \le 1$ *and* $q > 0$,

$$n(|\alpha| - S_{(\mu,k)}) \left\| P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n} \right\|_q \le \left\| 1 + e^{i\theta} S_{(\mu,k)} \right\|_q \left\| D_\alpha P(e^{i\theta}) - \frac{m\beta}{k^n} D_\alpha e^{in\theta} \right\|_\infty .$$

Remark 3.7. If we divide by $|\alpha|$ in above inequality and making $|\alpha| \to \infty$, we obtain Theorem 1.2.

Theorem 3.8. *If* $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k} |P(z)|$, *then for every real or complex number* β *with* $|\beta| \le 1$ *and* $q > 0$,

$$n(|\alpha| - k^\mu) \left\| \frac{P(e^{i\theta}) - \frac{m\beta}{k^n} e^{in\theta}}{D_\alpha(P(e^{i\theta})) - \frac{m\beta}{k^n} D_\alpha(e^{in\theta})} \right\|_q \le \left\| 1 + k^\mu e^{i\theta} \right\|_q . \tag{3.11}$$

Proof. Let $F(z) = P(z) - \frac{m\beta z^n}{k^n}$, then

$$G(z) = Q(z) - \frac{m\beta}{k^n}.$$

It can be easily verified that for $|z| = 1$,

$$|F'(z)| = |nG(z) - zG'(z)|. \tag{3.12}$$

Since $F(z)$ has all its zeros in $|z| \le k \le 1$, therefore using inequality (6) in Lemma 2.2 we get

$$|G'(z)| \le k^\mu |nG(z) - zG'(z)| \tag{3.13}$$

for $|z| = 1$, $1 \le \mu \le n$. From (3.8), we conclude that the function

$$W(z) = \frac{zG'(z)}{k^\mu \left\{ nG(z) - zG'(z) \right\}},$$

which is analytic in $|z| \le 1$, $|W(z)| \le 1$, $|z| = 1$ and $W(0) = 0$. Thus the function $1 + k^\mu W(z)$ is subordinate to $1 + k^\mu z$ for $|z| \le k$. Hence by the well known property of subordination [4], we have for each $q > 0$,

$$\int_0^{2\pi} |1 + k^\mu W(e^{i\theta})|^q d\theta \le \int_0^{2\pi} |1 + k^\mu e^{i\theta}|^q d\theta. \tag{3.14}$$

Now

$$|1+k^{\mu}W(z)| = \frac{n|P(z)-\frac{m\beta}{k^n}z^n|}{|P'(z)-\frac{mn\beta}{k^n}z^{n-1}|}$$

this implies

$$|1+k^{\mu}W(z)| = n\frac{|F(z)|}{|F'(z)|}. \tag{3.15}$$

Using Lemma 2.4, we get

$$n(|\alpha|-k^{\mu})\int_0^{2\pi}\left|\frac{P(e^{i\theta})-\frac{m\beta}{k^n}e^{in\theta}}{D_\alpha(P(e^{i\theta}))-\frac{m\beta}{k^n}D_\alpha(e^{in\theta})}\right|^q d\theta \le \int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^q d\theta.$$

□

For $\beta = 0$, in Theorem 3.3, we have

Corollary 3.9. *If* $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k}|P(z)|$, *then for every* $q > 0$,

$$n(|\alpha|-k^{\mu})\left\|\frac{P(e^{i\theta})}{D_\alpha(P(e^{i\theta}))}\right\|_q \le \left\|1+k^{\mu}e^{i\theta}\right\|_q.$$

Choosing $\mu = 1$ in Corollary 3.9, we obtain

Corollary 3.10. *If* $P(z) = a_n z^n + \sum_{j=1}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k}|P(z)|$, *then for every* $q > 0$,

$$n(|\alpha|-k)\left\|\frac{P(e^{i\theta})}{D_\alpha(P(e^{i\theta}))}\right\|_q \le \left\|1+ke^{i\theta}\right\|_q.$$

Remark 3.11. Dividing numerator and denominator by $|\alpha|$ and letting $|\alpha| \to \infty$ in Theorem 3.8, we get Theorem 1.3.

Theorem 3.12. *If* $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k}|P(z)|$, *then for every real or complex number* β *with* $|\beta| \le 1$ *and* $q > 0$, $r > 1$, $s > 1$ *with*

$\frac{1}{r}+\frac{1}{s}=1$,

$$n(|\alpha|-k^{\mu})\left\|P(e^{i\theta})-\frac{m\beta e^{in\theta}}{k^n}\right\|_q \leq \left\|1+k^{\mu}e^{i\theta}\right\|_{qr}\left\|D_\alpha P(e^{i\theta})-\frac{m\beta}{k^n}D_\alpha(e^{in\theta})\right\|_{qs}. \tag{3.16}$$

Proof. Proceeding on the similar lines as in the proof of Theorem 3.3 and using Lemma 2.3

$$|D_\alpha P(z)| \geq (|\alpha|-k^{\mu})|P'(z)|,$$

$$n^q(|\alpha|-k^{\mu})\int_0^{2\pi}|Pe^{i\theta}-\frac{m\beta}{k^n}e^{in\theta}|^q d\theta$$
$$\leq \left\{\int_0^{2\pi}|1+k^{\mu}e^{in\theta}|^{qr}d\theta\right\}^{\frac{1}{r}}\left\{\int_0^{2\pi}|D_\alpha P(z)-\frac{m\beta}{k^n}D_\alpha(z^n)|^{qs}d\theta\right\}^{\frac{1}{s}}.$$

Equivalently,

$$n(|\alpha|-k^{\mu})\left\|P(e^{i\theta})-\frac{m\beta e^{in\theta}}{k^n}\right\|_q \leq \left\|1+k^{\mu}e^{i\theta}\right\|_{qr}\left\|D_\alpha P(e^{i\theta})-\frac{m\beta}{k^n}D_\alpha(e^{in\theta})\right\|_{qs}.$$

Hence this completes proof of Theorem 3.12. □

Letting $s\to\infty$ (so that $r\to 1$), Theorem 3.12, yields

Corollary 3.13. *If* $P(z)=a_nz^n+\sum_{j=\mu}^{n}a_{n-j}z^{n-j}$, $1\leq\mu\leq n$, *is a polynomial of degree* n *having all its zeros in* $|z|\leq k\leq 1$ *and* $m=\min_{|z|=k}|P(z)|$, *then for every real or complex number* β *with* $\beta\leq 1$ *and* $q>0$,

$$n(|\alpha|-k^{\mu})\left\|P(e^{i\theta})-\frac{m\beta e^{in\theta}}{k^n}\right\|_q \leq \left\|1+k^{\mu}e^{i\theta}\right\|_{q}\left\|D_\alpha P(e^{i\theta})-\frac{m\beta}{k^n}D_\alpha(e^{in\theta})\right\|_{\infty}.$$

Remark 3.14. Finally, If we divide by $|\alpha|$ in above inequality and making $|\alpha|\to\infty$, we obtain Theorem 1.4.

References

[1] A. Aziz, *Integral mean Estimates for Polynomials with Restricted Zeros*, Journal of Approximation Theory, **55** (1998), 232–239.

[2] A. Aziz and N.A. Rather, *Some Zygmund type Lq inequalities for polynomials*, J. Math. Anal. and Appl., **289** (2004), 14–29.

[3] K.K. Dewan, N. Singh, A. Mir and A. Bhat, *Some inequalities for the polar derivatives of a polynomial*, South East Asian Bulletin of Mathematics, **34** (2010), 69–77.

[4] N.K. Govil, *On derivative of a Polynomial*, Proc. Amer. Math. Soc., **41** (1973), 543–546.

[5] A. Liman, Ph. D Thesis, University of Kashmir, April 2006.

[6] M.A. Malik, *On the derivative of a polynomial*, J. London. Math. Soc., **1** (1969), 57–60.

[7] M.A Malik, *An integral mean estimate for polynomials*, Proc. Amer. Math. Soc., **91** (1984), 281-284.

[8] W.M. Shah, *A generalization of a theorem of paul Turan*, J. Ramanujan Math. Soc., **1** (1996), 67–72.

[9] A.E. Taylor, Introduction to Functional Analysis, John wiley and Sons Inc., New York, 1958.

[10] P. Turan, *Uber die ableitung von Polynomen*, Composition Math., **7** (1939), 89–95.