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CERTAIN NEW RESULTS ON RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Abstract. Let R_n be the space of rational functions with prescribed poles. If $r \in R_n$, does not vanish in $|z| < k$, then for $k = 1$

$$|r'(z)| \leq \frac{|B'(z)|}{2} \sup_{z \in T} |r(z)|,$$

where $B(z)$ is the Blaschke product.

In this paper, we consider a more general class of rational functions $rof \in R_{m^\star n}$, defined by

$$(rof)(z) = r(f(z)),$$

where $f(z)$ is a polynomial of degree $m^\star$ and prove a more general result of the above inequality for $k > 1$.

We also prove that

$$\sup_{z \in T} \left[\left| \frac{r^{*\prime}(f(z))}{B'(z)} \right| + \left| \frac{r'(f(z))}{B'(z)} \right| \right] = \sup_{z \in T} \left| \frac{(rof)(z)}{f'(z)} \right|,$$

and as a consequence of this result, we present a generalization of a theorem of O'Hara and Rodriguez for self-inverse polynomials. Finally, we establish a similar result when supremum is replaced by infimum for a rational function which has all its zeros in the unit circle.

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1. Introduction

Let $\mathbb{P}_n$ be the space of complex polynomials of degree at most n and $\mathbb{C}$ be the complex plane. Let $T = \{z \in \mathbb{C} : |z| = 1\}$ and D^- denotes the region inside of T and D^+ denotes the region outside of T.

Let $p \in \mathbb{P}_n$. Then concerning the estimate of p' on T, we have by a famous result due to Bernstein [4]:

$$\sup_{z\in T} |p'(z)| \leq n \sup_{z\in T} |p(z)|. \tag{1.1}$$

This result is sharp and equality holds for the polynomials having all zeros at origin.

As a refinement of result of Bernstein [4], we mention the following result due to Aziz [2];

Theorem 1.1. ([2]) *If* $p \in P_n$ *and* $p^*(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, *then*

$$\sup_{z\in T}[|(p^*(z))'| + |p(z)|] = n \sup_{z\in T} |p(z)|. \tag{1.2}$$

The next result was conjuctured by Erdös and later verified by Lax [6]: If $p \in \mathbb{P}_n$, and all the zeros of $p(z)$ lie in TUD^+, then for $z \in T$, we have

$$\sup_{z\in T} |p'(z)| \leq \frac{n}{2} \sup_{z\in T} |p(z)|.$$

Equality holds for $p(z) = \lambda z^n + \mu$, $|\lambda| = |\mu| = 1$.

Let $a_1, a_2, ..., a_n$ be n given points in D^+. We consider the following space of rational functions R_n with prescribed poles:

$$R_n := R_n(a_1, a_2, ..., a_n) = \left[\frac{p(z)}{w(z)} : p \in P_n\right],$$

where

$$w(z) = (z - a_1)(z - a_2)...(z - a_n).$$

Let

$$B(z) := \frac{z^n\overline{w(1/\bar{z})}}{w(z)} = \prod_{k=1}^{n} \frac{1 - \bar{a}_k z}{z - a_k},$$

where $B(z) \in R_n$ is called Blaschke product. Note that $|B(z)| = 1$, when $z \in T$.

Li, Mohapatra and Rodriguez [8] extended Bernstein inequality to rational functions $r \in R_n$ with prescribed poles $a_1, a_2, ..., a_n$ and replaced z^n by Blaschke product $B(z)$ and proved:

Theorem 1.2. ([8]) *If $z \in T$, then for any $r \in R_n$,*

$$|r'(z)| \leq |B'(z)| \sup_{z\in T} |r(z)|.$$

Further more, the inequality is sharp and equality holds for $r(z) = \alpha B(z)$ with $|\alpha| = 1$.

The next result is due to Borwein and Erdelyi [4]:

Theorem 1.3. ([4]) *If $r \in R_n$ and all the zeros of $r(z)$ lie in TUD^+, then for $z \in T$*

$$|r'(z)| \leq \frac{|B'(z)|}{2} \sup_{z\in T} |r(z)|.$$

Equality holds for $r(z) = \lambda B(z) + \mu$, $|\lambda| = |\mu| = 1$.

Now, we consider a class of rational functions $r(f(z))$ defined by

$$(rof)(z) = r(f(z)) = \frac{p(f(z))}{w(f(z))},$$

where $f(z)$ is a polynomial of degree $m^\star$ and $r(z)$ is a rational function of degree n, so that $rof \in R_{m^\star n}$, and

$$w(f(z)) = \prod_{j=1}^{m^\star n} (z - a_j).$$

Hence, in case of Balaschke product $B(z)$ is given by

$$B(z) := \frac{w^*(f(z))}{w(f(z))} = \frac{z^{m^\star n}\overline{w(f(\frac{1}{\bar{z}}))}}{w(f(z))} = \prod_{j=1}^{m^\star n}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right).$$

Now onwards, we shall always assume that all poles $a_1, a_2, ..., a_{m^\star n}$ of $r(f(z))$ lie in D^+. For the case when all poles are in D^-, we can obtain analogous results with suitable transformations.

2. Lemmas

The first two lemmas are due to Li, Mohapatra and Rodriguez [8].

Lemma 2.1. ([8]) *Suppose $\lambda \in T$. Then the equation $B(z) = \lambda$ has exactly n simple roots, say $t_1, t_2, ..., t_n$ and all lie on the unit circle T. Moreover*

$$\frac{t_k B'(t_k)}{\lambda} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2} \quad for\ k = 1, 2, 3, ..., n. \tag{2.1}$$

Lemma 2.2. ([8]) *If $|x| = |y| = 1$, then*

$$(x-y)^2 = -xy|x-y|^2. \tag{2.2}$$

Qasim [5] have proved:

Lemma 2.3. ([5]) *If $rof \in R_{m^\star n}$ and $z \in T$, then*

$$B'(z)r(f(z)) - f'(z)r'(f(z))[B(z) - \lambda] = \frac{B(z)}{z}\sum_{k=1}^{m^\star n} c_k r(f(t_k))\left|\frac{B(z)-\lambda}{z-t_k}\right|^2, \tag{2.3}$$

where $c_k = c_k(\lambda)$ is defined for $k = 1, 2, 3, ..., m^\star n$ by

$$c_k^{-1} = \sum_{j=1}^{m^\star n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2}. \tag{2.4}$$

Furthermore, for $z \in T$

$$\frac{zB'(z)}{B(z)} = \sum_{k=1}^{m^\star n} c_k \left|\frac{B(z)-\lambda}{z-t_k}\right|^2 \tag{2.5}$$

and also

$$|B'(z)| = \frac{zB'(z)}{B(z)} = \sum_{k=1}^{m^\star n} \frac{|a_k|^2 - 1}{|z - a_k|^2}, \tag{2.6}$$

where $t_k, k = 1, 2, 3, ..., m^\star n$ are defined in Lemma 2.1.

Aziz and Dawood [1] proved the following lemma :

Lemma 2.4. ([1]) *Let $p(z)$ be a polynomial of degree n, having all zeros in $T \cup D^-$. Then*

$$\inf_{z\in T} p'(z) \geq n \inf_{z\in T} p(z).$$

Next Lemma has been proved by Idrees Qasim [5]:

Lemma 2.5. ([5]) *Let $r(f(z)) \in R_{m^\star n}$. Then for $z \in T$,*

$$|r'(f(z))| + |r^{*\prime}(f(z))| \leq \frac{|B'(z)|}{m^\star m} \sup_{z\in T} |r(f(z))|, \tag{2.7}$$

where $r^*(f(z)) = B(z)\overline{r(f(\frac{1}{\bar{z}}))}$. The result is sharp and equality holds for $r(f(z)) = aB(z)$ with $a \in T$, where $f(z) = z^{m^\star}$.

Now, we shall prove the following lemma:

Lemma 2.6. *Let $r(f(z)) \in R_{m^\star n}$. Then for $z \in T$,*

$$Re\frac{z((wof)(z))'}{(wof)(z)} = \frac{n - |B'(z)|}{2}$$

and

$$Re\frac{z((wof)^*(z))'}{(wof)^*(z)} = \frac{n + |B'(z)|}{2}.$$

Proof. We have

$$(wof)^*(z) = z^n\overline{(wof)(1/\bar{z})}.$$

Also

$$z(wof)^*(z))' = nz^n\overline{(wof)(1/\bar{z})} - z^{n-1}f'(1/\bar{z})\overline{(w'(f(1/\bar{z})}.$$

Now for $z \in T$, we have

$$\frac{z((wof)^*(z))'}{(wof)^*(z)} = n - \frac{\overline{w'(f(z))f'(z)}}{\overline{(wof)(z)}}.$$

This gives

$$Re\frac{z((wof)^*(z))'}{(wof)^*(z)} + Re\frac{w'(f(z))f'(z)}{(wof)(z)} = n. \tag{2.8}$$

Also we know that

$$B(z) := \frac{w^*(f(z))}{w(f(z))} = \frac{z^{m^\star n}\overline{w(f(\frac{1}{\bar{z}}))}}{w(f(z))} = \prod_{j=1}^{m^*n}\left(\frac{1 - \bar{a}_j z}{z - a_j}\right).$$

Since $\frac{zlB'(z)}{B(z)}$ is a positive real number, this gives

$$\frac{zB'(z)}{B(z)} = \sum_{k=1}^{m^*n} z[\frac{-\bar{a}_j}{1 - \bar{a}_j z} - \frac{1}{z - a_j}] = \sum_{k=1}^{m^*n}\frac{(|a_j|^2 - 1)}{(\bar{z} - \bar{a}_j)(z - a_j)}$$

and

$$|B'(z)| = \frac{zB'(z)}{B(z)} = \sum_{k=1}^{m^\star n}\frac{|a_k|^2 - 1}{|z - a_k|^2}.$$

Again

$$\frac{zB'(z)}{B(z)} := \frac{z(\frac{(wof)^*(z)}{(wof)(z)})'}{(\frac{(wof)^*(z)}{(wof)(z)})} = z\frac{((wof)^*(z))'}{(wof)^*(z)} - \frac{((wof)(z))'}{(wof)(z)}.$$

Equivalently

$$\frac{zB'(z)}{B(z)} = z\frac{((wof)^*(z))'}{(wof)^*(z)} - \frac{(w'(f(z)))f'(z)}{(wof)(z)}.$$

Therefore, for $z \in T$, we get

$$|B'(z)| = Re\left[z\frac{((wof)^*(z))'}{(wof)^*(z)}\right] - Re\left[\frac{(w'(f(z)))f'(z)}{(wof)(z)}\right]. \tag{2.9}$$

From (2.8) and (2.9), we obtain for $z \in T$,

$$Re\frac{z((wof)(z))'}{(wof)(z)} = \frac{n - |B'(z)|}{2}$$

and

$$Re\frac{z((wof)^*(z))'}{(wof)^*(z)} = \frac{n + |B'(z)|}{2}.$$

This completes the proof. □

3. Main results

We first prove the following generalization of Theorem 1.3:

Theorem 3.1. *Let $rof \in R_{m^*n}$ and all zeros of $r(f(z))$ lie in $T_k \cup D_k^+$, where $k \geq 1$, T_k is circle of radius k (centre origin) and D_k^+ represents exterior of T_k. Then for $z \in T$,*

$$|r'(f(z))| \leq \frac{1}{2}\left[\frac{|B'(z)|}{mm^*} + \frac{n|rof|^2}{(\sup_{z\in T}|r(f(z))|)^2|(f'(z))^2|}\left(\frac{2m^*}{k+1} - 1\right)mm^*\right] \times \sup_{z\in T}|r(f(z))|,$$

where $m = \inf_{z\in T}|f(z)|$.

Proof. Let $(rof)(z) = \frac{(pof)(z)}{(wof)(z)} \in R_{m^*n}$. If $b_1, b_2, ..., b_{m^*n^*}$ are all zeros of $(pof)(z)$, then $m^*n^* \leq m^*n$, $|b_j| \geq k > 1$, $j = 1, 2, 3, ..., m^*n^*$ and we have

$$\begin{aligned} z(\frac{((rof)(z))'}{(rof)(z)} &= z\frac{(\frac{(pof)(z))}{(wof)(z)})'}{\frac{(pof)(z)}{(wof)(z)}} \\ &= z\left[\frac{p'(f(z))f'(z)}{(pof)(z)} - \frac{zw'(f(z))f'(z)}{(wof)(z)}\right] \\ &= \sum_{k=1}^{m^*n^*}\frac{z}{z-b_j} - \frac{zw'(f(z))f'(z)}{(wof)(z)}. \end{aligned}$$

For $z \in T$, this gives, with the help of Lemma 2.6, that

$$Re\left[z(\frac{((rof)(z))'}{(rof)(z)}\right] = Re\sum_{k=1}^{m^*n^*}\frac{z}{z-b_j} - Re\frac{zw'(f(z))f'(z)}{(wof)(z)}.$$

$$Re\left[z\frac{((rof)(z))'}{(rof)(z)}\right] = Re\sum_{k=1}^{m^*n^*}\frac{z}{z-b_j} - \left[\frac{n-|B'(z)|}{2}\right]. \tag{3.1}$$

Now it can be easily verified that for $z \in T$, $|b| \geq k > 1$, since $Re(z) \leq |z|$

$$Re\left(\frac{z}{z-b}\right) \leq \frac{1}{k+1}.$$

Using this in (3.1), we get for $z \in T$,

$$Re\left[z\frac{((rof)(z))'}{(rof)(z)}\right] \leq \sum_{k=1}^{m^*n^*}\frac{1}{k+1} - \left[\frac{n-|B'(z)|}{2}\right]$$

and

$$Re\left[z\frac{((rof)(z))'}{(rof)(z)}\right] \leq \frac{m^*n^*}{k+1} - \left[\frac{n-|B'(z)|}{2}\right].$$

Since $m^*n^* \leq m^*n)$, this gives

$$Re\left[z\frac{((rof)(z))'}{(rof)(z)}\right] \leq \frac{|B'(z)|}{2} + \frac{m^*n}{k+1} - \frac{n}{2}$$

or

$$Re\left[z\frac{((rof)(z))'}{(rof)(z)}\right] \leq \frac{|B'(z)|}{2} + n\left[\frac{m^*}{k+1} - \frac{1}{2}\right].$$

Since

$$\Big((rof)^*(z)\Big)' = \left[B'(z)\overline{\Big((rof)(\frac{1}{\bar{z}})\Big)}\right]$$

and

$$z\frac{\Big((rof)^*(z)\Big)'}{(rof)(z)} = \left[|B'(z)| - \frac{r'(f(z))f'(z)}{(rof)(z)}\right]B(z),$$

we have

$$\begin{aligned}\left|z\frac{\Big((rof)^*(z)\Big)'}{(rof)(z)}\right|^2 &= \left||B'(z)| - \frac{r'(f(z))f'(z)}{(rof)(z)}\right|^2 \\ &= |B'(z)|^2 + \left|\frac{r'(f(z))f'(z)}{(rof)(z)}\right|^2 \\ &\quad - 2|B'(z)|Re\left[\frac{zr'(f(z))f'(z)}{(rof)(z)}\right] \\ &\geq |B'(z)|^2 + \left|\frac{r'(f(z))f'(z)}{(rof)(z)}\right|^2 \\ &\quad - |B'(z)|\left[|B'(z)| + n\left(\frac{2m^*}{k+1} - 1\right)\right] \\ &\geq \left|\frac{r'(f(z))f'(z)}{(rof)(z)}\right|^2 - n\left(\frac{2m^*}{k+1} - 1\right)|B'(z)|.\end{aligned}$$

This implies, for $\in T$,

$$|((rof)^*(z))'| \geq \left[|r'(f(z))f'(z)|^2 - n|(rof)(z)|^2\left(\frac{2m}{k+1} - 1\right)|B'(z)|\right]^{\frac{1}{2}}$$

and

$$\begin{aligned}&\left[|r'(f(z))f'(z)|^2 + n|(rof)(z)|^2\left(\frac{2m^*}{k+1} - 1\right)|B'(z)|\right]^{\frac{1}{2}} + |r'(f(z))f'(z)| \\ &\leq |r^{*\prime}(f(z))f'(z)| + |r'(f(z))f'(z)|.\end{aligned}$$

Combining this with Lemma 2.5, we get

$$\begin{aligned}&\left[|r'(f(z))f'(z)|^2 - n|(rof)(z)|^2\left(\frac{2m^*}{k+1} - 1\right)|B'(z)|\right]^{\frac{1}{2}} + |r'(f(z))f'(z)| \\ &\leq |f'(z)|\frac{|B'(z)|}{mm^*}\sup_{z\in T}|r(f(z))|,\end{aligned}$$

where m^* is degree of the polynomial $f(z)$ and $m = \inf_{z\in T}|f(z)|$. This gives

$$\begin{aligned}&|r'(f(z))f'(z)|^2 + n|(rof)(z)|^2(\frac{2m^*}{k+1} - 1)|B'(z)| \\ &\leq \left[|f'(z)|\frac{|B'(z)|}{mm^*}\sup_{z\in T}|r(f(z))| - |r'(f(z))f'(z)|\right]^2.\end{aligned}$$

Equivalently

$$|r'(f(z))f'(z)|^2 - n|(rof)(z)|^2(\frac{2m^*}{k+1} - 1)|B'(z)|$$

$$\leq \left(|\frac{|f'(z)||B'(z)|}{mm^*}\sup_{z\in T}|r(f(z))|\right)^2 + |r^{'}(f(z))f^{'}(z)|^2$$
$$-2|f'(z)|\frac{|B'(z)|}{mm^*}\sup_{z\in T}|r(f(z))||r^{'}(f(z))f^{'}(z)|.$$

Which after simplification yields for $z \in T$,

$$|r^{'}(f(z))f^{'}(z)| \leq \frac{1}{2}\left[\frac{|B'(z)|}{mm^*} + \frac{n|rof|^2}{(\sup_{z\in T}|r(f(z))|)^2|(f'(z))^2|}\left(\frac{2m^*}{k+1}-1\right)mm^*\right] \times \sup_{z\in T}|r(f)|.$$

□

For $k=1$, we get the following:

Corollary 3.2. *Let $rof \in R_{m^*n}$ and all zeros of $r(f(z))$ lie in $T \cup D^+$. Then for $z \in T$,*

$$|r'(f(z))| \leq \frac{1}{2}\left[\frac{|B'(z)|}{mm^*} + \frac{n|rof|^2}{(\sup_{z\in T}|r(f(z))|)^2|(f'(z))^2|}(m^*-1)mm^*\right]\sup_{z\in T}|r(f(z))|.$$

Set $f(z)=z$, Theorem 3.1 yields the following:

Corollary 3.3. *Let $r \in R_n$ and all zeros of r lie in $T_k \cup D_k^+$, where $k \geq 1$. Then for $z \in T$,*

$$|r'(z)| \leq \frac{1}{2}\left[|B'(z)| - \frac{n(k-1)}{(k+1)}\frac{|r(z)|^2}{(\sup_{z\in T}|r(z)|)^2}\right]\sup_{z\in T}|r(z)|.$$

Equality holds for

$$r(z) = \left[\frac{(z+k)}{(z-a)}\right]^2,$$

where $a>1$, $k \geq 1$ and $B(z) = \left[\frac{(1-az)}{(z-a)}\right]^2$ and $z=1$.

Remark 3.4. For $k=1$, Corollary 3.2 reduces to Theorem 1.3.

Next, we shall prove the following result:

Theorem 3.5. *Let $rof \in R_{m^*n}$ and $r^*(f(z)) = B(z)\overline{r(f(\frac{1}{\bar{z}}))}$. Then*

$$\sup_{z\in T}\left[\left|\frac{r^{*\prime}(f(z))}{B'(z)}\right| + \left|\frac{r'(f(z))}{B'(z)}\right|\right] = \sup_{z\in T}\left|\frac{(rof)(z)}{f'(z)}\right|.$$

Furthermore, in above equation suprema on both sides is attained at the same point $z_0 \in T$.

Proof. Let $rof \in R_{m^*n}$. Then we have

$$(rof)^*(z) = B(z)\overline{r(f(\frac{1}{\bar{z}}))}.$$

Therefore,

$$\frac{((rof)^*(z))'}{(rof)^*(z)} = \frac{zB'(z)}{B(z)} - \frac{\overline{r'(f(\frac{1}{\bar{z}}))f'(\frac{1}{\bar{z}})}}{z\overline{(rof)(\frac{1}{\bar{z}})}}.$$

Since

$$|B'(z)| = \frac{zB'(z)}{B(z)},$$

for $z \in T$

$$\frac{((rof)^*(z))'}{(rof)^*(z)} = |B'(z)| - \frac{\overline{r'(f(\frac{1}{\bar{z}}))f'(\frac{1}{\bar{z}})}}{z\overline{(rof)(\frac{1}{\bar{z}})}}.$$

This gives

$$|B'(z)| = \left|\frac{((rof)^*(z))'}{(rof)^*(z)} + \frac{\overline{r'(f(\frac{1}{\bar{z}}))f'(\frac{1}{\bar{z}})}}{z\overline{(rof)(\frac{1}{\bar{z}})}}\right|$$

$$\leq \left|\frac{((rof)^*(z))'}{(rof)^*(z)}\right| + \left|\frac{\overline{zr'(f(\frac{1}{\bar{z}}))f'(\frac{1}{\bar{z}})}}{\overline{(rof)(\frac{1}{\bar{z}})}}\right|. \tag{3.2}$$

Since for $z \in T$, $|(rof)^*(z)| = |(rof)(z)|$, it follows from (2.2) that

$$|B'(z)||(rof)(z)| \leq |((rof)^*(z))'| + |r'(f(z))f'(z)|. \tag{3.3}$$

Equivalently

$$\sup_{z\in T}\left[\left|\frac{r^{*\prime}(f(z))}{B'(z)}\right| + \left|\frac{r'(f(z))}{B'(z)}\right|\right] \geq \sup_{z\in T}\left|\frac{(rof)(z)}{f'(z)}\right|. \tag{3.4}$$

From Lemma 2.5, we have

$$\sup_{z\in T}\left[\left|\frac{r^{*\prime}(f(z))}{B'(z)}\right| + \left|\frac{r'(f(z))}{B'(z)}\right|\right] \leq \sup_{z\in T}\left|\frac{(rof)(z)}{f'(z)}\right|. \tag{3.5}$$

From (3.4) and (3.5), we get

$$\sup_{z\in T}\left[\left|\frac{r^{*\prime}(f(z))}{B'(z)}\right| + \left|\frac{r'(f(z))}{B'(z)}\right|\right] = \sup_{z\in T}\left|\frac{(rof)(z)}{f'(z)}\right|.$$

This completes the proof. □

Set $f(z) = z$ in Theorem 3.5, we get the following result:

Corollary 3.6. *Let $r \in R_n$ and $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$. Then*

$$\sup_{z\in T}\left[\left|\frac{r^{*\prime}(z)}{B'(z)}\right| + \left|\frac{r'(z)}{B'(z)}\right|\right] = \sup_{z\in T}\left|\frac{r(z)}{B(z)}\right|.$$

Furthermore, in above equation suprema on both sides is attained at the same point $z_0 \in T$.

The rational function $r \in R_n$ is self-inversive if $r^*(z) = s\overline{r(\frac{1}{\bar{z}})}$, for $s \in T$. The following result, which is a generalization of Theorem 1 of [7] for self-inversive polynomials, follows from Theorem 3.5.

Corollary 3.7. *If $r \in R_n$ is self-inversive function, then*

$$2\sup_{z\in T}\left|\frac{r'(z)}{B'(z)}\right| = \sup_{z\in T}\left|\frac{r(z)}{B(z)}\right|.$$

Theorem 3.8. *Suppose $rof \in R_{m^*n}$ has m^*n zeros and all the zeros of rof lie in TUD^-. If*

$$(rof)^*(z) = B(z)\overline{r(f(\frac{1}{\bar{z}}))},$$

then for $\in T$,

$$\inf_{z\in T}\left[\left|\frac{r'(f(z))}{B'(z)}\right| - \left|\frac{r^{*\prime}(f(z))}{B'(z)}\right|\right] = \inf_{z\in T}\left|\frac{(rof)(z)}{f'(z)}\right|.$$

Moreover, in above equation infima on both sides is attained at the same point $z_0 \in T$.

Proof. Suppose all zeros m^*n of $rof \in R_{m^*n}$ lie in TUD^- and let $m_* = \inf_{z\in T}|(rof)(z)|$, then we have $m_* \leq |(rof)(z)|$, for $z \in T$. We show for any complex number α with $|\alpha| < 1$, then rational function $F(z) = c(z) + \alpha m_*$ has all its zeros in TUD^-. This is obvious if $m_* = 0$ i.e, if $(rof)(z)$ has a zero on T. So we suppose all the zeros of $(rof)(z)$ lie in D^- so that $m_* \neq 0$ and we have for $z \in T$

$$|\alpha m_*| < |m_*| \leq |(rof)(z)|.$$

Apply Rouche's theorem, it follows that

$$F(z) = c(z) + \alpha m_*$$

has all its zeros in D^-. Hence in any case $F(z)$ has all its zeros in TUD^- for every $\alpha, |\alpha| < 1$. Let

$$\begin{aligned} F^*(z) &= B(z)\overline{f(\frac{1}{\bar{z}})} \\ &= B(z)\overline{(rof)(\frac{1}{\bar{z}})} + \bar{\alpha}mB(z) \\ &= (rof)^*(z) + \bar{\alpha}mB(z). \end{aligned}$$

Then all the zeros of $F^*(z)$ lie in TUD^+. Now, it follows from Theorem 3.1 with $k = 1$, $z \in T$ and $(rof)^*$ replaced by F^*,

$$|(F^*(z))'| \leq |(F(z))'|.$$

This implies

$$|((rof)^*(z))' + \bar{\alpha} m B'(z)| \leq |((rof)(z))'|.$$

Choosing argument of α suitably, we get for $z \in T$

$$|((rof)^*(z))'| + m|\alpha||B'(z)| \leq |((rof)(z))'|.$$

Letting $|\alpha| \to 1$, we get

$$m|B'(z)| \leq |((rof)^*(z))'| - |((rof)(z))'|.$$

This implies

$$\inf_{z \in T} \left[\left| \frac{r'(f(z))}{B'(z)} \right| - \left| \frac{r^{*\prime}(f(z))}{B'(z)} \right| \right] \geq \inf_{z \in T} \left| \frac{(rof)(z)}{f'(z)} \right|. \tag{3.6}$$

Again

$$(rof)^*(z) = B(z) \overline{r\left(f(\frac{1}{\bar{z}}) \right)}.$$

Therefore,

$$|(r^*(f(z)))'| = \left| B'(z) \overline{r\left(f(\frac{1}{\bar{z}}) \right)} - \frac{1}{z^2} B(z) \overline{r'\left(f(\frac{1}{\bar{z}}) \right) f'\left(\frac{1}{\bar{z}} \right)} \right|$$

$$\geq \left| B'(z) \overline{r\left(f(\frac{1}{\bar{z}}) \right)} \right| - \left| B(z) \overline{r'\left(f(\frac{1}{\bar{z}}) \right) f'\left(\frac{1}{\bar{z}} \right)} \right|.$$

This implies

$$\inf_{z \in T} \left[\left| \frac{r'(f(z))}{B'(z)} \right| - \left| \frac{r^{*\prime}(f(z))}{B'(z)} \right| \right] \leq \inf_{z \in T} \left| \frac{(rof)(z)}{f'(z)} \right|. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\inf_{z \in T} \left[\left| \frac{r'(f(z))}{B'(z)} \right| - \left| \frac{r^{*\prime}(f(z))}{B'(z)} \right| \right] = \inf_{z \in T} \left| \frac{(rof)(z)}{f'(z)} \right|.$$

We now show that the infimum of both sides are attained at the same point $z_0 \in T$. Let

$$\inf_{z \in T} |(rof)(z)| = |(rof)(z_0)|.$$

Since

$$|(rof)(z)||B'(z)| \geq |((rof)(z))'| - |((rof)^*(z))'|, \tag{3.8}$$

we have

$$|(rof)(z_0)| \geq \left| \frac{((rof)(z_0))'}{B'(z_0)} \right| - \left| \frac{((rof)^*(z_0))'}{B'(z_0)} \right|. \tag{3.9}$$

From (3.8) and (3.9), we get

$$\inf_{z\in T}\left[\left|\frac{r^{*\prime}(f(z_0))}{B'(z_0)}\right| + \left|\frac{r'(f(z_0))}{B'(z_0)}\right|\right] = \inf_{z\in T}\left|\frac{(rof)(z_0)}{f'(z_0)}\right|.$$

This completes the proof. □

From Theorem 3.3 we can deduce the following:

Corollary 3.9. *Suppose $rof \in R_{m^*n}$ has m^*n zeros and all the zeros of rof lie in TUD^-. If*

$$(rof)^*(z) = \mathcal{B}(z)\overline{r(f(\frac{1}{\bar{z}}))},$$

then for $\in T$,

$$\inf_{z\in T}\left[\left|\frac{r'(f(z))}{\mathcal{B}'(z)}\right|\right] \geq \inf_{z\in T}\left|\frac{(rof)(z)}{f'(z)}\right|.$$

Set $f(z) = z$, we get the following result:

Corollary 3.10. *If $r \in R_n$ has n zeros and all the zeros of r lie in TUD^-. If*

$$(r)^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})},$$

hen for $\in T$,

$$\inf_{z\in T}\left[\left|\frac{r'(z)}{B'(z)}\right|\right] \geq \inf_{z\in T}\left|r(z)\right|.$$

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