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HALPERN'S ITERATION FOR APPROXIMATING FIXED POINTS OF A NEW CLASS OF ENRICHED NONSPREDING-TYPE MAPPINGS IN HILBERT SPACES WITH APPLICATIONS TO MINIMAX INEQUALITY PROBLEM

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Abstract. In this paper, we propose a modified Halpern's iterative scheme developed from a sequence of a new class of enriched nonspreading mappings and an enriched nonexpansive mapping in the setup of a real Hilbert space. Moreover, we prove strong convergence theorem of the proposed method under mild conditions on the control parameters. Also, we obtain some basic properties of our new class of enriched nonspreading mappings.

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1. Introduction

In this paper, let $\mathcal{H}$ be a real Hilbert space with inner product $\langle , . , \rangle$ which induces the norm $\|.\|$; $\emptyset \neq \mathcal{C} \subset \mathcal{H}$, $\mathbb{N}$ and $\mathbb{R}$ represent a nonempty closed and convex subset of $\mathcal{H}$, the set of all positive integers and the set of real numbers, respectively. If $\{\psi_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{C}$ and $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ is a nonlinear mapping, then $\rightharpoonup, \rightarrow$ and $F(\Im) = \{\psi \in \mathcal{C} : \Im\psi = \psi\}$ denote weak convergence, strong convergence and the set of fixed point of the mapping $\Im$, respectively.

A mapping $\Im$ is known as nonexpansive if it satisfies the inequality

$$\|\Im\psi - \Im\phi\| \leq \|\psi - \phi\|, \ \ \forall \psi, \phi \in \mathcal{C}; \tag{1.1}$$

The problem of investigating fixed points of nonexpansive mapping has been widely studied by several authors.

Halpern [9] first initiated the following iterative sequence:

$$\begin{cases} \psi_1 \in \mathcal{C} \\ \psi_{n+1} = \wp_n u + (1 - \wp_n)\Im\wp_n, \end{cases} \tag{1.2}$$

where $\{\wp_n\}_{n=1}^{\infty} \subset [0, 1]$ and $u \in \mathcal{C}$ is fixed. Moreover, Halpern established in [9] the following result on the convergence of (1.2) for appropriate conditions on $\{\wp_n\}_{n=1}^{\infty}$.

Theorem 1.1. ([9]) *Let $\mathcal{H}$ and $\mathcal{C}$ be as described above (with $\mathcal{C}$ bounded). Let $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be a nonexpansive mapping. For a fixed $u \in \mathcal{C}$, let the sequence $\{\psi_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} \psi_1 \in \mathcal{C} \\ \psi_{n+1} = \wp_n^{-\theta} u + (1 - \wp_n^{-\theta})\Im\wp_n, \end{cases} \tag{1.3}$$

where $\theta \in (0, 1)$. Then, $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to the element of $F(\Im)$ nearest to u.

He further demonstrated that the control conditions

$$C_1 : \lim_{n\to\infty} \wp_n = 0 \quad \text{and} \quad C_2 : \sum_{n=1}^{\infty} \wp_n = \infty$$

are necessary for convergence to the fixed point of $\Im$.

Thereafter, several investigation has been carried out to ascertain the implications of condition C_1 and C_2 on the convergence of (1.3).

In this direction, Chidume and Chidume [8] and [27] independently established that the conditions are sufficient to ensure strong convergence to the fixed point of the following iteration method:

$$\begin{cases} \psi_1 \in \mathcal{C} \\ \psi_{n+1} = \wp_n u + (1-\wp_n)[\lambda\psi_n + (1-\lambda)\Im\wp_n], \quad \forall n \geq 1. \end{cases}$$

In [24], Osilike and Isogugu studied Halpern-type for k-strictly pseudononspreading mappings $\Im$, which is more general that the class of nonspreadings. To get the strong convergence of (1.2), they replaced the mapping $\Im$ with the averaged type mapping $\Im_\delta$; that is, with the mapping

$$\Im_\delta = (1-\delta)I + \delta\Im, \quad \delta \in (0,1). \tag{1.4}$$

Recently, Kohasaka and Takahashi [11, 12] came up with an important class of nonlinear mappings which they referred to as nonspreading mappings. Let $\mathcal{E}$ be a real smooth, strictly convex and reflexive Banach space and denote by $j : \mathcal{E} \longrightarrow 2^{\mathcal{E}^\star}$ the duality mapping of $\mathcal{E}$. Let $\emptyset \neq K \subset \mathcal{E}$ be a closed and convex. A mapping $\Im : K \longrightarrow K$ is known as nonspreading if

$$\varphi(\Im\psi, \Im\phi) + \varphi(\Im\phi, \Im\psi) \leq \varphi(\Im\psi, \phi) + \varphi(\Im\phi, \psi), \tag{1.5}$$

for all $\psi, \phi \in \mathbb{K}$, where

$$\varphi(\psi, \phi) = \|\psi\|^2 - 2\langle\psi, j(\phi)\rangle + \|\phi\|^2, \quad \forall \psi, \phi \in \mathcal{E}. \tag{1.6}$$

Kohasaka and Takahashi considered the class of nonspreading mappings to study the resolvent of a maximum monotone operator in a real smooth, strictly convex and reflexive Banach space. This class of mappings was obtained from the class of firmly nonexpansive mappings (see, for example, [10, 11]). We mention in passing that if $\mathcal{E}$ is a real Hilbert space $\mathcal{H}$, then j is the identity and

$$\varphi(\psi, \phi) = \|\psi\|^2 - 2\langle\psi, \phi\rangle + \|\phi\|^2.$$

Consequently. if $\mathcal{H}$ and $\mathcal{C}$ are as described above, then the mapping $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ is nonspreading if

$$\|\Im\psi - \Im\phi\|^2 \leq \|\Im\psi - \phi\|^2 + \|\Im\phi - \psi\|^2, \quad \forall \psi, \phi \in \mathcal{C}. \tag{1.7}$$

It is established in [14] that (1.7) is equivalent to the inequality

$$\|\Im\psi - \Im\phi\|^2 \leq \|\psi - \phi\|^2 + \langle\psi - \Im\psi, -\phi - \Im\phi\rangle, \quad \forall \psi, \phi \in \mathcal{C}. \tag{1.8}$$

Remark 1.2. If $\Im$ is nonspreading (resp. nonexpansive) and $F(\Im) \neq \emptyset$, then $\Im$ is quasi-nonexpansive.

In [14], lemoto and Takahashi studied the iterative approximation of common fixed points of nonexpansive mapping $\eth$ and nonspreading mapping $\Im$ of $\mathcal{C}$ into itself in a real Hilbert space. They considered an iterative sequence akin to the one employed by Moudafi [16]. To be precise, they proved the following result:

Theorem 1.3. ([14, Theorem 4.1]) *Let $\mathcal{H}$ and $\mathcal{C}$ be as described above. Let $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be a nonspreadinmg mapping and $\eth : \mathcal{C} \longrightarrow \mathcal{C}$ be a nonexpansive mapping such that $F(\Im) \cap F(\eth) \neq \emptyset$. Let $\{\psi_n\}_{n=1}^{\infty}$ be a real sequence generated by*

$$\begin{cases} \psi_1 \in \mathcal{C} \\ \psi_{n+1} = (1-\wp_n)\psi_n + \wp_n[\mu_n\psi_n + (1-\mu_n)\Im\wp_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\wp_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty} \subset [0,1]$. Then, the following conclusions hold:

(1) *If $\sum\limits_{n=1}^{\infty} \wp_n(1-\wp_n) = \infty$ and $\sum\limits_{n=1}^{\infty}(1-\mu_n) < \infty$, then $\{\wp_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ converges weakly to $\nu \in F(\Im)$.*

(2) *If $\liminf\limits_{n\to\infty 1} \wp_n(1-\wp_n) > 0$ and $\sum\limits_{n=1}^{\infty} \mu_n < \infty$, then $\{\wp_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ converges weakly to $\nu \in F(\Im)$.*

(3) *If $\liminf\limits_{n\to\infty 1} \wp_n(1-\wp_n) > 0$ and $\liminf\limits_{n\to\infty 1} \mu_n(1-\mu_n) > 0$, then $\{\wp_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ converges weakly to $\nu \in F(\Im) \cap F(\eth)$.*

Searching for the fixed points of nonexpansive mappings, nonspreading mappings and strictly pseudononspreading mappings are vital topics in fixed point theory, and have been proven to be priceless in the applied areas of signal processing [34], the split feasibility problems [33] and convex feasibility problem [15]. Subsequently, as a worthy generations of the aforementioned mappings, the notion of enriched nonlinear mappings was initiated by Berinde [2, 3] in the setup of a real Hilbert space. This concept was later extended to the more general Banach space by Saleem [26].

Readers interested in this direction of research may consult [18, 19, 20, 21, 22, 25] and the references therein.

Definition 1.4. A mapping $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ is called $\Psi_{\Im}$-enriched Lipschitizian (or $(\sigma, \Psi_{\Im})$-enriched Lipshitizian) (see [26]) if for all $\psi, \phi \in \mathcal{C}$, there exist $\sigma \in [0, +\infty)$ and a continuous nondecreasing function $\Psi_{\Im} : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$, with $\Psi_{\Im}(0) = 0$ such that

$$\|\sigma(\psi - \phi) + \Im\psi - \Im\phi\| \leq (\sigma + 1)\Psi_{\Im}(\|\psi - \phi\|). \tag{1.9}$$

The following special cases due to inequality (1.9) are worth mentioning:

(1) if $\sigma = 0$, inequality (1.9) reduces to a class of mappings known as $\Psi_{\Im}$-Lipschitizian;

(2) if $\sigma = 0$ and $\Psi(t) = Lt$, for $L > 0$, then (1.9) reduces to a class of mappings called L-Lipschitizian with L as the Lipschitz constant. In a more special case where $\sigma = 0, \Psi_{\Im}(t) = Lt$ and $L = 1$, then $\Psi_{\Im}$-enriched Lipschitizian mapping immediately reduces to the class of nonexpansive mappings on $\mathcal{C}$;

(3) if $\Psi_{\Im}(s) = s$, then inequality (1.9) becomes

$$\|\sigma(\psi - \phi) + \Im\psi - \Im\phi\| \leq (\sigma + 1)\|\psi - \phi\| \tag{1.10}$$

and is known as σ-enriched nonexpansive mapping. This class of mappings was first considered by Berinde [2, 3] as a generalization of a well-known class of mappings called nonexpansive mapping.

Note that if $\Psi_{\Im}$ is not necessarily nondecreasing and satisfies the condition

$$\Psi_{\Im}(t) < t \text{ for } t > 0,$$

then we have the class of η-enriched contraction mappings.

Considering results of Lemoto and Takahashi [14], Berinde [3] and other results in the papers studied, the following question becomes necessary:

Question 1:

(1) Could there be a nonlinear mapping that contains the class of mapping defined by (1.7) for which we would obtain the results in [14] as special cases?
(2) Could it be possible to obtain strong convergence result for an averaged mapping in the setup of a real Hilbert space?

Lemoto and Takahashi considered the class of nonspreading mappings and proved weak convergence Theorem as their main result in [14]. The results they obtained (in conjunction with those of Kohasaka and Takahashi [12]) opened a new direction in fixed point theory.

In this paper, we first introduce a new class of nonlinear mapping called σ-enriched nonspreading mappings and give some nontrivial examples to demonstrate its existence. Further, we modify the iterative method studied in [14] and thereafter give an affirmative answer to Question 1.

The rest of the paper is organised as follows: Section 2 will consider preliminary results which will be needed in establishing our main results. Many lemmas and propositions which will serve as our main results (including some of their consequences) and the conclusion of results obtained in this paper will be considered in Section 3.

2. Preliminaries

In the sequel, we state some well-known definitions and lemmas which shall be helpful in establishing the proof of our main results.

Let $\mathcal{H}$ be a real Hilbert space and let $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{H}$. We shall represent weak convergence of $\{\psi_n\}_{n=1}^{\infty}$ to a point $\psi \in \mathcal{H}$ by $\psi_n \rightharpoonup \psi$ and the strong convergence of $\{\psi_n\}_{n=1}^{\infty}$ to a point $\psi \in \mathcal{H}$ by $\psi_n \to \psi$ as $n \to \infty$, respectively.

Lemma 2.1. ([24]) *Let $\mathcal{H}$ be a real Hilbert space. Then, the following well-researched results are valid:*

(*i*) *For all $\hbar, \wp \in \mathcal{H}$,*
$$\|\wp + \hbar\|^2 = \|\wp\|^2 + 2\langle \wp, \hbar\rangle + \|\hbar\|^2.$$
(*ii*) *For all $\hbar, \wp \in \mathcal{H}$,*
$$\|\hbar + \wp\|^2 \leq \|\hbar\|^2 + 2\langle \wp, \hbar + \wp\rangle.$$
(*iii*) *For $\wp, \hbar \in \mathcal{H}$ and for all $t \in [0,1]$.*
$$\|t\wp + (1-t)\hbar\|^2 = t\|\wp\|^2 + (1-t)\|\hbar\|^2 - t(1-t)|\wp - \hbar\|^2.$$
(*iv*) *If $\{\psi_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$ such that $\psi_n \rightharpoonup \wp \in \mathcal{H}$, then*
$$\limsup_{n\to\infty} \|\psi_n - \hbar\|^2 = \limsup_{n\to\infty} \|\psi_n - \wp\|^2 + \|\wp - \hbar\|^2.$$

Definition 2.2. ([24]) Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be closed and convex. The nearest point projection $P_{\mathcal{C}} : \mathcal{H} \longrightarrow \mathcal{C}$ defined from $\mathcal{H}$ onto $\mathcal{C}$ is a operator that assigns to each $\psi \in \mathcal{H}$ its nearest point represented with $P_{\mathcal{C}}\psi$ in Ω. Thus, $P_{\mathcal{C}}$ is the unique point in $\mathcal{C}$ such that
$$\|\psi - P_{\mathcal{C}}\psi\| \leq \|\psi - \hbar\|, \quad \forall \hbar \in \mathcal{C}.$$

Lemma 2.3. ([15]) *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be closed and convex and $P_{\mathcal{C}} : \mathcal{H} \longrightarrow \mathcal{C}$. be a metric projection. Then*

(*i*)
$$\|P_{\mathcal{C}}\wp - P_{\mathcal{C}}\hbar\| \leq \langle \wp - \hbar, P_{\mathcal{C}}\wp - P_{\mathcal{C}}\hbar\rangle, \quad \forall \wp, \hbar \in \mathcal{H};$$
(*ii*) *$P_{\mathcal{C}}$ is a nonexpansive mapping, that is, $\|P_{\mathcal{C}}\wp - P_{\mathcal{C}}\hbar\| \leq \|\wp - \hbar\|$;*
(*iii*) *given that $\wp \in \mathcal{H}$ and $z \in \mathcal{C}$, then $z = P_{\mathcal{C}}\wp$ if and only if*
$$\langle \wp - z, \hbar - z\rangle \leq 0, \quad \forall \hbar \in \mathcal{C}.$$

Lemma 2.4. ([30]) *Let $\mathcal{H}$ be a real Hilbert space and $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be closed and convex. Let $P_{\mathcal{C}} : \mathcal{H} \longrightarrow \Omega$ be the metric projection of $\mathcal{H}$ onto $\mathcal{C}$. Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{C}$ and let*
$$\|\psi_{n+1} - \vartheta\| \leq \|\psi_n - \vartheta\|$$
for all $\vartheta \in \mathcal{C}$. Then, $\{P_C\psi_n\}_{n=1}^{\infty}$ converges strongly.

Lemma 2.5. ([24]) *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be closed and convex and $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be nonexpansive. Then,*

(1) $I - \Im : \mathcal{C} \longrightarrow \mathcal{H}$ *is* $\frac{1}{2}$*- inverse strongly monotone; that is,*

$$\frac{1}{2}\|(I-\Im)\wp - (I-\Im)\hbar\|^2 \leq \langle \wp - \hbar, (I-\Im)\wp - (I-\Im)\hbar\rangle, \tag{2.1}$$

for all $\wp, \hbar \in \mathcal{C}$

(2) *In addition, if* $F(\Im) \neq \emptyset$*,* $(i - \Im)$ *is demiclosed; that is, for every sequence* $\{\wp_n\}_{n=1}^{\infty}$ *weakly convergent to* p *such that* $\wp_n - \Im\wp_n \to 0$ *as* $n \to \infty$*, it follows that* $p \in F(\Im)$*.*

In [24], Osilike and Isogugu defined the averaged mapping $\Im_\delta$ as follows

$$\Im_\delta = (1-\delta)I + \delta\Im = I - \delta(I - \Im), \tag{2.2}$$

where $\delta \in (0,1)$ and $\Im$ is a nonexpansive mapping. Furthermore, Bryne [5] and subsequently Moudafi [16] established some properties of the averaged mappings; in particular, they proved that $\Im_\delta$ is nonexpansive mapping. In this paper, inspired by [3, 14], we introduce an averaged type mapping $\Im_{\delta\beta}$ at below for nonlinear mapping $\Im : \mathcal{C} \longrightarrow \mathcal{C}$; we observe that $F(\Im) = F(\Im_\delta) = F(\Im_\delta\beta)$. Also, we observe that if $\Im$ is an σ-enriched nonspreading self-mapping of $\mathcal{C}$ and $F(\Im) \neq \emptyset$, then the averaged type mapping $\Im_{\delta\beta}$ is quasi-nonexpansive and as a consequence, the set of fixed points of $F(\Im_{\delta\beta})$ is closed and convex.

Lemma 2.6. ([1, 31]) *Let* $\{\nu_n\}_{n=1}^{\infty}$ *be a sequence of non-negative real real numbers validating the following inequality*

$$\nu_{n+1} \leq (1-\pi_n)\nu_n + \pi_n\mu_n + \vartheta_n,$$

where $\{\pi_n\}_{n=1}^{\infty}$ *and* $\{\mu_n\}_{n=1}^{\infty}$ *are real sequences such that*

(i) $\{\pi_n\}_{n=1}^{\infty} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \pi_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \mu_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\mu_n| < \infty$*;*

(iii) $\vartheta_n \geq 0$, $\sum_{n=0}^{\infty} \vartheta_n < \infty$.

Then, $\lim_{n\to\infty} \nu_n = 0$.

Finally, we state the following fundamental result proved by Mainge which played a pivotal role in establishing the proof of our main results.

Lemma 2.7. ([28]) *Let* $\{\gamma_n\}_{n=0}^{\infty}$ *be a sequence of real numbers for which we can find a subsequence* $\{\gamma_{n_k}\}_{k=0}^{\infty}$ *such that* $\gamma_k < \gamma_{k+1}$ *for all* $k \in \mathbb{N}$*. Consider the sequence of integers* $\{\tau(n)\}_{n=1}^{\infty}$ *given by*

$$\tau_n = \max\{i \leq n : \gamma_i \leq \gamma_{i+1}\}. \tag{2.3}$$

Then, $\{\tau(n)\}_{n=1}^{\infty}$ is a nondecreasing sequence, for all $n \geq n_0$, validating the following requirements

(i) $\lim_{n\to\infty} \tau(n) = \infty$;
(ii) $\gamma_{\tau(n)} < \gamma_{tau(n)}, \quad \forall n \geq n_0$;
(iii) $\gamma_n < \gamma_{\tau(n)}, \quad \forall n \geq n_0$.

Lemma 2.8. (Opial property [23]) *Let $\mathcal{H}$ be a real Hilbert space. Suppose $\wp_n \rightharpoonup \omega$. Then the inequality below holds:*

$$\liminf_{n\to\infty} \|\wp_n - \hbar\| > \liminf_{n\to\infty} \|\wp_n - \omega\|, \quad \forall \hbar \in \mathcal{H}, \hbar \neq \omega.$$

Lemma 2.9. ([14]) *Let $\mathcal{H}$ be a real Hilbert space. Then, the following identity holds.*

$$2\langle \psi - \phi, z - \omega\rangle = \|\psi - \omega\|^2 + \|\phi - z\|^2 - \|\psi - z\|^2 - \|\phi - \omega\|^2$$

for all $\psi, \phi, \omega, z \in \mathcal{H}$.

3. Results and discussion

Throughout this section, $\mathcal{H}, \mathcal{C}, F(\eth)$ and $F(\Im)$ shall denote a real Hilbert space, a nonempty closed and convex subset of $\mathcal{H}$, the set of fixed point of enriched nonexpansive mapping $\eth$ and the set of fixed point of enriched nonspreading mapping $\Im$, respectively.

In the sequel, we state the following definition.

Definition 3.1. ([13]) Let $\mathcal{H}$ be as described above. A mapping $\Im$ with dormain $D(\eth)$ and range $R(\Im)$ in $\mathcal{H}$ is known as σ-enriched nonspreading (σ-ESNM, for short) in the sense Kurokawa and Takahashi if there exists $\sigma \in [0, \infty)$ such that for all $(\psi, \phi) \in D(\Im)$, the inequality

$$2\|\sigma(\psi-\phi)+\Im\psi-\Im\phi\|^2 \leq \|\sigma(\psi-\phi)+\Im\psi-\phi\|^2+\|\sigma(\psi-\phi)+\psi-\Im\phi\|^2. \quad (3.1)$$

The following lemmas are some of the characterization of enriched nonspreading mapping.

Lemma 3.2. *Let $\mathcal{H}$ and $\mathcal{C}$ be as described above. Then a mapping $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ is an σ-enriched nonspreading if and only if*

$$2\|\sigma(\psi-\phi)+\Im\psi-\Im\phi\|^2 \leq \|\sigma(\psi-\phi)+\Im\psi-\phi\|^2+\|\sigma(\psi-\phi)+\psi-\Im\phi\|^2. \quad (3.2)$$

Proof. We have that

$$
\begin{aligned}
&2\|\sigma(\psi-\phi)+\Im\psi-\Im\phi\|^2\\
&\quad\le\|\sigma(\psi-\phi)+\Im\psi-\phi\|^2+\|\sigma(\psi-\phi)+\psi-\Im\phi\|^2\\
&\quad=\|[(\sigma+\Im)\psi-(\sigma+1)\psi]+(\sigma+1)(\psi-\phi)\|^2\\
&\qquad+\|(\sigma+1)\psi-(\sigma+\Im)\psi+(\sigma+\Im)\psi-(\sigma+\Im)\phi\|^2\\
&\quad=\|(\sigma+\Im)\psi-(\sigma+1)\psi\|^2+2(\sigma+1)\langle(\sigma+\Im)\psi-(\sigma+1)\psi,\psi-\phi\rangle\\
&\qquad+(\sigma+1)^2\|\psi-\phi\|^2+\|(\sigma+1)\psi-(\sigma+\Im)\psi\|^2\\
&\qquad+2\langle(\sigma+1)\psi-(\sigma+\Im)\psi,(\sigma+\Im)\psi-(\sigma+\Im)\phi\rangle\\
&\qquad+\|(\sigma+\Im)\psi-(\sigma+\Im)\phi\|^2
\end{aligned}
$$

if and only if

$$
\begin{aligned}
2\|\sigma(\psi-\phi)+\Im\psi-\Im\phi\|^2\le{}&(\sigma+1)^2\|\psi-\phi\|^2+2\|\Im)\psi-\psi\|^2\\
&+2(\sigma+1)\langle\Im\psi-\psi,\psi-\phi\rangle\\
&+2\langle\psi-\Im\psi,(\sigma+\Im)\psi-(\sigma+\Im)\phi\rangle\\
&+\|(\sigma+\Im)\psi-(\sigma+\Im)\phi\|^2
\end{aligned}
$$

if and only if

$$
\begin{aligned}
2\|\sigma(\psi-\phi)+\Im\psi-\Im\phi\|^2\le{}&(\sigma+1)^2\|\psi-\phi\|^2+2\|\Im)\psi-\psi\|^2\\
&+\|(\sigma+\Im)\psi-(\sigma+\Im)\phi\|^2\\
&-2\langle\psi-\Im\psi,(\psi-\Im\psi)-(y-\Im\phi)\rangle
\end{aligned}
$$

if and only if

$$\|\sigma(\psi-\phi)+\Im\psi-\Im\phi\|^2\le(\sigma+1)^2\|\psi-\phi\|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle.$$

This completes the proof. □

Now, by setting $\sigma=\dfrac{1}{\beta}-1$, for some $\beta\in(0,1]$, we obtain from (3.2) that

$$\|\sigma(\psi-\phi)+\Im\psi-\Im\phi\|^2\le(\sigma+1)^2\|\psi-\phi\|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle$$

if and only if

$$\left\|\Big(\frac{1}{\beta}-1\Big)(\psi-\phi)+\Im\psi-\Im\phi\right\|^2\leq\frac{1}{\beta^2}\|\psi-\phi\|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle$$

if and only if

$$\begin{aligned}\|\beta(\psi-\phi)+\beta\Im\psi-\beta\Im\phi\|^2\leq\|\psi-\phi\|^2+2\langle\psi-[(1-\beta)\psi\\ +\beta\Im\psi],\phi-[(1-\beta)\phi+\beta\Im\phi]\rangle.\end{aligned}\tag{3.3}$$

Remark 3.3. Observe that if $\sigma=0$ in (3.2) (or $\beta=1$ (3.3)), we obtain an important class nonspreading mappings studied in [13]. Again, if we take $\Im_\beta=(I-\beta)I+\beta\Im$, then (3.3) reduces to

$$\|\Im_\beta\psi-\Im_\beta\phi\|^2\quad\leq\quad\|\psi-\phi\|^2+2\langle\psi-\Im_\beta\psi,\phi-\Im_\beta\phi\rangle.\tag{3.4}$$

Therefore, the averaged operator $\Im_\beta$ is a nonspreading mapping whenever $\Im$ is an σenriched nonspreading mapping.

Remark 3.4. Any nonspreading mapping $\Im$ validating (3.1) with $\sigma=0$ is known as 0-enriched nonspreading.

Lemma 3.5. *Let $\mathcal{H}$ and $\mathcal{C}$ be as described above and let $\Im:\mathcal{C}\longrightarrow\mathcal{C}$ be an σ-enriched nonspreading mapping. Then, $F(\Im)$ is closed and convex.*

Proof. Let $\{\psi_n\}_{n=1}^\infty$ be a sequence in $F(\Im)$ which converges to ψ. We want to show that $\psi\in F(\Im)$. Now, since

$$\begin{aligned}\|\Im_\beta\psi-\psi\|&=\beta\|\Im\psi-\psi\|\\&\leq\beta\|\Im\psi-\Im\psi_n\|+\beta\|\psi_n-\psi\|\\&=\beta\|\sigma(\psi-\psi_n)+\Im\psi-\Im\psi_n-\sigma(\psi-\psi_n)\|+\beta\|\psi_n-\psi\|\\&\leq\beta\|\sigma(\psi-\psi_n)+\Im\psi-\Im\psi_n\|+\beta(\sigma+1)\|\psi_n-\psi\|\end{aligned}\tag{3.5}$$

and since $\Im$ is an σ-enriched nonspreading mapping, we have

$$\begin{aligned}\|\sigma(\psi-\psi_n)+\Im\psi-\Im\psi_n\|^2&\leq(\sigma+1)^2\|\psi_n-\psi\|^2\\&\quad+2\langle\psi-\Im\psi,\psi_n-\Im\psi_n\rangle\\&=(\sigma+1)^2\|\psi_n-\psi\|^2.\end{aligned}\tag{3.6}$$

(3.5) and (3.6) imply that

$$0\leq\|\Im\psi-\psi\|(\sigma+1)\|\psi_n-\psi\|\to0\quad\text{as}\quad n\to\infty.\tag{3.7}$$

Hence, $\psi\in F(\Im)$.

Next, let $\vartheta_1, \vartheta_2 \in F(\Im)$ and $\lambda \in [0,1]$. We prove that $\lambda\vartheta_1+(1-\lambda)\vartheta_2 \in F(\Im)$. Let $\psi = \lambda\vartheta_1+(1-\lambda)\vartheta_2$. Then, $\vartheta_1-\psi = (1-\lambda)(\vartheta_1-\vartheta_2)$ and $\vartheta_2-\psi = \lambda(\vartheta_2-\vartheta_1)$. Since

$$\begin{aligned}
&\beta^2\|\Im\psi - \psi\|^2 \\
&\quad = \|\psi - \Im_\beta\psi\|^2 \\
&\quad = \|\lambda\vartheta_1 + (1-\lambda)\vartheta_2 - \Im_\beta\psi\|^2 \\
&\quad = \|\lambda(\vartheta_1 - \Im_\beta\psi) + (1-\lambda)(\vartheta_2 - \Im_\beta\psi)\|^2 \\
&\quad = \lambda\|\vartheta_1 - \Im_\beta\psi\|^2 + (1-\lambda)\|\vartheta_2 - \Im_\beta\psi\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\quad = \lambda\|(1-\beta)\vartheta_1 + \beta\Im\vartheta_1 - [(1-\beta)\psi + \beta\Im\psi]\|^2 \\
&\qquad + (1-\lambda)\|(1-\beta)\vartheta_2 + \beta\Im\vartheta_2 - [(1-\beta)\psi + \beta\Im\psi]\|^2 \\
&\qquad - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\quad = \lambda\|(1-\beta)(\vartheta_1 - \psi) + \beta(\Im\vartheta_1 - \Im\psi)\|^2 \\
&\qquad + (1-\lambda)\|(1-\beta)(\vartheta_2 - \psi) + \beta(\Im\vartheta_2 - \Im\psi)\|^2 \\
&\qquad - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\quad = \frac{\lambda}{(\sigma+1)^2}\|\sigma(\vartheta_1 - \psi) + \Im\vartheta_1 - \Im\psi\|^2 \\
&\qquad + \frac{1-\lambda}{(\sigma+1)^2}\|\sigma(\vartheta_2 - \psi) + \Im\vartheta_2 - \Im\psi\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\quad \le \frac{\lambda}{(\sigma+1)^2}[(\sigma+1)^2\|\vartheta_1 - \psi\|^2 + 2\langle\vartheta_1 - \Im\vartheta_1, \psi - \Im\psi\rangle] \\
&\qquad + \frac{1-\lambda}{(\sigma+1)^2}[(\sigma+1)^2\|\vartheta_2 - \psi\|^2 + 2\langle\vartheta_2 - \Im\vartheta_2, \psi - \Im\psi]\rangle \\
&\qquad - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\quad = \lambda\|\vartheta_1 - \psi\|^2 + (1-\lambda)\|\vartheta_2 - \psi\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\quad = \lambda(1-\lambda)[1-\lambda+\lambda]\|\vartheta_1 - \vartheta_2\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2,
\end{aligned}$$

it follows that

$$\beta^2\|\Im\psi - \psi\|^2 \le 0.$$

Therefore, $\psi = \Im\psi$ implies that $\psi \in F(\Im)$ as required result. □

Proposition 3.6. *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ and $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be an σ-enriched nonspreading mapping. Then, $(I - \Im)$ is demiclosed at* 0.

Proof. Let $\{\psi_n\}_{n=0}^{\infty}$ be a sequence in Ω which converges weakly to ϑ and $\{\psi_n - \Im\psi_n\}_{n=0}^{\infty}$ converges strongly to 0. We want to show that $\vartheta \in F(\Im)$.

Now, since $\{\psi_n\}_{n=0}^{\infty}$ converges weakly, it is bounded.

For each $\psi \in \mathcal{H}$, define $f : \mathcal{H} \longrightarrow [0, \infty)$ by

$$f(\psi) = \limsup_{n\to\infty} \|\psi_n - \psi\|^2.$$

Then, using Lemma 2.1 (iv), we get

$$f(\psi) = \limsup_{n\to\infty} \|\psi_n - \vartheta\|^2 + \|\vartheta - \psi\|^2, \quad \forall \psi \in \mathcal{H}.$$

As consequence,

$$f(\psi) = f(\vartheta) + \|\vartheta - \psi\|^2, \quad \forall \psi \in \mathcal{H}$$

and

$$\begin{aligned} f(\Im_\beta) &= f(\vartheta) + \|\vartheta - \Im_\beta \vartheta\|^2 \\ &= f(\vartheta) + \frac{1}{(\sigma+1)^2}\|\vartheta - \Im\vartheta\|^2, \quad \forall \psi \in \mathcal{H}. \end{aligned} \tag{3.8}$$

Observe that

$$\begin{aligned} f(\Im_\beta) &= \limsup_{n\to\infty} \|\psi_n - \Im_\beta \vartheta\|^2 \\ &= \limsup_{n\to\infty} \|\psi_n - \Im_\beta\psi_n + \Im_\beta\psi_n - \Im_\beta\vartheta\|^2 \\ &= \limsup_{n\to\infty} \|\psi_n - [(1-\beta)\psi_n + \beta\Im\psi_n] + (1-\beta)\psi_n \\ &\quad + \beta\Im\psi_n - [(1-\beta)\vartheta + \beta\Im\vartheta]\|^2 \\ &= \limsup_{n\to\infty} \|\beta(\psi_n - \Im\psi_n) + (1-\beta)(\psi_n - \vartheta) + \beta(\Im\psi_n - \Im\vartheta)\|^2 \\ &= \limsup_{n\to\infty} \left\| \frac{\sigma}{\sigma+1}(\psi_n - \vartheta) + \frac{1}{\sigma+1}(\Im\psi_n - \Im\vartheta)\right\|^2 \\ &= \frac{1}{(\sigma+1)^2} \limsup_{n\to\infty} \|\sigma(\psi_n - \vartheta) + \Im\psi_n - \Im\vartheta\|^2 \\ &\le \frac{1}{(\sigma+1)^2} \limsup_{n\to\infty} [(\sigma+1)^2\|\psi_n - \vartheta\|^2 + 2\langle \psi_n - \Im\psi_n, \vartheta - \Im\vartheta\rangle] \\ &= \limsup_{n\to\infty} \|\psi_n - \vartheta\|^2 = f(\vartheta). \end{aligned} \tag{3.9}$$

(3.8) and (3.9) imply

$$\|\vartheta - \Im\vartheta\| \le 0,$$

so that $\vartheta \in F(\Im)$ as required. □

Proposition 3.7. *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \ne \mathcal{C} \subset \mathcal{H}$ and $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be an σ-enriched nonspreading mapping such that $F(\Im) \ne \emptyset$. Then, the averaged type operator*

$$\Im_{\delta\beta} = I - \delta(I - \Im_\beta) = (1-\delta)I + \delta[(1-\beta) + \beta\Im]$$

is quasi-nonexpansive mapping, where I is the identity mapping.

Proof. Set $\Im_\beta = (1-\beta) + \beta\Im$ so that by Lemma 2.1(iii), we have

$$\begin{aligned}
\|\Im_{\delta\beta}\psi - \Im_{\delta\beta}\phi\|^2 &= \|(1-\delta)(\psi-\phi) + \delta(\Im_\beta\psi - \Im\phi)\|^2 \\
&= (1-\delta)\|\psi-\phi\|^2 + \delta\|\Im_\beta\psi - \Im_\beta\phi\|^2 \\
&\quad - \delta(1-\delta)\|(\psi - \Im_\beta\psi) - (\phi - \Im_\beta\phi)\|^2 \\
&= (1-\delta)\|\psi-\phi\|^2 + \delta\|(1-\beta)(\psi-\phi) + \Im\psi - \Im\beta\phi\|^2 \\
&\quad - \delta(1-\delta)\|(\psi - \Im_\beta\psi) - (\phi - \Im_\beta\phi)\|^2 \\
&= (1-\delta)\|\psi-\phi\|^2 + \frac{\delta}{(\sigma+1)^2}\|\sigma(\psi-\phi) + \Im\psi - \Im\phi\|^2 \\
&\quad - \frac{\delta(1-\delta)}{(\sigma+1)^2}\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 \\
&= (1-\delta)\|\psi-\phi\|^2 + \frac{\delta}{(\sigma+1)^2}\|\sigma(\psi-\phi) + \Im\psi - \Im\phi\|^2 \\
&\quad - \frac{\delta(1-\delta)}{(\sigma+1)^2}\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 \\
&\le (1-\delta)\|\psi-\phi\|^2 \\
&\quad + \frac{\delta}{(\sigma+1)^2}[(\sigma+1)^2\|\psi-\phi\|^2 + 2\langle\psi - \Im\psi, \phi - \Im\phi\rangle] \\
&\quad - \frac{\delta(1-\delta)}{(\sigma+1)^2}\|\psi - \Im\psi - (\phi - \Im\phi)\|^2 \\
&= (1-\delta)\|\psi-\phi\|^2 + \frac{2\delta}{(\sigma+1)^2}\langle\psi - \Im\psi, \phi - \Im\phi\rangle \\
&\quad - \frac{\delta(1-\delta)}{(\sigma+1)^2}\|\psi - \Im\psi - (\phi - \Im\phi)\|^2. \qquad (3.10)
\end{aligned}$$

Since

$$\begin{aligned}
\langle\psi - \Im_{\delta\beta}\psi, \phi - \Im_{\delta\beta}\phi\rangle &= \langle\psi - [\delta\psi + (1-\delta)\Im_\beta\psi], \phi - [\delta\phi + (1-\delta)\Im_\beta\phi]\rangle \\
&= (1-\delta)^2\langle\psi\Im_\beta\psi, \phi - \Im_\beta\phi\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle\psi - \Im_\beta\psi, \phi - \Im_\beta\phi\rangle &= \langle\psi - [(1-\beta)\psi + \beta\Im], \phi - [(1-\beta)\phi + \beta\Im\phi]\rangle \\
&= \beta^2\langle\psi - \Im\psi, \phi - \Im\phi\rangle,
\end{aligned}$$

it follows that

$$\begin{aligned}\langle \psi - \Im\psi, \phi\Im\phi\rangle &= \frac{1}{\beta^2(1-\delta)^2}\langle \psi - \Im_{\delta\beta}\psi, \phi - \Im_{\delta\beta}\phi\rangle \\ &= \frac{(\sigma+1)^2}{(1-\delta)^2}\langle \psi - \Im_{\delta\beta}\psi, \phi - \Im_{\delta\beta}\phi\rangle. \qquad (3.11)\end{aligned}$$

(3.10) and (3.11) imply

$$\|\Im_{\delta\beta}\psi - \Im_{\delta\beta}\phi\|^2 \leq \|\psi - \phi\|^2 + \frac{2\delta}{(1-\delta)}\langle \psi\Im_{\delta\beta}\psi, \phi - \Im_{\delta\beta}\phi\rangle. \qquad (3.12)$$

In particular, choosing $\phi = \vartheta$, where $\vartheta \in F(\Im) = F(\Im_\beta) = F(\Im_{\delta\beta})$, we get

$$\|\Im_{\delta\beta}\psi - \Im_{\delta\beta}\phi\| \leq \|\psi - \phi\| \qquad (3.13)$$

as required result. □

Lemma 3.8. *Let $\mathcal{H}$ and $\mathcal{C}$ be as described above. Let $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be an σ-enriched nonspreading and let*

$$\mathcal{A} = (\sigma+1)I - (\sigma + \Im).$$

Then,

$$\|\mathcal{A}\psi - \mathcal{A}\phi\|^2 \leq (\sigma+1)\langle \psi - \phi, \mathcal{A}\psi - \mathcal{A}\phi\rangle + \frac{1}{2}(\|\mathcal{A}\psi\|^2 + \|\mathcal{A}\phi\|^2)$$

for all $\psi, \phi \in \mathcal{C}$.

Proof. Put $A = (\sigma+1)I - (\sigma+\Im)$. Then, for any $\psi, \phi \in \mathcal{C}$, we obtain

$$\begin{aligned}\|\mathcal{A}\psi - \mathcal{A}\phi\|^2 &= \langle \mathcal{A}\psi - \mathcal{A}\phi, \mathcal{A}\psi - \mathcal{A}\phi\rangle \\ &= \langle (\sigma+1)(\psi-\phi) - [(\sigma+\Im)\psi - (\sigma+\Im)\phi], \mathcal{A}\psi - \mathcal{A}\phi\rangle \\ &= (\sigma+1)\langle \psi - \phi, \mathcal{A}\psi - \mathcal{A}\phi\rangle \\ &\quad - \langle (\sigma+\Im)\psi - (\sigma+\Im)\phi, \mathcal{A}\psi - \mathcal{A}\phi\rangle. \qquad (3.14)\end{aligned}$$

Also, from Lemma 2.9, we obtain

$$\begin{aligned}&2\langle (\sigma+\Im)\psi - (\sigma+\Im)\phi, \mathcal{A}\psi - \mathcal{A}\phi\rangle \\ &\quad = 2\langle (\sigma+\Im)\psi - (\sigma+\Im)\phi, (\sigma+1)(\psi-\phi) - (\sigma+\Im)\psi \\ &\qquad - (\sigma+\Im)\phi\rangle \\ &\quad = 2\langle (\sigma+\Im)\psi - (\sigma+\Im)\phi, (\sigma+1)\psi - (\sigma+1)\phi\rangle \\ &\qquad - \|\sigma(\psi-\phi) + \Im\psi - \Im\phi\|^2\end{aligned}$$

$$
\begin{aligned}
&\geq \|(\sigma+\Im)\psi-(\sigma+1)\phi\|^2+\|(\sigma+\Im)\phi-(\sigma+1)\psi\|^2\\
&\quad-\|(\sigma+\Im)\psi-(\sigma+1)\psi\|^2-\|(\sigma+\Im)\phi-(\sigma+1)\phi\|^2\\
&\quad-[\|(\sigma+\Im)\psi-(\sigma+1)\phi\|^2+\|(\sigma+1)\psi-(\sigma+\Im)\phi\|^2]\\
&=\|(\sigma+\Im)\psi-(\sigma+1)\phi\|^2+\|(\sigma+\Im)\phi-(\sigma+1)\psi\|^2\\
&\quad-\|(\sigma+\Im)\psi-(\sigma+1)\psi\|^2-\|(\sigma+\Im)\phi-(\sigma+1)\phi\|^2\\
&\quad-[\|(\sigma+\Im)\psi-(\sigma+1)\phi\|^2+\|-[(\sigma+\Im)\phi-(\sigma+1)\psi]\|^2]\\
&=\|(\sigma+\Im)\psi-(\sigma+1)\phi\|^2+\|(\sigma+\Im)\phi-(\sigma+1)\psi\|^2\\
&\quad-\|(\sigma+\Im)\psi-(\sigma+1)\psi\|^2-\|(\sigma+\Im)\phi-(\sigma+1)\phi\|^2\\
&\quad-[\|(\sigma+\Im)\psi-(\sigma+1)\phi\|^2+\|(\sigma+\Im)\phi-(\sigma+1)\psi\|^2]\\
&=-\|-[(\sigma+1)\psi-(\sigma+\Im)\psi]\|^2\\
&\quad-\|-[(\sigma+1)\phi-(\sigma+\Im)\phi]\|^2\\
&=-\|(\sigma+1)\psi-(\sigma+\Im)\psi\|^2-\|(\sigma+1)\phi-(\sigma+\Im)\phi\|^2\\
&=-\|\mathcal{A}\psi\|^2-\|\mathcal{A}\phi\|^2.
\end{aligned}
$$

So, we obtain

$$
\|\mathcal{A}\psi-\mathcal{A}\phi\|^2 \leq (\sigma+1)\langle\psi-\phi,\mathcal{A}\psi-\mathcal{A}\phi\rangle+\frac{1}{2}(\|\mathcal{A}\psi\|^2+\|\mathcal{A}\phi\|^2).
$$

This completes the proof. □

Example 3.9. Let $X=\mathbb{R}$ and $\mathcal{C}=\left[\frac{1}{2},k\right]$, where $\frac{1}{2}<k<1$. For each $\psi\in\mathcal{C}$, let $\Im:\mathcal{C}\longrightarrow\mathcal{C}$ be defined by

$$
\Im\psi=\frac{k}{2k-1}(k-\psi).
$$

Then, $\Im$ is an enriched nonspreading mapping. To see this, observe that since

$$
\begin{aligned}
|\sigma(\psi-\phi)+\Im\psi-\Im\phi|^2 &= \left|\sigma(\psi-\phi)+\frac{k}{2k-1}(k-\psi)-\frac{k}{2k-1}(k-\phi)\right|^2\\
&=\left|\sigma(\psi-\phi)+\frac{k}{2k-1}(\psi-\phi)\right|^2\\
&=\left(\sigma-\frac{k}{2k-1}\right)^2|\psi-\phi|^2,
\end{aligned}
$$

$$
\begin{aligned}
\langle\psi-\Im\psi,\phi-\Im\phi\rangle &= \frac{1}{2k-1}\langle 2k\psi-\psi-k^2+k\psi, 2k\phi-\phi-k^2+k\phi\rangle\\
&=\frac{1}{2k-1}\langle(3k-1)\psi-k^2,(3k-1)\phi-k^2\rangle>0
\end{aligned}
$$

and

$$\Big(\sigma - \frac{k}{2k-1}\Big)^2|\psi - \phi|^2 < (\sigma+1)^2|\psi - \phi|^2,$$

it follows that

$$(\sigma+1)^2|\psi-\phi|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle > |\sigma(\psi-\phi) + \Im\psi - \Im\phi|^2,$$

which in turn satisfies condition (3.2).

Now, by noticing that $\frac{k}{2k-1} > 1$, $\frac{1}{2k-1} > 1$, that is, $k < 1$, we have, for any $\psi, \phi \in \mathcal{C}$ and $\sigma = 0$, that

$$|\sigma(\psi-\phi) + \Im\psi - \Im\phi| = \frac{(\sigma+1)k}{2k-1}|\psi - \phi| > (\sigma+)|\psi-\phi|.$$

Thus, $\Im$ is not L-Lipschitzian with $L = 1$.

Example 3.10. Let $B_\rho\{\psi \in \mathcal{H} : \|\psi\| \leq \rho\}$ for $\rho > 0$ and $\mathcal{C} = B_2 \subset \mathcal{H}$ and define a mapping $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\Im\psi = \begin{cases} \psi, & \psi \in B_2 \\ P_{B_1}\psi, & \psi \in \backslash B_2, \end{cases}$$

where P_A is a metric projection of $\mathcal{H}$ onto $\mathcal{A}$. Then, $\Im$ is an enriched nonspreading mappings which does not admit continuity. Obviously, $F(\Im) = B_2$.

Let $\psi, \phi \in \mathcal{C}$. It suffices to check the case $\psi \in \mathcal{C} \setminus B_2, \phi \in B_2$. Now, since P_{B_1} is nonexpansive (and hence 0-enriched nonexpansive) and $\phi - \Im\phi = 0$, it follows that

$$\begin{aligned} \|\sigma(\psi-\phi) + \Im\psi - \Im\phi\|^2 &= \|\sigma(\psi-\phi) + P_{B_1}\psi - \phi\|^2 \\ &= \|\sigma(\psi-\phi) + P_{B_1}\psi - P_{B_1}\phi\|^2 \\ &\leq (\sigma+)^2\|\psi-\phi\|^2 \\ &= (\sigma+)^2\|\psi-\phi\|^2 + 2\langle\psi - \Im\psi, \phi - \Im\phi\rangle. \end{aligned}$$

Therefore, $\Im$ is an enriched nonspreading mapping. Clearly, $\Im$ is not continuous. In fact, for $\psi_0 \in \partial B_2, \phi_0 \in \partial\mathcal{C}$, consider $\psi_n = \big(1 - \frac{1}{n}\big)\psi_0 + \frac{1}{n}\psi_0 \in \mathcal{C}$ for each ≥ 1. Then, $\psi_n \to \psi_0$ but $\Im\psi_n = P_{B_1}\psi_n \nrightarrow \Im\psi_0$ because $\|P_{B_1}\psi_n = 1$ and $\|\psi_0\| = 2$.

Remark 3.11. Note that $\Im$ is not continuous in the last example. Hence, $\Im$ is not uniformly continuous. In other words, the class of enriched nonspreading mappings is generally not Lipschitzian.

The following examples demonstrates the fact that the class of enriched nonexpansive and the class enriched nonspreading mappings are independent.

Example 3.12. Let $\mathbb{R} \supset \mathcal{C} = \left[\frac{1}{2}, 2\right]$ be endowed with the usual norm and let $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be define by $\Im\psi = \frac{1}{\psi}$ for all $\psi \in \mathcal{C}$. Then,

(i) $\Im$ is not nonexpansive,
(ii) $\Im$ is a $\frac{3}{2}$-enriched nonexpansive,
(iii) $F(\Im) = \{1\}$,
(iv) $\Im$ is not $\frac{3}{2}$-enriched nonspreading.

To validate (i)-(iv),

(i) assume $\Im$ is nonexpansive. Then, by the definition of nonexpansive mapping, we should have

$$|\Im\psi - \Im\phi| = \left|\frac{\phi - \psi}{\psi\phi}\right| \leq |\psi - \phi|, \quad \forall \psi, \phi \in \mathcal{C},$$

which, when $\psi = \frac{1}{2}$ and $\phi = 1$, yields a contradiction.

(ii) for all $\forall \psi, \phi \in \mathcal{C}$,

$$\begin{aligned} |\sigma(\psi - \phi) + \Im\psi - \Im\phi| &= \left|\sigma(\psi - \phi) + \frac{1}{\psi} - \frac{1}{\phi}\right| \\ &= \left|\sigma(\psi - \phi) + \frac{\phi - \psi}{\psi\phi}\right| \\ &= \left(\sigma - \frac{1}{\psi\phi}\right)|\psi - \phi|. \end{aligned}$$

Observe that for any $\sigma \geq \frac{3}{2}$, the last identity becomes

$$|\sigma(\psi - \phi) + \Im\psi - \Im\phi| = (\sigma + 1)|\psi - \phi|, \quad \forall \psi, \phi \in \mathcal{C},$$

and as such validates our conclusion that $\Im$ is a $\frac{3}{2}$-enriched nonexpansive

(iii) $F(\Im) = \{1\}$ is not difficult to see.

(iv) since every σ-enriched nonexpansive mapping satisfies σ-enriched Lipschitz condition (see, for instance, [26])

$$\|\sigma(\psi - \phi) + \Im\psi - \Im\phi\| \quad = \quad (\sigma + 1)L\|\psi - \phi\|, \quad \forall \psi, \phi \in \mathcal{C},$$

where L is the Lipschitz constant, and since every σ-enriched nonspreading mapping is generally not Lipschitzian, it follows from (ii) that $\Im$ is not an σ-enriched nonspreading mapping.

Example 3.13. Let $X = \mathbb{R}$ denote the set of real numbers with the usual norm. For each $\psi \in \mathbb{R}$, let the mapping $\Im$ be given by

$$\Im\psi = \begin{cases} 0, & \text{if} \quad \psi \in (-\infty, 2] \\ 1, & \text{if} \quad \psi \in (2, \infty). \end{cases}$$

Then, for all $\psi, \phi \in (-\infty, 2]$ and for all $\sigma \in [0, \infty)$, we have

$$\begin{aligned}(\sigma+1)^2|\psi-\phi|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle &= (\sigma^2 + 2\sigma + 1)|\psi - \phi|^2 + 2\psi\phi \\ &= (\sigma^2 + 2\sigma)|\psi - \phi|^2 + \psi^2 + \phi^2 \\ &\geq \sigma^2|\psi - \phi|^2 \\ &= |\sigma(\psi - \phi) + \Im\psi - \Im\phi|^2.\end{aligned}$$

Also, for all $\psi, \phi \in (2, \infty)$ and for all $\sigma \in [0, \infty)$, we have

$$\begin{aligned}(\sigma+1)^2|\psi-\phi|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle &= (\sigma+1)^2|\psi - \phi|^2 + 2(\psi - 1)(\phi - 1) \\ &> \sigma^2|\psi - \phi|^2 \\ &= |\sigma(\psi - \phi) + \Im\psi - \Im\phi|^2.\end{aligned}$$

Finally, if $\psi \in (-\infty, 2]$ and $\phi \in (2, \infty)$, then for all $\sigma \in [0, \infty)$, we get

$$\begin{aligned}(\sigma+1)^2|\psi-\phi|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle &= (\sigma^2 + 2\sigma)|\psi - \phi|^2 + \psi^2 + \phi^2 - 2\psi \\ &> |\sigma(\psi - \phi) - 1|^2 \\ &= |\sigma(\psi - \phi) + \Im\psi - \Im\phi|^2.\end{aligned}$$

Thus, for all $\psi, \phi \in X$ for all for all $\sigma \in [0, \infty)$, we obtain

$$|\sigma(\psi - \phi) + \Im\psi - \Im\phi|^2 \leq (\sigma+1)^2|\psi-\phi|^2 + 2\langle \psi - \Im\psi, \phi - \Im\phi\rangle.$$

Hence, $\Im$ is σ-enriched nonspreading. Since every σ-enriched nonexpansive mapping $\Im$ must satisfy σ-enriched Lipschitz condition (see, for instance, [26])

$$\|\sigma(\psi - \phi) + \Im\psi - \Im\phi\| \quad = \quad (\sigma+1)L\|\psi - \phi\|, \quad \forall \psi, \phi \in \mathcal{C},$$

where L is the Lipschitz constant, it is not difficult to see that $\Im$ is not σ-enriched nonexpansive mapping.

The next example shows that an σ-enriched nonspreading mapping needs not be nonspreading so that the class of σ-enriched nonspreading mappings properly contains the class of nonspreading mappings.

Example 3.14. Let $\mathbb{R}$ denote the reals with the usual norm and suppose the mapping $\Im : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$\Im\psi = -\psi.$$

Then, it is an σ-enriched nonspreading mapping. Indeed, for all $\psi, \phi \in \mathbb{R}$ with

$$|\sigma(\psi-\phi)+\Im\psi-\Im\phi|^2 \quad = \quad (\sigma-1)^2|\psi-\phi|^2$$

and

$$\begin{aligned}(\sigma+1)^2|\psi-\phi|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle &= (\sigma+1)^2|\psi-\phi|^2+2\langle 2\psi,2\phi\rangle\\ &= (\sigma+1)^2|\psi-\phi|^2+8\psi\phi,\end{aligned}$$

there exists an $\sigma \in [1,\infty)$ such that

$$|\sigma(\psi-\phi)+\Im\psi-\Im\phi|^2 \le (\sigma+1)^2|\psi-\phi|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle.$$

However, $\Im$ is not nonspreading, for if $\psi \neq 0$ and $\phi = -\psi$, then

$$|\Im\psi-\Im\phi|^2 = 4\psi^2 > -4\psi^2 = |\psi-\phi|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle.$$

Remark 3.15. If $F(\Im) \neq \emptyset$ in (3.2), then we obtain a class of mapping called σ-enriched quasi-nonexpansive mappings. The example below shows that this class of mappings properly contains the classes of σ-enriched nonspreading mappings and σ-enriched nonexpansive mappings.

Example 3.16. Let $\mathbb{R}$ denotes the reals and $\mathbb{R} \supset \mathcal{C} = [-\pi,\pi]$. Let a mapping $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be given by

$$\Im\psi = \psi\cos(\psi), \quad \psi \in \mathcal{C}.$$

Observe that $F(\Im)\{0\}$. Also, for all $\psi \in \mathcal{C}$,

$$\begin{aligned}|\sigma(\psi-0)+\Im\psi-0| &= |\sigma\psi+\psi\cos(\psi)|\\ &\le \sigma|\psi|+|\psi\cos(\psi)|\\ &\le \sigma|\psi|+|\psi|\\ &= (\sigma+1)|\psi-0|.\end{aligned}$$

Hence, $\Im$ is an σ-enriched quasi-nonexpansive mapping. However, $\Im$ is neither σ-enriched nonspreading mappings nor σ-enriched nonexpansive mappings. To see this, take $\psi = \pi$ and $\phi = \frac{\pi}{2}$. Then,

$$\begin{aligned}|\sigma(\psi-\phi)+\Im\psi-\Im\phi|^2 &= \left|\frac{\sigma\pi}{2}-\pi\right|^2 = \frac{1}{4}(\sigma+1)^2\pi^2\\ &> \frac{1}{4}(\sigma+1)^2\pi^2-2\pi^2\\ &= (\sigma+1)^2|\psi-\phi|^2+2\langle\psi-\Im\psi,\phi-\Im\phi\rangle\end{aligned}$$

and

$$\begin{aligned}|\sigma(\psi-\phi)+\Im\psi-\Im\phi| &= \frac{1}{2}(\sigma+2)\pi n\\ &> \frac{1}{2}(\sigma+1)\pi\\ &= (\sigma+1)|\psi-\phi|.\end{aligned}$$

Therefore, $\Im$ is neither σ-enriched nonspreading mappings nor σ-enriched nonexpansive mappings, respectively.

Lemma 3.17. *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \mathcal{C} \subset \mathcal{H}, u \in \mathcal{C}$ fixed, $\Im$ an σ-enriched nonspreading mapping from $\mathcal{C}$ into itself and $\eth$ an σ-enriched nonexpansive mapping such that $F(\Im) \cap F(\eth) \neq \emptyset$. Consider a bounded sequence $\{\phi_n\}_{n=1}^{\infty}$. Then,*

(1) *if $\|\phi_n - \eth\phi_n\| \to 0$ as $n \to \infty$, then*

$$\limsup_{n\to\infty}\langle u-\vartheta, \phi_n-\vartheta\rangle \leq 0, \tag{3.15}$$

where $\vartheta \in P_{F(\eth)}u$ is the unique fixed point in $F(\eth$ that satisfies the variational inequality

$$\langle u-\vartheta, \psi-\vartheta\rangle \leq 0, \quad \forall \psi \in F(\eth). \tag{3.16}$$

(2) *if $\|\phi_n - \Im\phi_n\| \to 0$ as $n \to \infty$, then*

$$\limsup_{n\to\infty}\langle u-\vartheta, \phi_n-\vartheta\rangle \leq 0, \tag{3.17}$$

where $\vartheta \in P_{F(\Im)}u$ is the unique fixed point in $F(\Im)$ that satisfies the variational inequality

$$\langle u-\vartheta, \psi-\vartheta\rangle \leq 0, \quad \forall \psi \in F(\Im). \tag{3.18}$$

(3) *if $\|\phi_n - \eth\phi_n\| \to 0$ as $n \to \infty$ and $\|\phi_n - \Im\phi_n\| \to 0$ as $n \to \infty$, then*

$$\limsup_{n\to\infty}\langle u-\vartheta_0, \phi_n-\vartheta_0\rangle \leq 0, \tag{3.19}$$

where $\vartheta_0 \in P_{F(\eth)\cap F(\Im)}u$ is the unique fixed point in $F(\Im)$ that satisfies the variational inequality

$$\langle u-\vartheta_0, \psi-\vartheta_0\rangle \leq 0, \quad \forall \psi \in F(\Im). \tag{3.20}$$

Proof. (1) Let ϑ satisfy (3.16). Let $\{\phi_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{\phi_n\}_{n=1}^{\infty}$ such that $\phi_{n_k} \rightharpoonup \nu$ as $k \to \infty$ (this is possible by the boundedness of $\{\phi_n\}_{n=1}^{\infty}$). Applying the hypothesis $\|\phi_n - \eth\phi_n\| \to 0$ as $n \to \infty$, and by the demiclosedness of $I - \eth$ at 0, we obtain that $\nu \in F(\eth)$ and

$$\begin{aligned}\limsup_{n\to\infty}\langle u-\vartheta,\phi_n-\vartheta\rangle &= \lim_{k\to\infty}\langle u-\vartheta,\phi_{n_k}-\vartheta\rangle\\ &= \langle u-\vartheta,\nu-\vartheta\rangle\\ &\le 0.\end{aligned}$$

(2) The proof is the same as in (1) since $\Im$ is demiclosed at 0.

(3) $\{\phi_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{\phi_n\}_{n=1}^{\infty}$ such that $\phi_{n_k} \rightharpoonup \omega$ as $k \to \infty$. Then, by the demiclosedness of $I - \eth$ and $I - \Im$ at 0, and applying the hypotheses $\|\phi_n - \eth\phi_n\| \to 0$ as $n \to \infty$ and $\|\phi_n - \Im\phi_n\| \to 0$ as $n \to \infty$, we get that $\omega \in F(\eth) \cap F(\Im)$. So,

$$\begin{aligned}\limsup_{n\to\infty}\langle u-\vartheta_0,\phi_n-\vartheta_0\rangle &= \lim_{k\to\infty}\langle u-\vartheta_0,\phi_{n_k}-\vartheta_0\rangle\\ &= \langle u-\vartheta_0,\omega-\vartheta_0\rangle\\ &\le 0.\end{aligned}$$

□

Lemma 3.18. *Let $u \in \mathcal{C}$ be an anchor and $\{\psi_n\}_{n=1}^{\infty}$ be the sequence given by*

$$\psi_{n+1} = \wp_n u + (1-\wp_n) Z_n \psi_n, \tag{3.21}$$

where $Z_n = \mu_n \eth_{\delta\beta} + (1-\mu_n)\Im_{\delta\beta}$. Then,

(1) *Z_n is quasi-nonexpansive for all $n \in \mathbb{N}$.*

(2) *$\{\psi_n\}_{n=1}^{\infty}, \{\eth\psi_n\}_{n=1}^{\infty}, \{\Im\psi_n\}_{n=1}^{\infty}, \{\eth\beta\psi_n\}_{n=1}^{\infty}, \{\Im_\beta\psi_n\}_{n=1}^{\infty}, \{\eth_{\delta\beta}\psi_n\}_{n=1}^{\infty}, \{\Im_{\delta\beta}\psi_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ are bounded sequences.*

Proof. Since $\Im_{\delta\beta}$ and $\eth_{\delta\beta}$ are quasi-nonexpansive, it follows that Z_n is quasi-nonexpansive giving the fact convex combination quasi-nonexpansive mappings is quasi-nonexpansive. The boundedness of $\{\psi_n\}_{n=1}^{\infty}$ follows directly from the nonexpansivity of Zn. Indeed, let $\vartheta \in F(\eth) \cap F(\Im)$. Then, from (3.21), we have

$$\begin{aligned}\|\psi_{n+1}-\vartheta\| &= \|\wp_n(u-\vartheta) + (1-\wp_n)(Z_n\psi_n - \vartheta)\|\\ &\le \wp_n\|u-\vartheta\| + (1-\wp_n)\|Z_n\psi_n-\vartheta\|\\ &\le \wp_n\|u-\vartheta\| + (1-\wp_n)\|\psi_n-\vartheta\|.\end{aligned} \tag{3.22}$$

Using the fact that

$$\|\psi_1-\vartheta\| = \max\{\|u-\vartheta\|, \|\psi_1-\vartheta\|\},$$

and, by induction,

$$\|\psi_n-\vartheta\| = \max\{\|u-\vartheta\|, \|\psi_1-\vartheta\|\},$$

it follows from (3.22) that

$$\begin{aligned}\|\psi_{n+1}-\vartheta\| &\leq \wp_n\|u-\vartheta\|+(1-\wp_n)\max\{\|u-\vartheta\|,\|\psi_1-\vartheta\|\}\\ &\leq \max\{\|u-\vartheta\|,\|\psi_1-\vartheta\|\}+(1-\wp_n)\max\{\|u-\vartheta\|,\|\psi_1-\vartheta\|\}\\ &= \max\{\|u-\vartheta\|,\|\psi_1-\vartheta\|\}.\end{aligned}$$

Consequently, $\{\psi_n\}_{n=1}^{\infty}$ is bounded. The boundedness of the rest of the sequences $[\{\eth\psi_n\}_{n=1}^{\infty}, \{\Im\psi_n\}_{n=1}^{\infty}, \{\eth\beta\psi_n\}_{n=1}^{\infty}, \{\Im_\beta\psi_n\}_{n=1}^{\infty}\{\eth_{\delta\beta}\psi_n\}_{n=1}^{\infty}, \{\Im_{\delta\beta}\psi_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}]$ follows directly from the boundedness of $\{\psi_n\}_{n=1}^{\infty}$ and by the quasi-nonexpansivity of involved mappings. □

Now, we establish our strong convergence theorem.

Theorem 3.19. *Let $\mathcal{H}$ be a real Hilbert space and let $\emptyset \neq \mathcal{C}$ be closed and convex. Let $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be σ-enriched nonspreading mapping and $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be σ-enriched nonexpansive mapping such that $F(\Im) \cap F(\eth) \neq \emptyset$. Let $\Im_{\delta\beta}$ and $\eth_{\delta\beta}$ be averaged type mappings, where $\Im_{\delta\beta} = (1-\delta) + \delta\Im_\beta$ and $\eth_{\delta\beta} = (1-\delta) + \delta\eth_\beta$. Suppose that $\{\wp_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ satisfying*

(i) $\lim\limits_{n\to\infty} \wp_n = 0$,

(ii) $\sum\limits_{n=1}^{\infty} \wp_n = \infty$.

Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ and define the sequence $\{\psi_n\}_{n=1}^{\infty}$ as follows

$$\psi_{n+1} = \wp_n u + (1-\wp_n)[\mu_n\eth_{\delta\beta}\psi_n + (1-\mu_n)\Im_{\delta\beta}\psi_n]. \tag{3.23}$$

Then, the following results hold:

(1) *If $\sum\limits_{n=1}^{\infty}(1-\mu_n) < \infty$, then $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to $\vartheta = P_{F(\eth)}u$ which is the unique solution in $F(\eth)$ of the variational inequality $\langle u - \vartheta, \psi - \vartheta\rangle \leq 0$ for all $\psi \in F(\eth)$.*

(2) *If $\sum\limits_{n=1}^{\infty}\mu_n < \infty$, then $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to $\vartheta = P_{F(\Im)}u$ which is the unique solution in $F(\Im)$ of the variational inequality $\langle u-\vartheta, \psi-\vartheta\rangle \leq 0$ for all $\psi \in F(\Im)$.*

(3) *If $\liminf\limits_{n\to\infty}\mu_n(1-\mu_n) > 0$, then $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to $\vartheta_0 = P_{F(\eth)\cap F(\Im)}u$ which is the unique solution in $F(\eth)\cap F(\Im)$ of the variational inequality $\langle u - \vartheta_0, \psi - \vartheta_0\rangle \leq 0$ for all $\psi \in F(\eth)\cap F(\Im)$.*

Proof. (1) Set $Q_n = (1-\wp_n)(\Im_{\delta\beta}\psi_n + \eth_{\delta\beta}\psi_n)$ so that (3.23) becomes

$$\psi_{n+1} = \wp_n u + (1-\wp_n)\eth_{\delta\beta}\psi_n + (1-\mu_n)Q_n, \tag{3.24}$$

where Q_n is bounded, that is, $\|Q_n\| \leq O(1)$. Firstly, we prove that $\lim\limits_{n\to\infty}\|\psi_n - \eth_{\delta\beta}\psi_n\| = 0$. Let $\vartheta \in F(\eth) = F(\eth_\beta) = F(\eth_{\delta\beta})$ be the unique solution in $F(\eth)$

of the variational inequality

$$\langle u-\vartheta, \psi-\vartheta\rangle \leq 0. \tag{3.25}$$

Then, we get from (3.24) that

$$\begin{aligned}\|\psi_{n+1}-\vartheta\|^2 &= \|\wp_n u + (1-\wp_n)\eth_{\delta\beta}\psi_n + (1-\mu_n)Q_n - \vartheta\|^2 \\ &= \|[(1-\wp_n)(\eth_{\delta\beta}\psi_n - \psi_n) + \psi_n - \vartheta] \\ &\quad + [\wp_n(u-\psi_n) + (1-\mu_n)Q_n]\|^2.\end{aligned}$$

The last identity together with Lemma 2.1 gives

$$\begin{aligned}\|\psi_{n+1}-\vartheta\|^2 &\leq \|(1-\wp_n)\delta\beta(\eth\psi_n - \psi_n) + \psi_n - \vartheta\|^2 \\ &\quad + 2\langle \wp_n(u-\psi_n) + (1-\mu_n)Q_n, \psi_{n+1}-\vartheta\rangle \\ &\leq \|(1-\wp_n)\delta\beta(\eth\psi_n - \psi_n) + \psi_n - \vartheta\|^2 \\ &\quad + 2\wp_n\langle (u-\psi_n), \psi_{n+1}-\vartheta\rangle + 2(1-\mu_n)\langle Q_n, \psi_{n+1}-\vartheta\rangle \\ &\leq (1-\wp_n)^2\delta^2\beta^2\|\eth\psi_n - \psi_n\|^2 + \|\psi_n-\vartheta\|^2 \\ &\quad - 2(1-\wp_n)\delta\beta\langle \psi_n-\vartheta, \psi_n - \eth\psi_n\rangle \\ &\quad + 2\wp_n\|u-\psi_n\|\|\psi_{n+1}-\vartheta\| + 2(1-\mu_n)\|Q_n\|\|\psi_{n+1}-\vartheta\| \\ &\leq (1-\wp_n)^2\delta^2\beta^2\|\eth\psi_n - \psi_n\|^2 + \|\psi_n-\vartheta\|^2 \\ &\quad - 2(1-\wp_n)\delta\beta\langle \psi_n-\vartheta, (I-\eth)\psi_n - (I-\eth)\vartheta\rangle \\ &\quad + 2\wp_n\|u-\psi_n\|\|\psi_{n+1}-\vartheta\| \\ &\quad + 2(1-\mu_n)\|Q_n\|\|\psi_{n+1}-\vartheta\| \\ &\leq \|\psi_n-\vartheta\|^2 + (1-\wp_n)^2\delta^2\beta^2\|\eth\psi_n - \psi_n\|^2 \\ &\quad - (1-\wp_n)\delta\beta\|(I-\eth)\psi_n - (I-\eth)\vartheta\| \\ &\quad + \wp_n O(1) + (1-\mu_n)O(1) \\ &= \|\psi_n-\vartheta\|^2 + (1-\wp_n)\delta\beta[1-(1-\wp_n)\delta\beta]\|\eth\psi_n - \psi_n\|^2 \\ &\quad + \wp_n O(1) + (1-\mu_n)O(1).\end{aligned} \tag{3.26}$$

and from which we obtained

$$\begin{aligned}0 &\leq (1-\wp_n)\delta\beta[1-(1-\wp_n)\delta\beta]\|\eth\psi_n - \psi_n\|^2 \\ &\leq (\|\psi_n-\vartheta\|^2 - \|\psi_{n+1}-\vartheta\|^2) + \wp_n O(1) + (1-\mu_n)O(1).\end{aligned} \tag{3.27}$$

In view of the fact that the sequence $\{\|\psi_n-\vartheta\|\}_{n=1}^{\infty}$ is monotone, we consider the following two cases:

Case 1: $\{\|\psi_n-\vartheta\|\}_{n=1}^{\infty}$ is monotonically non-increasing.

In fact, since $\{\|\psi_n-\vartheta\|\}_{n=1}^{\infty}$ is monotonically non-increasing, $\lim_{n\to\infty}\|\psi_n-\vartheta\|$ exists. Using the fact that $\lim_{n\to\infty}\wp_n = 0$ and $\sum_{n=1}^{\infty}(1-\mu_n) < \infty$, we obtain from

(3.27) that

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}[(1-\wp_n)\delta\beta[1-(1-\wp_n)\delta\beta]\|\eth\psi_n-\psi_n\|^2]\\ &\leq \limsup_{n\to\infty}(\|\psi_n-\vartheta\|^2-\|\psi_{n+1}-\vartheta\|^2+\wp_n O(1)+(1-\mu_n)O(1))\\ &= 0. \qquad (3.28)\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\|\eth\psi_n-\psi_n\|=0 \qquad (3.29)$$

and

$$\lim_{n\to\infty}\|\psi_n-\eth_{\delta\beta}\psi_n\|=\delta\beta\lim_{n\to\infty}\|\psi_n-\eth\psi_n\|=0. \qquad (3.30)$$

It therefore follows from Lemma 3.17 that

$$\limsup_{n\to\infty}\langle u-\vartheta,\psi_n-\vartheta\rangle\leq 0. \qquad (3.31)$$

Finally, we show that $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to ϑ. Now, from (3.24) and Lemma 2.1, we get

$$\begin{aligned}\|\psi_{n+1}-\vartheta\|^2 &= \|\wp_n(u-\vartheta)+(1-\wp_n)(\eth_{\delta\beta}\psi_n-\vartheta)+(1-\mu_n)Q_n\|^2\\ &\leq \|\wp_n(u-\vartheta)+(1-\wp_n)(\eth_{\delta\beta}\psi_n-\vartheta)\|^2+2(1-\mu_n)\langle Q_n,\psi_{n+1}\rangle\\ &\leq \wp_n^2\|u-\vartheta\|^2+(1-\wp_n)^2\|\eth_{\delta\beta}\psi_n-\vartheta\|^2\eth_{\delta\beta}\psi_n-\vartheta\rangle\\ &\quad +2\wp_n(1-\wp_n)\langle u-\vartheta,+(1-\mu_n)O(1). \qquad (3.32)\end{aligned}$$

Since $\eth$ is σ-enriched nonexpansive, it follows from

$$\begin{aligned}\|\eth_{\delta\beta}\psi_n-\vartheta\| &= \|(1-\delta)(\psi_n-\vartheta)+\delta(\eth_\beta\psi_n-\vartheta)\|\\ &\leq (1-\delta)\|\psi_n-\vartheta\|+\delta\|(1-\beta)\psi_n+\beta\eth\psi_n-\vartheta\|\\ &= (1-\delta)\|\psi_n-\vartheta\|+\frac{\delta}{\sigma+1}\|\sigma(\psi_n-\vartheta)+\eth\psi_n-\eth\vartheta\|\\ &\leq (1-\delta)\|\psi_n-\vartheta\|+\frac{\delta}{\sigma+1}[(\sigma+1)\|\psi_n-\vartheta\|]\\ &= \|\psi_n-\vartheta\|,\end{aligned}$$

$$\begin{aligned}\eth_{\delta\beta}\psi_n-\vartheta &= (1-\delta)\psi_n+\delta\eth_\beta\psi_n-\vartheta\\ &= (1-\delta)\psi_n+\delta[(1-\beta)\psi_n+\beta\eth\psi_n]-\vartheta\\ &= \psi_n-\vartheta+\delta\beta(\eth\psi_n-\psi_n)\\ &= \psi_n-\vartheta+\frac{\delta}{\sigma+1}(\eth\psi_n-\psi_n)\end{aligned}$$

and (3.32) that

$$\begin{aligned}\|\psi_{n+1}-\vartheta\|^2 &\le (1-\wp_n)^2\|\psi_n-\vartheta\|^2+\wp_n^2 O(1)\\ &\quad + 2\wp_n(1-\wp_n)\langle u-\vartheta, \psi_n-\vartheta+\frac{\delta}{\sigma+1}(\eth\psi_n-\psi_n)\rangle+(1-\mu_n)O(1)\\ &\le (1-\wp_n)^2\|\psi_n-\vartheta\|^2+\wp_n^2 O(1)+2\wp_n(1-\wp_n)\langle u-\vartheta,\psi_n-\vartheta\rangle\\ &\quad +\frac{2\wp_n(1-\wp_n)\delta}{\sigma+1}\langle u-\vartheta,\eth\psi_n-\psi_n\rangle+(1-\mu_n)O(1)\\ &\le (1-\wp_n)^2\|\psi_n-\vartheta\|^2+\wp_n^2 O(1)+2\wp_n(1-\wp_n)\langle u-\vartheta,\psi_n-\vartheta\rangle\\ &\quad +\wp_n O(1)\|\eth\psi_n-\psi_n\|+(1-\mu_n)O(1). \end{aligned} \tag{3.33}$$

If we set $\xi_n = \wp_n)O(1)+)O(1)\|\eth\psi_n-\psi_n\|+2\wp_n(1-\wp_n))\langle u-\vartheta,\psi_n-\vartheta\rangle$ and $\varphi_n=(1-\mu_n))O(1)$, then we obtain from (3.33) that

$$\|\psi_{n+1}-\vartheta\|^2 \;\le\; (1-\wp_n)^2\|\psi_n-\vartheta\|^2+\wp_n\xi_n+\varphi_n. \tag{3.34}$$

Thus, from assumptions $\sum_{n=1}^{\infty}(1-\mu_n)<\infty$ and $\sum_{n=1}^{\infty}\mu_n<\infty$, and from the fact that $\limsup_{n\to\infty}\langle\vartheta-\vartheta,\psi_n-\vartheta\rangle\ge 0$, we obtain Lemma 2.6.

Case 2: We can find a subsequence $\{\psi_{n_k}\}_{k=1}^{\infty}$ such that

$$\|\psi_{n_k}-\vartheta\|<\|\psi_{n_k+1}-\vartheta\|,\quad \forall k\in\mathbb{N}. \tag{3.35}$$

Then, by Lemma 2.7, we can find a sequence $\{\tau(n)\}_{n=1}^{\infty}$ that satisfies the following requirements:

Requirement (Z)

(i) $\{\tau(n)\}_{n=1}^{\infty}$ is s nondecreasing;;

(ii) $\lim_{n\to\infty}\tau(n)=\infty$;

(iii) $\|\psi_{\tau(n)}-\vartheta\|<\|\psi_{\tau(n+1)}-\vartheta\|,\quad \forall n\ge n_0$;

(iv) $\|\psi_n-\vartheta\|<\|\psi_{\tau(n+1)}-\vartheta\|,\quad \forall n\ge n_0$.

As a consequence, and following the σ-enriched nonexpansivity of $\eth$, we have

$$\begin{aligned} 0 &\le \liminf_{n\to\infty}(\|\psi_\tau(n+1)-\vartheta\|-\|\psi_{\tau(n)}-\vartheta\|)\\ &\le \limsup_{n\to\infty}(\|\psi_{\tau(n+1)}-\vartheta\|-\|\psi_{\tau(n)}-\vartheta\|)\\ &= \limsup_{n\to\infty}[\|\wp_{\tau(n)}(u-\eth_{\delta\beta}\psi_{\tau(n)})+\eth_{\delta\beta}\psi_{\tau(n)}-\vartheta+(1-\mu_{\tau(n)})Q_{\tau(n)}\|\\ &\quad -\|\psi_{\tau(n)}-\vartheta\|]\\ &\le \limsup_{n\to\infty}[\wp_{\tau(n)}\|u-\eth_{\delta\beta}\psi_{\tau(n)}\|+\|\eth_{\delta\beta}\psi_{\tau(n)}-\vartheta\|+(1-\mu_{\tau(n)})\|Q_{\tau(n)}\|\\ &\quad -\|\psi_{\tau(n)}-\vartheta\|] \end{aligned}$$

$$\begin{aligned}&\le \limsup_{n\to\infty}[\wp_{\tau(n)}\|u-\eth_{\delta\beta}\psi_{\tau(n)}\|+(1-\delta)\|\psi_{\tau(n)}-\vartheta\|\\&\quad+\frac{\delta}{\sigma+1}\|\sigma(\psi_{\tau(n)}-\vartheta)+\eth\psi_{\tau(n)}-\eth\vartheta\|\\&\quad+(1-\mu_{\tau(n)})\|Q_{\tau(n)}\|-\|\psi_{\tau(n)}-\vartheta\|]\\&\le \limsup_{n\to\infty}[\wp_{\tau(n)}\|u-\eth_{\delta\beta}\psi_{\tau(n)}\|+\|\psi_{\tau(n)}-\vartheta\|+(1-\mu_{\tau(n)})\|Q_{\tau(n)}\|\\&\quad-\|\psi_{\tau(n)}-\vartheta\|]=0,\end{aligned} \tag{3.36}$$

so that

$$\lim_{n\to\infty}(\|\psi_{\tau(n+1)}-\vartheta\|-\|\psi_{\tau(n)}-\vartheta\|)=0. \tag{3.37}$$

Since from (3.27),

$$\begin{aligned}0&\le(1-\wp_{\tau(n)})\delta\beta[1-(1-\wp_{\tau(n)})\delta\beta]\|\eth\psi_{\tau(n)}-\psi_{\tau(n)}\|^2\\&\le(\|\psi_{\tau(n)}-\vartheta\|^2-\|\psi_{\tau(n)+1}-\vartheta\|^2)\\&\quad+\wp_{\tau(n)}O(1)+(1-\mu_{\tau(n)})O(1),\end{aligned}$$

it follows from (3.37), $\sum_{n=1}^{\infty}(1-\mu_{\tau(n)})<\infty$ and $\lim_{n\to\infty}\wp_{\tau(n)}=0$ that

$$\lim_{n\to\infty}\|\eth\psi_{\tau(n)}-\psi_{\tau(n)}\|=0. \tag{3.38}$$

By Lemma 3.17, we get

$$\limsup_{n\to\infty}\langle u-\vartheta,\psi_{\tau(n)}-\vartheta\rangle=0. \tag{3.39}$$

Finally, we prove that $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to ϑ. Following the same approach as in Case 1, we obtain

$$\lim_{n\to\infty}\|\psi_{\tau(n)}-\vartheta\|=0$$

and from Requirement Z(iv) and (3.37), we reach the conclusion that

$$\lim_{n\to\infty}\|\psi_n-\vartheta\|=0. \tag{3.40}$$

(2) Now, we rewrite (3.23) in the form

$$\psi_{n+1}=\wp_n u+(1-\wp_n)\Im_{\delta\beta}\psi_n+\mu_n Q_n, \tag{3.41}$$

where $Q_n=(1-\wp_n)(\eth_{\delta\beta}\psi_n-\Im_{\delta\beta}\psi_n)$ is bounded, that is, $\|Q_n\|\le O(1)$. Firstly, we show that $\lim_{n\to\infty}\|\psi_n-\Im_{\delta\beta}\psi_n\|=0$.

Let $\vartheta\in F(\eth)=F(\eth_\beta)=F(\eth_{\delta\beta})$ be the unique solution in $F(\Im)$ of the variational inequality

$$\langle u-\vartheta,\psi-\vartheta\rangle\le 0. \tag{3.42}$$

Then, we get from (3.41) that

$$\begin{aligned}\|\psi_{n+1}-\vartheta\|^2 &= \|\wp_n u+(1-\wp_n)\Im_{\delta\beta}\psi_n+\mu_n Q_n-\vartheta\|^2\\ &= \|[(1-\wp_n)(\Im_{\delta\beta}\psi_n-\psi_n)+\psi_n-\vartheta]+[\wp_n(u-\psi_n)+\mu_n Q_n]\|^2\end{aligned}$$

From the last identity (together with Lemma 2.1), we obtain

$$\begin{aligned}\|\psi_{n+1}-\vartheta\|^2 &\le \|(1-\wp_n)\delta\beta(\Im\psi_n-\psi_n)+\psi_n-\vartheta\|^2\\ &\quad+2\langle\wp_n(u-\psi_n)+\mu_n Q_n,\psi_{n+1}-\vartheta\rangle\\ &\le \|(1-\wp_n)\delta\beta(\Im\psi_n-\psi_n)+\psi_n-\vartheta\|^2\\ &\quad+2\wp_n\langle(u-\psi_n),\psi_{n+1}-\vartheta\rangle+2\mu_n\langle Q_n,\psi_{n+1}-\vartheta\rangle\\ &\le (1-\wp_n)^2\delta^2\beta^2\|\Im\psi_n-\psi_n\|^2+\|\psi_n-\vartheta\|^2\\ &\quad-2(1-\wp_n)\delta\beta\langle\psi_n-\vartheta,\psi_n-\Im\psi_n\rangle+\wp_n O(1)+\mu_n O(1).\end{aligned} \tag{3.43}$$

Since $\vartheta\in F(\eth)=F(\eth_\beta)$ implies that

$$\vartheta=\Im\vartheta=\Im_\beta\vartheta,$$

which further implies that

$$0=(\sigma+1)\vartheta-(\sigma+\Im)\vartheta,$$

it follows from (3.43) and lemma 3.8 that

$$\begin{aligned}&\|\psi_{n+1}-\vartheta\|^2\\ &\quad\le (1-\wp_n)^2\delta^2\beta^2\|\Im\psi_n-\psi_n\|^2+\|\psi_n-\vartheta\|^2-\frac{2(1-\wp_n)\delta\beta(\sigma+1)}{\sigma+1}\\ &\quad\quad\times\langle\psi_n-\vartheta,(\sigma+1)\psi_n-(\sigma+\Im)\psi_n-[(\sigma+1)\vartheta-(\sigma+\Im)\vartheta]\rangle\\ &\quad\quad+\wp_n O(1)+\mu_n O(1)\\ &\quad\le (1-\wp_n)^2\delta^2\beta^2\|\Im\psi_n-\psi_n\|^2+\|\psi_n-\vartheta\|^2-\frac{2(1-\wp_n)\delta\beta}{\sigma+1}\\ &\quad\quad\times\Bigg\{\|(\sigma+1)\psi_n-(\sigma+\Im)\psi_n-[(\sigma+1)\vartheta-(\sigma+\Im)\vartheta]\|^2\\ &\quad\quad-\frac{1}{2}[\|(\sigma+1)\psi_n-(\sigma+\Im)\psi_n\|^2\|(\sigma+1)\vartheta-(\sigma+\Im)\vartheta\|^2]\Bigg\}\\ &\quad\quad+\wp_n O(1)+\mu_n O(1)\end{aligned}$$

$$
\begin{aligned}
&= \|\psi_n - \vartheta\|^2 + (1-\wp_n)^2\delta^2\beta^2\|\Im\psi_n - \psi_n\|^2 \\
&\quad - \frac{(1-\wp_n)\delta\beta}{\sigma+1}\|\psi_n - \Im\psi_n\|^2 + \wp_n O(1) + \mu_n O(1) \\
&= \|\psi_n - \vartheta\|^2 - \frac{(1-\wp_n)\delta\beta}{\sigma+1}[1-(1-\wp_n)\delta\beta(\sigma+1)]\|\psi_n - \Im\psi_n\|^2 \\
&\quad + \wp_n O(1) + \mu_n O(1).
\end{aligned} \tag{3.44}
$$

Hence,

$$
\begin{aligned}
0 &\le \frac{(1-\wp_n)\delta\beta}{\sigma+1}[1-(1-\wp_n)\delta\beta(\sigma+1)]\|\psi_n - \Im\psi_n\|^2 \\
&\le (\|\psi_n - \vartheta\|^2 - \|\psi_{n+1} - \vartheta\|^2) + \wp_n O(1) + (1-\mu_n)O(1).
\end{aligned} \tag{3.45}
$$

Again, we consider the monotony of the sequence $\{\|\psi_n - \vartheta\|\}_{n=1}^{\infty}$ in the following two cases:

Case 1: $\{\|\psi_n - \vartheta\|\}_{n=1}^{\infty}$ is monotonically non-increasing. Since $\{\|\psi_n - \vartheta\|\}_{n=1}^{\infty}$ is monotonically non-increasing, $\lim_{n\to\infty}\|\psi_n - \vartheta\|$ exists. Using the fact that $\lim_{n\to\infty}\wp_n = 0$ and $\sum_{n=1}^{\infty}(1-\mu_n) < \infty$, we obtain from (3.27) that

$$
\begin{aligned}
0 &\le \limsup_{n\to\infty}\left[\frac{(1-\wp_n)\delta\beta}{\sigma+1}[1-(1-\wp_n)\delta\beta(\sigma+1)]\|\psi_n - \Im\psi_n\|^2\right] \\
&\le \limsup_{n\to\infty}(\|\psi_n - \vartheta\|^2 - \|\psi_{n+1} - \vartheta\|^2 + \wp_n O(1) + (1-\mu_n)O(1)) = 0
\end{aligned}
$$

and as a consequence

$$
\limsup_{n\to\infty}\|\psi_n - \Im\psi_n\| = 0 \tag{3.46}
$$

and

$$
\lim_{n\to\infty}\|\psi_n - \Im_{\delta\beta}\psi_n\| = \delta\beta\lim_{n\to\infty}\|\psi_n - \Im\psi_n\| = 0. \tag{3.47}
$$

It therefore follows from Lemma 3.17 that

$$
\limsup_{n\to\infty}\langle u - \vartheta, \psi_n - \vartheta\rangle \le 0. \tag{3.48}
$$

Finally, we show that $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to ϑ. Now, from (3.41) and Lemma 2.1, we get

$$
\begin{aligned}
\|\psi_{n+1}-\vartheta\|^2 &= \|\wp_n(u-\vartheta)+(1-\wp_n)(\Im_{\delta\beta}\psi_n-\vartheta)+\mu_n Q_n\|^2 \\
&\le \|\wp_n(u-\vartheta)+(1-\wp_n)(\Im_{\delta\beta}\psi_n-\vartheta)\|^2+2\mu_n\langle Q_n,\psi_{n+1}\rangle \\
&\le \wp_n^2\|u-\vartheta\|^2+(1-\wp_n)^2\|\Im_{\delta\beta}\psi_n-\vartheta\|^2 \\
&\quad +2\wp_n(1-\wp_n)\langle u-\vartheta,\Im_{\delta\beta}\psi_n-\vartheta\rangle+\mu_n O(1) \\
&\le \wp_n^2\|u-\vartheta\|^2+(1-\wp_n)^2\|\Im_{\delta\beta}\psi_n-\vartheta\|^2\Im_{\delta\beta}\psi_n-\psi_n\rangle+ \\
&\quad +2\wp_n\langle u-\vartheta,\Im_{\delta\beta}\psi_n-\psi_n\rangle+2\wp_n\langle u-\vartheta,\psi_n-\vartheta\rangle+\mu_n O(1) \\
&\le (1-\wp_n)\|\psi_n-\vartheta\|^2+\wp_n^2 O(1)+\wp_n O(1)\|\Im\psi_n-\psi_n\| \\
&\quad +2\wp_n(1-\wp_n)\langle u-\vartheta,\psi_n-\vartheta\rangle+\mu_n O(1) \qquad (3.49)
\end{aligned}
$$

Set $\xi_n=\wp_n O(1)+)O(1)\|\Im\psi_n-\psi_n\|+2\wp_n(1-\wp_n))\langle u-\vartheta,\psi_n-\vartheta\rangle$ and $\varphi_n=\mu_n O(1)$, then (3.49) becomes

$$\|\psi_{n+1}-\vartheta\|^2 \quad\le\quad (1-\wp_n)\|\psi_n-\vartheta\|^2+\wp_n\xi_n+\varphi_n. \qquad (3.50)$$

Thus, from assumptions $\sum_{n=1}^{\infty}\mu_n<\infty$ and $\sum_{n=1}^{\infty}\wp_n=\infty$, and from the fact that $\limsup_{n\to\infty}\langle\vartheta-\vartheta,\psi_n-\vartheta\rangle\ge 0$, we can employ Lemma 2.6 and obtain the required result.

Case 2: We can find a subsequence $\{\psi_{n_k}\}_{k=1}^{\infty}$ such that

$$\|\psi_{n_k}-\vartheta\|<\|\psi_{n_k+1}-\vartheta\|,\quad \forall k\in\mathbb{N}. \qquad (3.51)$$

Then, by Lemma 2.7, we can find a sequence $\{\tau(n)\}_{n=1}^{\infty}$ that satisfies the following requirements:

Requirement (Z)

(i) $\{\tau(n)\}_{n=1}^{\infty}$ is s nondecreasing;
(ii) $\lim_{n\to\infty}\tau(n)=\infty$;
(iii) $\|\psi_{\tau(n)}-\vartheta\|<\|\psi_{\tau(n+1)}-\vartheta\|,\quad \forall n\ge n_0$;
(iv) $\|\psi_n-\vartheta\|<\|\psi_{\tau(n+1)}-\vartheta\|,\quad \forall n\ge n_0$.

Consequently,

$$
\begin{aligned}
0 &\le \liminf_{n\to\infty}(\|\psi_\tau(n+1)-\vartheta\|-\|\psi_{\tau(n)}-\vartheta\|) \\
&\le \limsup_{n\to\infty}(\|\psi_{\tau(n+1)}-\vartheta\|-\|\psi_{\tau(n)}-\vartheta\|) \\
&\le \limsup_{n\to\infty}(\|\psi_{n+1}-\vartheta\|-\|\psi_n-\vartheta\|) \\
&= \limsup_{n\to\infty}[\|\wp_n(u-\Im_{\delta\beta}\psi_n)+\Im_{\delta\beta}\psi_n-\vartheta+\mu_n Q_n\|-\|\psi_n-\vartheta\|] \\
&\le \limsup_{n\to\infty}[\wp_n\|u-\Im_{\delta\beta}\psi_n\|+\|\Im_{\delta\beta}\psi_n-\vartheta\|+\mu_n\|Q_n\|-\|\psi_n-\vartheta\|] \\
&\le \limsup_{n\to\infty}[\wp_n O(1)+\|\psi_n-\vartheta\|+\mu_n\|Q_n\|-\|\psi_n-\vartheta\|]=0.
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty}(\|\psi_{\tau(n+1)}-\vartheta\|-\|\psi_{\tau(n)}-\vartheta\|)=0. \tag{3.52}$$

Since from (3.45),

$$\begin{aligned}0 &\leq \frac{(1-\wp_{\tau(n)})\delta\beta}{\sigma+1}[1-(1-\wp_{\tau(n)})\delta\beta]\|\psi_{\tau(n)}-\Im\psi_{\tau(n)}\|^2\\ &\leq (\|\psi_{\tau(n)}-\vartheta\|^2-\|\psi_{\tau(n)+1}-\vartheta\|^2)+\wp_{\tau(n)}O(1)+\mu_{\tau(n)}O(1),\end{aligned}$$

it follows from (3.52), $\sum_{n=1}^{\infty}\mu_{\tau(n)}<\infty$ and $\lim_{n\to\infty}\wp_{\tau(n)}=0$ that

$$\lim_{n\to\infty}\|\psi_{\tau(n)}-\Im\psi_{\tau(n)}\|=0. \tag{3.53}$$

By Lemma 3.17, we get

$$\limsup_{n\to\infty}\langle u-\vartheta,\psi_{\tau(n)}-\vartheta\rangle=0. \tag{3.54}$$

Finally, we prove that $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to ϑ.

Following the same approach as in Case 1, we obtain

$$\lim_{n\to\infty}\|\psi_{\tau(n)}-\vartheta\|=0$$

and from Requirement Z(iv) and (3.53), we conclude that

$$\lim_{n\to\infty}\|\psi_n-\vartheta\|=0. \tag{3.55}$$

(3) Set $Z_n=\mu_n\eth_{\delta\beta}\psi_n+(1-\mu_n)\Im_{\delta\beta}\psi_n$ so that (3.23) becomes

$$\psi_{n+1}=\wp_n u+(1-\wp_n)Z_n. \tag{3.56}$$

Firstly, we prove that $\lim_{n\to\infty}\|\psi_n-Z_n\|=0$.

Let $\vartheta_0\in F(\eth)\cap F(\Im)$ be the unique solution of the variational inequality

$$\langle u-\vartheta_0,\psi-\vartheta_0\rangle\leq 0 \tag{3.57}$$

for all $\psi \in F(\eth) \cap F(\Im)$. Then, we estimate as follows:

$$
\begin{aligned}
&\|Zn\psi_n - \vartheta_0\|^2 \\
&\quad = \|\mu_n(\eth_{\delta\beta}\psi_n - \vartheta_0) + (1-\mu_n)(\Im_{\delta\beta}\psi_n - \vartheta_0)\|^2 \\
&\quad = \mu_n\|\eth_{\delta\beta}\psi_n - \vartheta_0\|^2 + (1-\mu_n)\|\Im_{\delta\beta}\psi_n - \vartheta_0\|^2 \\
&\qquad - \mu_n(1-\mu_n)\|\eth_{\delta\beta}\psi_n - \Im_{\delta\beta}\|^2 \\
&\quad \le \mu_n\|\psi_n - \vartheta_0\|^2 + (1-\mu_n)\left[\|\psi_n - \vartheta_0\|^2 - \frac{2(\sigma+1)}{\sigma}\|\psi_n - \Im_{\delta\beta}\psi_n\|^2\right] \\
&\qquad - \mu_n(1-\mu_n)\|\eth_{\delta\beta}\psi_n - \Im_{\delta\beta}\|^2 \quad (\text{by} \quad (\mathbf{??})) \\
&\quad = \|\psi_n - \vartheta_0\|^2 - \frac{2(1-\mu_n)(\sigma+1)}{\sigma}\|\psi_n - \Im_{\delta\beta}\psi_n\|^2 \\
&\qquad - \mu_n(1-\mu_n)\|\eth_{\delta\beta}\psi_n - \Im_{\delta\beta}\|^2. \qquad (3.58)
\end{aligned}
$$

But,

$$
\begin{aligned}
\|\psi_{n+1} - \vartheta_0\|^2 &= \|Z_n\psi_n - \vartheta_0 + \wp_n(u - Z_n\psi_n)\|^2 \\
&= \|Z_n\psi_n - \vartheta_0\|^2 + \wp_n[\wp_n\|u - Z_n\psi_n\|^2 \\
&\quad + 2\|Z_n\psi_n - \vartheta_0\|\|u - Z_n\psi_n\|] \\
&= \|Z_n\psi_n - \vartheta_0\|^2 + \wp_n O(1) \\
&\le \|\psi_n - \vartheta_0\|^2 - \frac{2(1-\mu_n)(\sigma+1)}{\sigma}\|\psi_n - \Im_{\delta\beta}\psi_n\|^2 \\
&\quad - \mu_n(1-\mu_n)\|\eth_{\delta\beta}\psi_n - \Im_{\delta\beta}\|^2 + \wp_n O(1).
\end{aligned}
$$

The last inequality implies that

$$
\frac{2(1-\mu_n)(\sigma+1)}{\sigma}\|\psi_n - \Im_{\delta\beta}\psi_n\|^2 \le \|\psi_n - \vartheta_0\|^2 - \|\psi_{n+1} - \vartheta_0\|^2 + \wp_n O(1) \qquad (3.59)
$$

and

$$
\mu_n(1-\mu_n)\|\eth_{\delta\beta}\psi_n - \Im_{\delta\beta}\psi_n\|^2 \le \|\psi_n - \vartheta_0\|^2 - \|\psi_{n+1} - \vartheta_0\|^2 + \wp_n O(1). \qquad (3.60)
$$

Now, we consider the following cases.

Case 1: Since $\{\|\psi_n - \vartheta_0\|\}_{n=1}^{\infty}$ is monotone non-increasing, $\lim_{n\to\infty}\|\psi_n - \vartheta_0\|$ exists. Also, since $\lim_{n\to\infty}\wp_n = 0$ and $\liminf_{n\to\infty}\mu_n(1-\mu_n) > 0$, it follows from (3.59) that

$$
\lim_{n\to\infty}\|\psi_n - \Im_{\delta\beta}\psi_n\| = \lim_{n\to\infty}\|\psi_n - \Im\psi_n\| = 0. \qquad (3.61)
$$

Moreover, from (3.60), we obtain

$$
\lim_{n\to\infty}\|\eth_{\delta,\beta}\psi_n - \Im_{\delta\beta}\psi_n\| = \lim_{n\to\infty}\|\eth\psi_n - \Im\psi_n\| = 0. \qquad (3.62)
$$

Since
$$\|\psi_n - \eth\psi_n\| \leq \|\psi_n - \Im\psi_n\| + \|\Im\psi_n - \eth\psi_n\|,$$
it follows from (3.61) and (3.62) that
$$\lim_{n\to\infty}\|\psi_n - \eth\psi_n\| = 0 \tag{3.63}$$
and
$$\begin{aligned}\lim_{n\to\infty}\|\psi_n - \eth_{\delta\beta}\psi_n\| &= \delta\lim_{n\to\infty}\|\psi_n - \eth_{\beta}\psi_n\| \\ &= \delta\beta\lim_{n\to\infty}\|\psi_n - \eth\psi_n\| \\ &= 0. \end{aligned}\tag{3.64}$$

Now, since
$$\|Z_n\psi_n - \psi_n\| \leq \mu_n\|\eth_{\delta\beta}\psi_n - \psi_n\| + (1-\mu_n)\|\Im_{\delta\beta}\psi_n - \psi_n\|,$$
it follows from (3.61) and (3.64) that
$$\lim_{n\to\infty}\|\psi_n - Z_n\psi_n\| = 0. \tag{3.65}$$

Finally, we show that $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to ϑ_0. For this, using the quasi-nonexpansivity property of Z_n, we estimate as follows. Now, from (3.56) and Lemma 2.1, we get
$$\begin{aligned}\|\psi_{n+1} - \vartheta_0\|^2 &= \|(1-\wp_n)(Z_n\psi_n - \vartheta_0) + \wp_n(u-\vartheta_0)\|^2 \\ &\leq (1-\wp_n)^2\|Z_n\psi_n - \vartheta_0\|^2 + \wp_n^2\|u-\vartheta_0\|^2 \\ &\quad + 2\wp_n(1-\wp_n)\langle Z_n\psi_n - \vartheta_0, u-\vartheta_0\rangle \\ &\leq (1-\wp_n)^2\|Z_n\psi_n - \vartheta_0\|^2 + \wp_n^2\|u-\vartheta_0\|^2 \\ &\quad + 2\wp_n(1-\wp_n)\langle Z_n\psi_n - \psi_n, u-\vartheta_0\rangle \\ &\quad + 2\wp_n(1-\wp_n)\langle \psi_n - \vartheta, u-\vartheta_0\rangle \\ &\leq (1-\wp_n)\|\psi_n - \vartheta_0\|^2 + \wp_n^2 O(1) + \wp_n O(1)\|Z_n\psi_n - \psi_n\| \\ &\quad + 2\wp_n(1-\wp_n)\langle \psi_n - \vartheta_0, u-\vartheta_0\rangle. \end{aligned}\tag{3.66}$$

Set $\xi_n = \wp_n O(1)+)O(1)\|Z_n\psi_n - \psi_n\| + 2(1-\wp_n))\langle u-\vartheta, \psi_n - \vartheta\rangle$ and $\varphi_n = \mu_n O(1)$, then (3.66) becomes
$$\|\psi_{n+1} - \vartheta\|^2 \leq (1-\wp_n)\|\psi_n - \vartheta\|^2 + \wp_n\xi_n + \varphi_n. \tag{3.67}$$
Thus, from assumptions $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\wp_n = \infty$, and from the fact that $\limsup_{n\to\infty}\langle\vartheta - \vartheta, \psi_n - \vartheta\rangle \geq 0$, we can employ Lemma 2.6 and obtain the required result.

Case 2: There exists a subsequence $\{\psi_{n_k}\}_{k=1}^{\infty}$ for which
$$\|\psi_{n_k} - \vartheta\| < \|\psi_{n_k+1} - \vartheta\|, \quad \forall k \in \mathbb{N}.$$

Then, by Lemma 2.7, we can find a sequence $\{\tau(n)\}_{n=1}^{\infty}$ that satisfies the following requirements:

Requirement (Z)

(i) $\{\tau(n)\}_{n=1}^{\infty}$ is s nondecreasing;

(ii) $\lim\limits_{n\to\infty} \tau(n) = \infty$;

(iii) $\|\psi_{\tau(n)} - \vartheta_0\| < \|\psi_{\tau(n+1)} - \vartheta_0\|, \quad \forall n \geq n_0$;

(iv) $\|\psi_n - \vartheta_0\| < \|\psi_{\tau(n+1)} - \vartheta_0\|, \quad \forall n \geq n_0$.

Consequently,

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty}(\|\psi_{\tau(n+1)} - \vartheta_0\| - \|\psi_{\tau(n)} - \vartheta_0\|) \\ &\leq \limsup_{n\to\infty}(\|\psi_{\tau(n+1)} - \vartheta_0\| - \|\psi_{\tau(n)} - \vartheta_0\|) \\ &\leq \limsup_{n\to\infty}(\|\psi_{n+1} - \vartheta_0\| - \|\psi_n - \vartheta_0\|) \\ &= \limsup_{n\to\infty}[\|\wp_n(u - \vartheta_0) + (1-\wp_n)(Z_n\psi_n - \vartheta_0)\| - \|\psi_n - \vartheta\|] \\ &\leq \limsup_{n\to\infty}[\wp_n\|u - \vartheta_0\| + (1-\wp_n)\|Z_n\psi_n - \vartheta_0\| - \|\psi_n - \vartheta\|] \\ &\leq \limsup_{n\to\infty}[\wp_n\|u - \vartheta_0\| + (1-\wp_n)\|\psi_n - \vartheta_0\| - \|\psi_n - \vartheta\|] \\ &\leq \limsup_{n\to\infty}[\wp_n O(1) + \|\psi_n - \vartheta\| - \|\psi_n - \vartheta\|] = 0. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}(\|\psi_{\tau(n+1)} - \vartheta\| - \|\psi_{\tau(n)} - \vartheta\|) = 0. \tag{3.68}$$

By (3.59),

$$\begin{aligned} 0 &\leq \frac{(1-\wp_{\tau(n)})\delta\beta}{\sigma+1}[1 - (1-\wp_{\tau(n)})\delta\beta]\|\psi_{\tau(n)} - \Im\psi_{\tau(n)}\|^2 \\ &\leq (\|\psi_{\tau(n)} - \vartheta\|^2 - \|\psi_{\tau(n)+1} - \vartheta\|^2) + \wp_{\tau(n)}O(1) + \mu_{\tau(n)}O(1) \end{aligned}$$

and by (3.60), we have

$$\begin{aligned} \mu_{\tau(n)}(1-\mu_{\tau(n)})\|\eth_{\delta\beta}\psi_{\tau(n)} - \Im_{\delta\beta}\psi_{\tau(n)}\|^2 \leq \|\psi_{\tau(n)} - \vartheta_0\|^2 - \|\psi_{\tau(n)+1} - \vartheta_0\|^2 \\ + \wp_{\tau(n)}O(1). \end{aligned}$$

Using the same argument as in Case 1, we obtain that

$$\lim_{n\to\infty}\|\psi_{\tau(n)} - \Im\psi_{\tau(n)}\| = 0, \tag{3.69}$$

$$\lim_{n\to\infty}\|\psi_{\tau(n)} - \eth\psi_{\tau(n)}\| = 0 \tag{3.70}$$

and

$$\lim_{n\to\infty}\|\psi_{\tau(n)} - Z_{\tau_n}\psi_{\tau(n)}\| = 0. \tag{3.71}$$

Thus, by Lemma 3.17, (3.69), (3.70) and (3.71), we get

$$\limsup_{n\to\infty}\langle u-\vartheta, \psi_{\tau(n)}-\vartheta\rangle = 0. \tag{3.72}$$

Finally, we prove that $\{\psi_n\}_{n=1}^{\infty}$ converges strongly to ϑ.

Using (3.71), the assumption that $\sum_{n=1}^{\infty}\wp_n = \infty$, (3.72) and following the same approach as in Case 1, we obtain

$$\lim_{n\to\infty}\|\psi_{\tau(n)}-\vartheta\| = 0.$$

From Requirement Z(iv) and (3.72), we conclude that

$$\lim_{n\to\infty}\|\psi_n-\vartheta\| = 0.$$

□

Remark 3.20. (1) The main result of this paper provides an affirmative answer to the question of Kurokawa and Takahashi; see Remark of page 1567 in [13].

(2) Note that [14, Theorem 4.1] is a weak convergence result and that our Theorem 3.19 is a strong convergence result. However, it worths mentioning that the method of proving Theorem 3.19 is very different from the one employed in proving Theorem 4.1.

(3) In most cases, strong convergence is more desirable than weak convergence.

4. Applications

In this section, we present some applications of Theorem 3.19 for the Ky Fan minimax inequality and other related Problems.

We know that Ky Fan minimax inequality problem is to search for $\psi \in \mathcal{C}$ that guarantees

$$(EP) \quad g(\psi,\phi) \geq 0 \quad \text{for each} \quad \phi \in \mathcal{C},$$

where $g: \mathcal{C}\times\mathcal{C} \longrightarrow \mathbb{R}$ is a bifunction. This problem includes the following problems as special cases: optimization problems, Nash equilibrium problems, variational inequality problems, fixed point problems, minimax inequalities, and saddle point problems (see, for instance, [4] and other related literature for more detail). The solution of Ky Fan minimax inequality problem (EP, for short) is represented with $EP(\mathcal{C}, g)$.

To solve the Ky Fan minimax inequality problem, the following assumptions are imposed on g.

Assumption:

(A_1) $g(\psi, \psi) \geq 0$ for each $\psi \in \mathcal{C}$;
(A_2) g is monotone, that is, $g(\psi, \phi) + g(\phi, \psi) \leq 0$ for any $\psi, \phi \in \mathcal{C}$;
(A_3) for each $\psi, \phi, z \in \mathcal{C}$, $\limsup_{t\to\infty} g(t\psi + (1-t)\phi, z) \leq g(\psi, .)$;
(A_4) for each $\psi \in \mathcal{C}$, the scalar function $\phi \to g(\psi, \phi)$ is convex and lower semicontinuous.

Theorem 4.1. *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be closed and convex and $G : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ be a function satisfying Assumption. Let $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be an σ-enriched nonspreading mapping and let*

$$T_r^G \psi = \left\{ \phi \in \mathcal{C} : G(\phi, e) + \frac{1}{r}\langle e - \phi, \phi - \psi \rangle \geq 0, \quad \forall y \in \mathcal{C} \right\}$$

for all $\psi \in \mathcal{H}$. Let $\beta, \delta \in (0,1)$, $\Im_{\delta\beta}$ be an averaged type mappings, where $\Im_{\delta\beta} = (1-\delta)I + \delta\Im_\beta$ and $\Im_\beta = (1-\beta)I + \beta\Im$. Suppose that $\Gamma = Fix(\Im) \cap EP(\mathcal{C}, G) \neq \emptyset$. Let $\{\wp_n\}_{n=1}^\infty$ be a real sequence in $(0,1)$ satisfying

(1) $\lim_{n\to\infty} \wp_n = 0$,
(2) $\sum_{n=1}^\infty \wp_n = \infty$.

Let $\psi_0 \in \mathcal{C}$ $[0,1]$ and define $\{\psi_n\}_{n=1}^\infty$ by

$$\psi_{n+1} = \wp_n \psi_0 + (1-\wp_n)\Im_{\delta\beta}(T_r^G \psi_n). \tag{4.1}$$

Then, $\lim_{n\to\infty} P_\Gamma \psi_0$ exist and convergent to a point in Γ.

Theorem 4.2. *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset \neq \mathcal{C} \subset \mathcal{H}$ be closed and convex and $G : \mathcal{C} \times \mathcal{C} \longrightarrow \mathbb{R}$ be a function satisfying Assumption. Let $\Im : \mathcal{C} \longrightarrow \mathcal{C}$ be an σ-enriched nonspreading mapping. Let $\beta, \delta \in (0,1)$, $\Im_{\delta\beta}$ be an averaged type mappings, where $\Im_{\delta\beta} = (1-\delta)I + \delta\Im_\beta$ and $\Im_\beta = (1-\beta)I + \beta\Im$. Suppose that $\Gamma = Fix(\Im) \neq \emptyset$. Let $\{\wp_n\}_{n=1}^\infty$ be a real sequence in $(0,1)$ satisfying*

(1) $\lim_{n\to\infty} \wp_n = 0$,
(2) $\sum_{n=1}^\infty \wp_n = \infty$.

Let $\psi_0 \in \mathcal{C}$ $[0,1]$ and define $\{\psi_n\}_{n=1}^\infty$ by

$$\psi_{n+1} = \wp_n \psi_0 + (1-\wp_n)\Im_{\delta\beta}\psi_n. \tag{4.2}$$

Then, $\lim_{n\to\infty} P_\Gamma \psi_0$ exist and convergent to a point in Γ.

Another worthy application of Theorem 3.19 is in the study of split feasibility problem (shortly, SFP). In [7], Censor and Elfving commenced an investigation on SFP for modeling inverse problems that emanates from medical image reconstruction. Since then, SFP has found numerous applications in such fields as bimedical engineering, control theory, geophysics, approximation theory, image processing, communications, etc; see [6], [17] for more details. The SFP is presented as follows:

$$\text{Search for} \quad \wp^\star \in \mathcal{C} \quad \text{whic guarantees that} \quad \hbar^\star = \Im\wp^\star \in \mathcal{Q}, \tag{4.3}$$

where $\emptyset \neq \mathcal{C}, \mathcal{Q}$ are closed and convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $\Im : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a bounded linear operator.

Theorem 4.3. *Let $\Im : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ be an σ-enriched nonspreading mapping with $F(\Im) \neq \emptyset$. and $\eth : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ be σ-enriched nonspreading mapping with $F(\eth) \neq \emptyset$. $\mathcal{A} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator with $\|\mathcal{A}\| > 0$. Suppose that $\Gamma = \{\psi \in Fix(\Im), \mathcal{A}\psi \in Fix(\eth)\} \neq \emptyset$. Let $\delta, \beta \in (0,1)$ and $V = I_1 - \frac{1}{\|\mathcal{A}\|^2}\mathcal{A}^\star(I_2 - \eth_{\delta\beta})\mathcal{A}$, where I_1 and I_2 are identity mappings on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\Im_{\delta\beta}$ and $\eth_{\delta\beta}$ be averaged type mappings, where $\Im_{\delta\beta} = (1-\delta)I_1 + \delta\Im_\beta$, $\eth_{\delta\beta} = (1-\delta)I_2 + \delta\eth_\beta$, $\Im_\beta = (1-\beta)I_1 + \beta\Im$ and $\eth_\beta = (1-\beta)I_2 + \beta\eth_\beta$. Suppose that $\{\wp_n\}_{n=1}^\infty$ is a real sequence in $(0,1)$ satisfying*

(1) $\lim_{n\to\infty} \wp_n = 0$,
(2) $\sum_{n=1}^\infty \wp_n = \infty$.

Let $\psi_0 \in \mathcal{C}$ and define the sequence $\{\psi_n\}_{n=1}^\infty$ by

$$\psi_{n+1} = \wp_n\psi_0 + (1-\wp_n)\Im_{\delta\beta}V\psi_n. \tag{4.4}$$

Then, $\lim_{n\to\infty} P_\Gamma\psi_0$ exist and convergent to a point in Γ.

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