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MONOTONE METHOD FOR NONLINEAR HILFER FRACTIONAL REACTION-DIFFUSION EQUATIONS

Pandurang D. Kundgar[1], Jagdish A. Nanware[2] and Gunvant A. Birajdar[3]

[1]Department of PG Studies and Research in Mathematics,
Shrikrishana Mahavidyalaya, Gunjoti, Dist.Dharashiv-413606(MS), India
e-mail: pandurang.kundgar30@gmail.com

[2]Department of PG Studies and Research in Mathematics,
Shrikrishana Mahavidyalaya, Gunjoti, Dist.Dharashiv-413606(MS), India
e-mail: jag_skmg91@rediffmail.com

Department of Mathematics, Institute of Chemical Technology,
Mumbai (MS), 400 019, India
e-mail: gabirajdar11@gmail.com

Abstract. In this paper, we developed the existence and uniqueness results by monotone method for non-linear fractional reaction-diffusion equation together with initial and boundary conditions. In this text the Hilfer fractional derivative is used to denote the time fractional derivative. The employment of monotone method generates two sequences of minimal and maximal solutions which converges to lower and upper solutions respectively.

1. Introduction

In recent times, there has been a significant rise in the study of fractional calculus in mathematics, as well as in other areas of science and engineering. Researchers have been focusing on studying various fractional differential operators with different characteristics, such as generalization and qualitative properties. Some of these operators include Riemann-Liouville (R-L), Caputo,

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[0]Corresponding author: Gunvant A. Birajdar(gabirajdar11@gmail.com).

Hadamard, Hilfer, ψ-Caputo, ψ-Hilfer, and more. Among these, the Hilfer fractional differential operator has been found to be useful in many physical situations, which has attracted researchers to work on it.

Over the last decade, several authors have made significant progress in this field, as described in articles like [1, 2, 4, 5, 10, 11, 15, 23]. These articles provide preliminary properties and basic results in this area. Researchers have also studied initial boundary value problems (IBVP) for fractional diffusion, and monotone iterative techniques have proven to be useful in dealing with such problems.

For instance, Furati et al. [9] considered an initial value problem for a class of Hilfer fractional differential equations. The monotone method combining lower and upper solutions for R-L IBVP for reaction-diffusion equations is exhibited in literature [7, 22, 28, 29]. The developments of such problems using the monotone method have been extensively studied in [8, 17, 16, 18, 19, 20, 21, 26]. However, more literature is needed on IBVP for nonlinear Hilfer fractional diffusion equations by the monotone method.

This has motivated us to develop a monotone method with lower and upper solutions. Two monotone convergent sequences are constructed, which converge to the minimal and maximal solutions to the problem. The application of the method existence and uniqueness of the IBVP discussed in detail.

Consider the nonlinear Hilfer fractional reaction-diffusion equations with initial and boundary conditions

$$\partial_t^{\alpha,\beta} u - k u_{xx} = f(x,t,u) \ \ on \ \ Q_T, \tag{1.1}$$

$$u(0,t) = A(t), u(L,t) = B(t) \ \ in \ \ \Gamma_T, \text{(boundary conditions)}, \tag{1.2}$$

$$\Gamma(\alpha) t^{1-\alpha} u(x,t) \mid_{t=0} = f^0(x) \ \ x \in \Omega, \text{(initial condition)}\ , \tag{1.3}$$

where $f \in C[[0,L] \times [0,T] \times \mathbb{R}, \mathbb{R}]$, $\Omega = [0,L]$, $J = (0,T]$, $Q_T = J \times \Omega$, $k > 0$ and $\Gamma_T = (0,T) \times \partial\Omega$. The equation (1.1)- together with initial and boundary conditions (1.2)-(1.3) is called IBVP for nonlinear fractional reaction diffusion equation. Here $\partial_t^{\alpha,\beta}$ is partial Hilfer fractional derivative with respect to time t of order $0 \le \alpha \le 1$ and $0 < \beta < 1$.

The paper is organized as follows: Section 2 provides the basic definitions, while Section 3 presents the comparison results. In section 4, we discuss the monotone method and the existence and uniqueness of the solution of IBVP (1.1)-(1.3). Finally, the paper concludes with a summary.

2. Preliminaries

In this section, we will review some definitions and results that will be helpful in developing our main findings.

Definition 2.1. ([25]) The Riemann-Liouville fractional integral of order α of a function $u(t)$ is defined as

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \quad 0 < \alpha \leq 1.$$

Definition 2.2. ([25]) The Riemann-Liouville fractional derivative of order α for a function $u(t)$ is defined as

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{\alpha-1}u(s)ds, \quad t > 0, \ \ n-1 < \alpha < n.$$

Definition 2.3. ([25]) The Caputo fractional derivative of order α for a function $u(t)$ is defined as

$${}^cD_{0+}^{\alpha}u(t) = D_{0+}^{\alpha}\left[u(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}u^k(0)\right], \quad t > 0, \ \ n-1 < \alpha < n,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.4. ([11]) (Hilfer fractional derivative) The generalized Riemann-Liouville fractional derivative of order $0 \leq \alpha \leq 1$ and $0 < \beta < 1$ with lower limit a is defined as

$$D_{a+}^{\alpha,\beta}u(t) = I_{a+}^{\alpha(1-\beta)}\frac{d}{dt}I_{a+}^{(1-\alpha)(1-\beta)}u(t)$$

for functions such that the expression on the right-hand side exists.

Definition 2.5. ([25]) The two parameter Mittag-Leffler function is defined as

$$E_{\alpha,r}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty}\frac{(\lambda t^{\alpha})^k}{\Gamma(\alpha k + r)}.$$

Definition 2.6. ([11]) A function $\phi(t) \in C(J,\mathbb{R})$ is a C_p continuous function, if $t^{1-\alpha}\phi(t) \in C(J_0,\mathbb{R})$, where $p = 1-\alpha$, $J = (0,T]$, $J_0 = [0,T]$. The set of C_p continuous functions is denoted by $C_p(J,\mathbb{R})$. Further, given a function $\phi(t) \in C_p(J,\mathbb{R})$, we call the function $t^{1-\alpha}\phi(t)$, the continuous extension of $\phi(t)$.

Remark 2.7. Note that any continuous function in J_0 is also a C_p continuous function.

Lemma 2.8. ([8]) (Comparison Result) *Let $\eta \in C_p[J_0,\mathbb{R}]$ be such that for some $t_1 \in (0,T]$, $\eta(t_1) = 0$, and $t^{1-\alpha}\eta(t) \leq 0$ on $[0,t_1]$. Then $D^{\alpha}\eta(t_1) \geq 0$.*

3. Auxiliary results

In this section, we obtain a comparison results for the nonlinear Hilfer fractional reaction-diffusion equations with initial and boundary conditions (1.1)-(1.3). The comparison theorem is with respect to the lower and upper solutions when the nonlinear term is of the form $f(x,t,u)$, where $f(x,t,u)$ satisfies one sided Lipschitz condition. In this case, we assume the nonlinear function in $u(x,t)$ for (x,t) in $[0,L]\times[0,T]$.

Consider the non-linear Hilfer fractional diffusion equations

$$\partial_t^{\alpha,\beta}u - ku_{xx} = f(x,t,u) \quad on \quad Q_T, \tag{3.1}$$

$$u(0,t) = A(t), u(L,t) = B(t) \quad in \quad \Gamma_T, \tag{3.2}$$

$$\Gamma(\alpha)t^{1-\alpha}u(x,t)\mid_{t=0} = f^0(x) \quad x\in\Omega, \tag{3.3}$$

where $f\in C[[0,L]\times[0,T]\times\mathbb{R},\mathbb{R}]$, $\Omega=[0,L]$, $J=(0,T]$, $Q_T=J\times\Omega$, $k>0$ and $\Gamma_T=(0,T)\times\partial\Omega$.

For compatible conditions for IBVP (3.1)-(3.3), we assume that

$$f^0(0) = A(0) = f^0(L) = B(0) = 0,\ \ \Gamma(\alpha)t^{1-\alpha}u(x,t)\mid_{t=0} = f^0(x).$$

In the throughout of this work, we assume that initial and boundary condition satisfy the compatibility conditions. Using the method of eigenfunction expansion, the solution of (3.1) is of the form:

$$u(x,t) = \sum_{n=0}^{\infty} b_n(t)\phi_n(x), \tag{3.4}$$

where the eigenfunctions of the related homogeneous problem are known to be $\phi_n(x) = sin\frac{n\pi x}{L}$ and its corresponding eigenvalues are $\lambda_n = \left[(\frac{n\pi}{L})^2\right]$. Using the standard arguments, one can compute $b_n(t)$ as follows:

$$\begin{aligned} b_n(t) = b_n^0 t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n t^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n t^\alpha)q_n(\tau) \\ + k\frac{2n\pi}{L^2}[A(\tau)-(-1)^n B(\tau)]d\tau, \end{aligned}$$

where

$$b_n^0 = \frac{2}{L}\int_0^L f^0(y)\phi_n(y)dy$$

and

$$q_n(t) = \frac{2}{L}\int_0^L f(x,t,u)\phi_n(y)dy.$$

Therefore,

$$\begin{aligned} b_n(t) =& \frac{2}{L}\int_0^L f^0(y)\phi_n(y)dy t^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n t^\alpha) \\ &+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n t^\alpha)\frac{2}{L}\int_0^L f(y,t,u)\phi_n(y)dyd\tau \\ &+ k\frac{2n\pi}{L^2}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n t^\alpha)[A(\tau)-(-1)^n B(\tau)]d\tau. \end{aligned}$$

So, using $b_n(t)$ in above equation, the solution $u(x,t)$, we have

$$\begin{aligned} u(x,t) =& \int_0^L t^{\alpha-1}\left[\sum_{n=1}^\infty \frac{2}{L}E_{\alpha,\alpha}(-k\lambda_n t^\alpha)\phi_n(x)\phi_n(y)\right] f^0(y)dy \\ &+ \int_0^t\int_0^L \left[\sum_{n=1}^\infty \frac{2}{L}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n (t-\tau)^\alpha)\phi_n(x)\phi_n(y)\right] \\ &\quad \times f(y,t,u)dyd\tau \\ &+ k\int_0^t \left[\frac{2n\pi}{L^2}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n(t-\tau)^\alpha)\phi_n(x)\right] A(\tau)d\tau \\ &- k\int_0^t \left[\frac{2n\pi}{L^2}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-k\lambda_n(t-\tau)^\alpha)\phi_n(x)\right] B(\tau)d\tau. \end{aligned}$$

After, simplifying we get

$$\begin{aligned} u(x,t) =& \int_0^L t^{\alpha-1}G(x,y,t)f^0(y)dy + \int_0^t\int_0^L G(x,y,t-\tau)f(y,t,u)dyd\tau \\ &+ k\int_0^t G_y(x,0,t-\tau)A(\tau)d\tau - k\int_0^t G_y(x,L,t-\tau)B(\tau)d\tau, \end{aligned}$$

where

$$G(x,y,t) = \sum_{n=0}^\infty \frac{2}{L}E_{\alpha,\alpha}(-k\lambda_n t^\alpha)\phi_n(x)\phi_n(y).$$

This result is useful in our main result for computing the linear approximations of the monotone iterates. We recall lemmas regarding the Mittage-Leffler function in the series form.

Lemma 3.1. ([6]) *Let $E_{\alpha,1}(-\lambda t^\alpha)$ be the Mittage-Leffler function of order α, where $0<\alpha\le 1$. Then, $\frac{E_{\alpha,1}(-\lambda_1 t^\alpha)}{E_{\alpha,1}(-\lambda_2 t^\alpha)} < 1$, where $\lambda_1,\lambda_2>0$ such that $\lambda_1=\lambda_2+c$ for $c>0$.*

Lemma 3.2. ([6]) *Let $E_{\alpha,\alpha}(-\lambda t^{\alpha})$ be the Mittage-Leffler function of order α, where $0 < \alpha \leq 1$. Then, $\frac{E_{\alpha,\alpha}(-\lambda_1 t^{\alpha})}{E_{\alpha,\alpha}(-\lambda_2 t^{\alpha})} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + c$ for $c > 0$.*

Now, we show the convergence of the above solutions using Lemma 3.1 and Lemma 3.2 above. We can split the solution of (3.1) as $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ respectively as follows:

(a) $u_1(x,t)$ is the solution of (3.1), when $f(x,t,u) = 0$, $A(t) = 0 = B(t)$,
(b) $u_2(x,t)$ is the solution of (3.1), when $A(t) = 0 = B(t)$, $f^0 = 0$,
(c) $u_3(x,t)$ is the solution of (3.1), when $f(x,t,u) = 0$, $f^0 = 0$.

Theorem 3.3. ([6]) *$u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ converge when $|f^0(x)| < N_1$, $N_1 > 0$, $|f(x,t,u)| < N_2$, $N_2 > 0$; $|A(t)| < M_1$, $M_1 > 0$ and $|B(t)| < M_2$, $M_1, M_2 > 0$, respectively.*

Definition 3.4. ([6]) The functions $v(x,t)$, $w(x,t) \in C^{2,\alpha}[Q_T, \mathbb{R}]$ are called the natural lower and upper solutions of (3.3) if

$$\begin{aligned} \partial_t^{\alpha,\beta} v - k v_{xx} \leq f(x,t,v) \quad on \quad Q_T, \\ v(0,t) \leq A(t), v(L,t) \leq B(t) \quad in \quad \Gamma_T, \\ \Gamma(\alpha) t^{1-\alpha} v(x,t) \mid_{t=0} \leq f^0(x) \quad x \in \Omega \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \partial_t^{\alpha,\beta} w - k w_{xx} \geq f(x,t,w) \quad on \quad Q_T, \\ w(0,t) \geq A(t), w(L,t) \geq B(t) \quad in \quad \Gamma_T, \\ \Gamma(\alpha) t^{1-\alpha} w(x,t) \mid_{t=0} \geq f^0(x) \quad x \in \Omega. \end{aligned} \tag{3.6}$$

The next result is a comparison result relative to lower and upper solutions of (3.3).

Theorem 3.5. *Assume that*

(i) *$v(x,t), w(x,t) \in C^{2,\alpha}[Q_T, \mathbb{R}]$ are natural lower and upper solutions of (3.3), respectively and*

$$\Gamma(\alpha) t^{1-\alpha} v(x,t) \mid_{t=0} \leq \Gamma(\alpha) t^{1-\alpha} w(x,t) \mid_{t=0},$$
$$v(0,t) \leq w(0,t), \quad v(L,t) \leq w(L,t).$$

(ii) *$f(x,t,u)$ satisfies the one sided Lipschitz condition*

$$f(x,t,u) - f(x,t,u^*) \leq M(u_1 - u_2^*),$$

whenever $u \leq u^*$ and $M > 0$.

Then $v(x,t) \leq w(x,t)$ on $J \times \Omega$.

Proof. The primary aim to prove the theorem is to first show that one of the inequalities in (i) is strictly satisfied.

For this purpose, consider $m(x,t) = v(x,t) - w(x,t)$. We claim that $m(x,t) < 0$, $(x,t) \in \Omega \times J$. Suppose that the conclusion is not true. Then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{1-\alpha} m(x_1,t_1) < 0$ on $[0,t_1)$, $m(x_1,t_1) = 0$. It easy to check $m_x(x_1,t_1) = 0$ and $m_{x,x}(x_1,t_1) = 0$. Then, using Lemma 2.1, we get $\partial_t^{\alpha,\beta} m(x,t) \geq 0$

From the hypothesis, we also have

$$\begin{aligned}
\partial_t^{\alpha,\beta} m(x_1,t_1) &= \partial_t^{\alpha,\beta} v(x_1,t_1) - \partial_t^{\alpha,\beta} w(x_1,t_1) \\
&< k\frac{\partial^2 v(x_1,t_1)}{\partial x^2} + f(x_1,t_1,v) - k\frac{\partial^2 w(x_1,t_1)}{\partial x^2} - f(x_1,t_1,w) \\
&< f(x_1,t_1,v) - f(x_1,t_1,w) \\
&= 0,
\end{aligned}$$

which is a contradiction. Therefore, $v(x,t) < w(x,t)$ on $\overline{Q_T}$.

In order to prove the theorem for the non strict inequalities, let

$$\begin{aligned}
w^*(x,t) &= w(x,t) + \epsilon t^{\alpha-1} E_{\alpha,\alpha}[2Mt^\alpha], \\
v^*(x,t) &= v(x,t) - \epsilon t^{\alpha-1} E_{\alpha,\alpha}[2Mt^\alpha].
\end{aligned}$$

From this, it follows

$$\begin{aligned}
w^*(0,t) > v^*(0,t), \quad w(L,t) > v(L,t), \\
\Gamma(\alpha) t^{1-\alpha} w^*(x,t)\mid_{t=0} > \Gamma(\alpha) t^{1-\alpha} w(x,t)\mid_{t=0} > \Gamma(\alpha) t^{1-\alpha} v(x,t)\mid_{t=0} \\
> \Gamma(\alpha) t^{1-\alpha} v^*(x,t)\mid_{t=0}.
\end{aligned}$$

Then

$$\begin{aligned}
\partial_t^{\alpha,\beta} w(x,t) - k\frac{\partial^2 w^*(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta} w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} + \epsilon t^{\alpha-1} E_{\alpha,\alpha}[2Mt^\alpha] \\
&\geq f(x,t,w) + \epsilon t^{\alpha-1} 2M E_{\alpha,\alpha}[2Mt^\alpha] \\
&= f(x,t,w) + M\epsilon t^{\alpha-1} E_{\alpha,\alpha}[2Mt^\alpha] \\
&= f(x,t,w^*) + \epsilon M t^{\alpha-1} E_{\alpha,\alpha}[2Mt^\alpha] \\
&> f(x,t,w^*) \quad on \quad Q_T.
\end{aligned}$$

Similarly,

$$\partial_t^{\alpha,\beta} v^*(x,t) - k\frac{\partial^2 v^*(x,t)}{\partial x^2} > f(x,t,v^*) \quad \text{on} \quad Q_T.$$

By the strict inequality result, $v^* < w^*$ on $\overline{Q_T}$. Letting $\epsilon \to 0$ we have $v \leq w$ on $\overline{Q_T}$. □

The next result is the maximum principle for the Hilfer parabolic equation in one dimensional space which will be useful in proving the uniqueness of the solution.

Corollary 3.6. *Let*

$$\begin{aligned} \partial_t^{\alpha,\beta} m(x,t) - k\frac{\partial^2 m(x,t)}{\partial x^2} &\leq 0 \quad on \quad Q_T, \\ m(0,t) \leq 0, m(L,t) &\leq 0 \quad on \quad \Gamma_T, \\ \Gamma(\alpha) t^{1-\alpha} m(x,t)\mid_{t=0} &\leq 0 \quad on \quad \overline{\Omega}. \end{aligned}$$

Then $m(x,t) \leq 0$ on Q_T.

Proof. Suppose $m(x,t)$ has positive maximum at (x_1,t_1). Let $m(x_1,t_1) = K$. Let $\overline{m}(x,t) = m(x,t) - K$. Then $t^{1-\alpha}\overline{m}(x,t) \leq 0$ on $(0,t_1]$ and $\overline{m}(x_1,t_1) = 0$. Using Lemma 2.1, we get $\partial_t^{\alpha,\beta} m(x_1,t_1) \geq 0$. Also $\partial_t^{\alpha,\beta} m(x_1,t_1) \leq 0$. Combining these two, we get

$$\partial_t^{\alpha,\beta} m(x_1,t_1) - k\frac{\partial^2 m(x_1,t_1)}{\partial x^2} \geq 0.$$

Also, we have

$$\begin{aligned} \partial_t^{\alpha,\beta}\overline{m}(x,t) - K\frac{\partial^2 \overline{m}(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta} m(x,t) - K\frac{\partial^2 m(x,t)}{\partial x^2} - K\frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &< \partial_t^{\alpha,\beta} m(x,t) - K\frac{\partial^2 m(x,t)}{\partial x^2} \\ &< 0, \end{aligned}$$

which gives a contradiction. Hence, $m(x,t) \leq 0$. □

4. Main results

This section, to develop monotone method for Hilfer fractional reaction-diffusion equation (3.3) using lower and upper solutions. Also obtained existence and uniqueness of solution of problem (3.3).

Theorem 4.1. (i) *Let $f(x,t,u)$ in $C^{2,\alpha}[\Omega \times J \times \mathbb{R}, \mathbb{R}]$ be nondecreasing.*

(ii) *Let (v_0, w_0) be the lower and upper solutions of (3.3) such that $t^{1-\alpha}v_0 \leq t^{1-\alpha}w_0$ on $\overline{Q_T}$.*

(iii) *Let $f(x,t,u)$ satisfies the one sided Lipschitz condition*

$$f(x,t,u) - f(x,t,u^*) \geq -M(u-u^*),$$

whenever $u^* \leq u$ and $M > 0$.

Then there exist monotone sequences $\{t^{1-\alpha}v_n(x,t)\}$ and $\{t^{1-\alpha}w_n(x,t)\}$ such that $t^{1-\alpha}v_n(x,t) \to t^{1-\alpha}\rho(x,t)$ and $t^{1-\alpha}w_n(x,t) \to t^{1-\alpha}\gamma(x,t)$ uniformly and *monotonically on* $\overline{Q_T}$*, where* $\rho(x,t)$ *and* $\gamma(x,t)$ *are minimal and maximal solutions of* (3.3) *respectively.*

Proof. We construct the sequences $\{v_n(x,t)\}$ and $\{w_n(x,t)\}$ as follows:

$$\partial_t^{\alpha,\beta} v_n(x,t) - k\frac{\partial^2 v_n(x,t)}{\partial x^2} = f(x,t,v_{n-1}(x,t))), \ \ on \ \ Q_T, \tag{4.1}$$
$$\Gamma(\alpha)(t)^{1-\alpha} v_n(x,t)\mid_{t=0}= f^0(x), \ \ x\in\overline{\Omega},$$
$$v_n(0,t)=A(t), v_n(L,t)=B(t) \quad in \quad \Gamma_T$$

and

$$\partial_t^{\alpha,\beta} w_n(x,t) - k_i\frac{\partial^2 w_n(x,t)}{\partial x^2} = f(x,t,w_{n-1}(x,t)) \ \ on \ \ Q_T, \tag{4.2}$$
$$\Gamma(\alpha)(t)^{1-\alpha} w_n(x,t)\mid_{t=0}= f_0(x), \ \ x\in\overline{Q_T},$$
$$w_n(0,t)=A(t), w_n(L,t)=B(t) \ \ in \ \ \Gamma_T.$$

It is easy to observe that $v_1(x,t)$ and $w_1(x,t)$ exist and unique by the representation form of linear equation and Corollary 3.6. By induction and the assumptions on $f(x,t,u)$, we prove that the solution $v_n(x,t)$ and $w_n(x,t)$ exist and unique by Corollary 3.6, for any n. Let us prove first $v_0\le v_1$ and $w_1\le w_0$ on Q_T. Let $\rho(x,t)=v_0(x,t)-v_1(x,t)$. Then

$$\begin{aligned}\partial_t^{\alpha,\beta}\rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta} v_0(x,t) - k\frac{\partial^2 v_0(x,t)}{\partial x^2}\\ &\quad - \left[\partial_t^{\alpha,\beta} v_1(x,t) - k\frac{\partial^2 v_1(x,t)}{\partial x^2}\right]\\ &\le f(x,t,v_0) - [f(x,t,v_0]\\ &= 0,\end{aligned}$$

where $\rho(0,t)=0, \rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $\rho(x,t)\le 0$ on $\overline{Q_T}$ and $t^{1-\alpha}v_0(x,t)\le t^{1-\alpha}v_1(x,t)$ on $\overline{Q_T}$.

Assume that $v_{k-1}\le v_k$. Now we show $v_k\le v_{k+1}$. Let $\rho(x,t)=v_k(x,t)-v_{k+1}(x,t)$. Then

$$\begin{aligned}\partial_t^{\alpha,\beta}\rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta} v_k(x,t) - k\frac{\partial^2 v_k(x,t)}{\partial x^2}\\ &\quad - \left[\partial_t^{\alpha,\beta} v_i^{k+1}(x,t) - k\frac{\partial^2 v_{k+1}(x,t)}{\partial x^2}\right]\\ &\le f(x,t,v_k) - [f(x,t,v_{k+1})]\\ &\le M(v_{k+1}-v_k)\\ &\le -M\rho(x,t),\end{aligned}$$

where $\rho(0,t)=0, \rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $\rho(x,t)\leq 0$ on $\overline{Q_T}$ and $t^{1-\alpha}v_k(x,t)\leq t^{1-\alpha}v_{k+1}(x,t)$ on $\overline{Q_T}$. Hence by mathematical induction, we have

$$t^{1-\alpha}v_0 \leq t^{1-\alpha}v_1 \leq ... \leq t^{1-\alpha}v_k \leq t^{1-\alpha}v_{k+1} \leq ... \leq t^{1-\alpha}v_{n-1} \leq t^{1-\alpha}v_n. \quad (4.3)$$

We show that $w_1(x,t)\leq w_0(x,t)$ on $\overline{Q_T}$. Let $\rho(x,t)=w_1(x,t)-w_0(x,t)$. Then

$$\begin{aligned}\partial_t^{\alpha,\beta}\rho(x,t)-k_i\frac{\partial^2\rho(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta}w_1(x,t)-k\frac{\partial^2 w_1(x,t)}{\partial x^2}\\ &\quad -\left[\partial_t^{\alpha,\beta}w_0(x,t)-k\frac{\partial^2 w_0(x,t)}{\partial x^2}\right]\\ &\leq f(x,t,w_0)-[f(x,t,w_0]\\ &=0,\end{aligned}$$

where $\rho(0,t)=0, \rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $\rho(x,t)\leq 0$ on $\overline{Q_T}$ and $t^{1-\alpha}w_0(x,t)\leq t^{1-\alpha}w_1(x,t)$ on $\overline{Q_T}$.

Assume that $w_k(x,t)\leq w_{k-1}(x,t)$. To show that $w_{k+1}(x,t)\leq w_k(x,t)$. Let $\rho(x,t)=w_{k+1}(x,t)-w_k(x,t)$. Then

$$\begin{aligned}\partial_t^{\alpha,\beta}\rho(x,t)-k\frac{\partial^2\rho(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta}w_{k+1}(x,t)-k\frac{\partial^2 w_{k+1}(x,t)}{\partial x^2}\\ &\quad -\left[\partial_t^{\alpha,\beta}w_k(x,t)-k\frac{\partial^2 w_k(x,t)}{\partial x^2}\right]\\ &\leq f(x,t,w_k)-[f(x,t,w_{k+1})]\\ &\leq M(w_k-w_{k+1})\\ &\leq -M\rho(x,t),\end{aligned}$$

where $\rho(0,t)=0$, $\rho(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}\rho(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $\rho(x,t)\leq 0$ on $\overline{Q_T}$ and $t^{1-\alpha}w_{k+1}(x,t)\leq t^{1-\alpha}v_k(x,t)$ on $\overline{Q_T}$. Hence by mathematical induction, we have

$$t^{1-\alpha}w_n \leq t^{1-\alpha}w_{n-1} \leq ... \leq t^{1-\alpha}w_{k+1} \leq t^{1-\alpha}w_k \leq ... \leq t^{1-\alpha}w_1 \leq t^{1-\alpha}w_0. \quad (4.4)$$

Then, we prove that $v_1(x,t)\leq w_1(x,t)$. Let $\rho(x,t)=v_1(x,t)-w_1(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\partial_t^{\alpha,\beta}\rho(x,t) - k\frac{\partial^2\rho(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta}v_1(x,t) - k\frac{\partial^2 v_1(x,t)}{\partial x^2}\\ &\quad - \left[\partial_t^{\alpha,\beta}w_1(x,t) - k\frac{\partial^2 w_1(x,t)}{\partial x^2}\right]\\ &\le f(x,t,v_1) - [f(x,t,w_1]\\ &\le M(v_1 - w_1)\\ &\le -M\rho(x,t),\end{aligned}$$

where $\rho(0,t) = 0, \rho(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}\rho(x,t)\mid_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $\rho(x,t) \le 0$ on $\overline{Q_T}$ and $t^{1-\alpha}v_1(x,t) \le t^{1-\alpha}w_1(x,t)$ on $\overline{Q_T}$. Hence,

$$t^{1-\alpha}v_0(x,t) \le t^{1-\alpha}v_1(x,t) \le t^{1-\alpha}w_1(x,t) \le t^{1-\alpha}w_0(x,t)$$

on $\overline{Q_T}$. By mathematical induction and equations (4.3), (4.4) we have

$$t^{1-\alpha}v_0 \le ... \le t^{1-\alpha}v_n \le t^{1-\alpha}w_n \le ... \le t^{1-\alpha}w_0$$

on $\overline{Q_T}$ for all n.

Furthermore, if $t^{1-\alpha}v_0 \le t^{1-\alpha}u \le t^{1-\alpha}w_0$ on $\overline{Q_T}$, then for any $u(x,t)$ of (3.3), we establish the following inequality by the method of induction.

$$t^{1-\alpha}v_0 \le ... \le t^{1-\alpha}v_n \le t^{1-\alpha}u \le t^{1-\alpha}w_n \le ... \le t^{1-\alpha}w_0 \tag{4.5}$$

on $\overline{Q_T}$ for all n. It is certainly true for $n = 0$, by hypothesis. Assume the inequality (4.3) to be true for $n = k$, that is,

$$t^{1-\alpha}v_0 \le ... \le t^{1-\alpha}v_k \le t^{1-\alpha}u \le t^{1-\alpha}w_k \le ... \le t^{1-\alpha}w_0 \tag{4.6}$$

on $\overline{Q_T}$ for all n.

Let $\rho(x,t) = v_{k+1}(x,t) - u(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\partial_t^{\alpha,\beta}\rho(x,t) - k\frac{\partial^2\rho(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta}v_{k+1}(x,t) - k\frac{\partial^2 v_{k+1}(x,t)}{\partial x^2}\\ &\quad - \left[\partial_t^{\alpha,\beta}u(x,t) - k\frac{\partial^2 u(x,t)}{\partial x^2}\right]\\ &\ge f(x,t,v_{k+1}) - [f(x,t,u)]\\ &\ge -M(v_{k+1} - u)\\ &\ge -M\rho(x,t),\end{aligned}$$

where $\rho(0,t) = 0, \rho(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}\rho(x,t)\mid_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $\rho(x,t) \ge 0$ on $\overline{Q_T}$. Therefore, $t^{1-\alpha}v_{k+1}(x,t) \le t^{1-\alpha}u(x,t)$ on $\overline{Q_T}$.

Similarly, we can show that $t^{1-\alpha}u(x,t) \le t^{1-\alpha}w_{k+1}(x,t)$ on $\overline{Q_T}$.

Let $\rho(x,t) = u(x,t) - w_{k+1}(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\partial_t^{\alpha,\beta}\rho(x,t) - k\frac{\partial^2\rho(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta}u(x,t) - k\frac{\partial^2 u(x,t)}{\partial x^2} \\ &\quad - \left[\partial_t^{\alpha,\beta}w_{k+1}(x,t) - k\frac{\partial^2 w_{k+1}(x,t)}{\partial x^2}\right] \\ &\geq f(x,t,u) - [f(x,t,w_{k+1})] \\ &\geq -M(u - w_{k+1}) \\ &\geq -M\rho(x,t),\end{aligned}$$

where $\rho(0,t) = 0$, $\rho(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}\rho(x,t)\ |_{t=0}= 0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $\rho(x,t) \geq 0$ on $\overline{Q_T}$. Therefore, $t^{1-\alpha}u(x,t) \leq t^{1-\alpha}w_{k+1}(x,t)$ on $\overline{Q_T}$.

Hence we constructed the monotone sequence $\{v_n(x,t)\}$, $\{w_n(x,t)\}$ of lower and upper solutions of integral representation of linear problem.

Now, we show that the sequences $\left\{t^{1-\alpha}v_n(x,t)\right\}$ and $\left\{t^{1-\alpha}w_n(x,t)\right\}$ are uniformly bounded and equicontinuous. Using the Ascoli-Arzela theorem, we obtain subsequences of $\left\{t^{1-\alpha}v_n(x,t)\right\}$ and $\left\{t^{1-\alpha}w_n(x,t)\right\}$ which converge uniformly and monotonically on $\overline{Q_T}$. Since the sequences $\left\{t^{1-\alpha}v_n(x,t)\right\}$ and $\left\{t^{1-\alpha}w_n(x,t)\right\}$ are monotone, the entire sequence $\left\{t^{1-\alpha}v_n(x,t)\right\}$ and $\left\{t^{1-\alpha}w_n(x,t)\right\}$ converges to $t^{1-\alpha}\rho(x,t)$ and $t^{1-\alpha}\gamma(x,t)$, respectively. From this it follows that

$$\begin{aligned}t^{1-\alpha}v_0 \leq t^{1-\alpha}v_1 \leq ... \leq t^{1-\alpha}v_n \leq ... \leq t^{1-\alpha}\rho \leq t^{1-\alpha}u \\ \leq t^{1-\alpha}\gamma \leq ... \leq t^{1-\alpha}w_n \leq ... \leq t^{1-\alpha}w_0 \quad on \quad \overline{Q_T}.\end{aligned}$$

Consequently, $\rho(x,t)$ and $\gamma(x,t)$ are minimal and maximal solutions of (3.3) since

$$t^{1-\alpha}v_0 \leq t^{1-\alpha}\rho \leq t^{1-\alpha}u \leq t^{1-\alpha}\gamma \leq t^{1-\alpha}w_0 \quad on \quad \overline{Q_T}.$$

We prove the uniqueness of the solution of (3.3)in the following. □

Theorem 4.2. *Let all the assumptions of Theorem* 4.1 *hold. Further, let* $f(x,t,u)$ *satisfy the one sided Lipschitz condition*

$$f(x,t,u) - f(x,t,u^*) \leq M(u - u^*),\ \ M > 0.$$

Then the solution $u(x,t)$ *of* (3.3) *exists and is unique.*

Proof. We have already proved (ρ,γ) are minimal and maximal solutions of (3.3) on $\overline{Q_T}$. Hence, it is enough to show that $\gamma(x,t) \leq \rho(x,t) \quad on \quad \overline{Q_T}$.
It is known from Theorem 4.1 that $\gamma(x,t) \leq \rho(x,t)$ *on* $\overline{Q_T}$. Let $p(x,t) = \gamma(x,t) - \rho(x,t)$. By the hypothesis, we get

$$\begin{aligned}\partial_t^{\alpha,\beta}p(x,t)-k\frac{\partial^2 p(x,t)}{\partial x^2} &= \partial_t^{\alpha,\beta}\gamma(x,t)-k\frac{\partial^2\gamma(x,t)}{\partial x^2}\\ &\quad -\left[\partial_t^{\alpha,\beta}\rho(x,t)-k\frac{\partial^2\rho(x,t)}{\partial x^2}\right]\\ &\leq f(x,t,\gamma(x,t))-[f(x,t,\rho(x,t))]\\ &\leq M\mid\gamma(x,t)-\rho(x,t)\mid\\ &\leq M\mid p(x,t)\mid,\end{aligned}$$

where $p(0,t)=0$, $p(L,t)=0$ on $\overline{\Omega}$ and $\Gamma(\alpha)t^{1-\alpha}p(x,t)\mid_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.6, it follows that $p(x,t)\leq 0$. This proves that $\gamma(x,t)=\rho(x,t)=u(x,t)$ on $\overline{Q_T}$ and proof is complete. □

5. Conclusion

In this work, initially we have investigated the solutions of nonlinear Hilfer fractional reaction-diffusion equations of IBVP (1.1)-(1.2) using maximal principle and comparison theorem on Q_T. By applying the comparison result as a tool, we have developed a monotone method for the nonlinear Hilfer fractional reaction-diffusion equations of IBVP (1.1)-(1.2). The monotone method yields monotone sequences which converge uniformly and monotonically to minimal and maximal solutions of IBVP (1.1)-(1.2). We have proved that the uniqueness solution of $u(x,t)$ of the problem.

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