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FIXED POINT THEOREM OF ψ_s-RATIONAL TYPE CONTRACTIONS ALONG WITH ALTERING DISTANCE FUNCTIONS IN b-METRIC SPACES

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Abstract. In this paper, we introduce the new concept of ψ_s-rational type contractive mapping in the sense of b-metric spaces. Also, we obtain some fixed point results for these contractive mappings in complete b-metric spaces. Our main results generalize, extend and improve the corresponding results on the topics given in the literature. Finally, we also give some examples to illustrate our main results.

1. Introduction

In 1922, Polish mathematician, Banach [3] proved a very important result regarding a contractive mapping, which is well known as Banach's contraction principle, as follows:

Theorem 1.1. ([3]) *Let (X, d) be a complete metric space and $T : X \to X$ be a contractive mapping, that is,*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

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Theorem 1.1 was used to establish the existence of a solution for integral equations. Since then, because of its simplicity and usefulness, it has become a very famous and popular tool in solving the existence problems in many branches of mathematical analysis.

Recently, Khojasteh et al. [14] introduce two new types of fixed point theorems in the collection of multivalued and single-valued mappings in complete metric spaces.

On the other hand, Bakhtin [2] and Czerwik [8] presented a generalization of the well known Banach contraction principle in so called b-metric spaces. Afterward, many mathematicians studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, such as, Rao et al. [17] proved some fixed point results for the self mappings satisfying a generalized weak contractive condition in a complete partially ordered b-metric space. A few examples are illustrated to support findings. Gautam et al. [10] have settled the fixed point results in a quasi-partial b-metric space by adopting a new type of mapping into consideration, which is a combination of F and Kannan contractive type mapping. The existence of a fixed point in F-expanding type mapping is also proved. Applications of the results are obtained to find the solution of a functional equation. Petrusel et al. [16] prove some fixed point theorems for graphical contractions in complete b-metric spaces are given. Then, some common fixed point results for a pair of mappings in complete b-metric spaces are deduced. Samphavat et al. [18] extended fixed point theorems, which were originally provided by Kirk and Shahzad and were later extended by Pongsriiam and Termwuttipong, in this journal by considering all functions that are weakly separated from 0 (see [4, 5, 6, 7, 9, 12, 15, 19, 20, 21]) and references therein.

Moreover, Khan et al. [13] established some fixed point theorems in complete metric spaces and compact metric spaces by using the concept of an altering distance function.

In this paper, we introduce the concept of ψ_s-rational type contractive mapping in the sense of b-metric spaces. Also, we show the existence theorems of fixed point for these mappings in complete b-metric spaces. Our main results generalize, extend and improve the corresponding results on the topics given in the literature. Finally, we also give some examples to illustrate our main results which is not applied by the results of Khojasteh et al. [14], but can be applied to our results.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

In 1984, Khan et al. [13] introduced the concept of an altering distance function as follows:

Definition 2.1. ([13]) The function $\varphi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties hold:

(1) φ is continuous and non-decreasing;
(2) $\varphi(t) = 0$ if and only if $t = 0$.

Bakhtin [2] (see also Czerwik [8]) presented a generalization of metric space so called b-metric space.

Definition 2.2. ([8]) Let X be a nonempty set and $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions: for all $x, y, z, \in X$,

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

Then (X,d) is called a b-metric space with coefficient s.

It is clear that the definition of b-metric space is an extension of ordinary metric space. Also, if we give $s = 1$, the definition of b-metric space reduces to metric space. By this reason, our results are more general than the same results in metric space.

We give some example of b-metric spaces as follows:

Example 2.3. ([11]) Let $X = \mathbb{N} \cup \{\infty\}$ and $d : X \times X \to [0,\infty)$ defined by

$$d(m,n) = \begin{cases} 0, \text{ if } m = n; \\ \left|\frac{1}{m} - \frac{1}{n}\right|, \text{ if one of } m,n \text{ is even and the other is even or } \infty; \\ 5, \text{ if one of } m,n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty; \\ 2, \text{ otherwise.} \end{cases}$$

Then (X,d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Example 2.4. ([11]) Let $X = \{0,1,2,3,4\}$ and $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ (x+y)^2, & \text{if } x \neq y. \end{cases}$$

It is easy to see that (X,d) is a b-metric space with coefficient $s = \frac{49}{25}$.

Example 2.5. ([20]) Let p be a given real number in the interval $(0,1)$. The space

$$L_p[0,1] := \left\{ x : [0,1] \to \mathbb{R} \mid \int_0^1 |x(t)|^p dt < 1 \right\}$$

together with the mapping $d : L_p[0,1] \times L_p[0,1] \to \mathbb{R}_+$ defined by

$$d(x,y) := \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}$$

for each $x, y \in L_p[0,1]$, is a b-metric space with constant $s = 2^{\frac{1}{p}} > 1$.

Next, we present the definitions of b-convergent sequence, b-Cauchy sequence and b-completeness in b-metric spaces.

Definition 2.6. ([4]) Let (X, d) be a b-metric space and $\{x_n\}$ be a sequence in X. If there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$, then a sequence $\{x_n\}$ is called b-convergent. In this case, we write $\lim_{n\to\infty} x_n = x$.

Definition 2.7. ([4]) Let (X, d) be a b-metric space and $\{x_n\}$ be a sequence in X. If $d(x_n, x_m) \to 0$ as $n, m \to \infty$, then a sequence $\{x_n\}$ is called b-Cauchy.

Remark 2.8. ([4]) In a b-metric space (X, d), the following assertions hold.

- (p_1)**:** A b-convergent sequence has a unique limit.
- (p_2)**:** Every b-convergent sequence is a b-Cauchy sequence.
- (p_3)**:** In general, a b-metric is not continuous.

From the fact that in (p_3), we need the following lemma about the b-convergent sequences for the proof of our results.

Lemma 2.9. ([1]) *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

Definition 2.10. ([4]) The b-metric spaces is said to be complete if every b-Cauchy sequence b-convergent.

3. Main results

Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be two self-mappings. Throughout this paper, unless otherwise stated, for all $x, y \in X$, let

$$M_s(x,y) := \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}.$$

If $s = 1$, we write $M(x, y)$ instead $M_s(x, y)$, that is,

$$M(x,y) := \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Definition 3.1. Let (X, d) be a b-metric space with coefficient $s \geq 1$. A mapping $T : X \to X$ is said to be a ψ_s-rational type contraction if the following condition holds:

$$\psi(d(Tx,Ty)) \leq \left(\frac{d(x,Ty)+d(y,Tx)}{s^2 d(x,Tx)+s^2 d(y,Ty)+s^2}\right)\cdot \psi(M_s(x,y)) \tag{3.1}$$

for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function.

Theorem 3.2. *Let (X, d) be a complete b-metric space with coefficient $s > 1$ and self mapping T on X be a ψ_s-rational type contraction mapping such that ψ is homogeneous and subadditive function. Then T has at least one fixed point $z \in X$ and if x, y are two distinct fixed points of T, then $d(x, y) \geq \frac{s^2}{2}$.*

Moreover, for each $x_0 \in X$, the Picard sequence $\{x_n\}$ in X which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point z.

Proof. Let x_0 be an arbitrary point in X and choose a sequence $\{x_n\}$ such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &= \psi(d(Tx_{n-1},Tx_n))\\ &\leq \left(\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{s^2 d(x_{n-1},Tx_{n-1})+s^2 d(x_n,Tx_n)+s^2}\right)\\ &\quad \times \psi(M_s(x_{n-1},x_n))\\ &\leq \left(\frac{d(x_{n-1},x_{n+1})}{s d(x_{n-1},x_{n+1})+s^2}\right)\cdot \psi(M_s(x_{n-1},x_n))\\ &= \delta_n \psi(M_s(x_{n-1},x_n)),\end{aligned} \tag{3.2}$$

where $\delta_n = \frac{d(x_{n-1},x_{n+1})}{sd(x_{n-1},x_{n+1})+s^2}$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}M_s(x_{n-1},x_n) &= \max\Big\{d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n),\\ &\qquad \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2s}\Big\}\\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}.\end{aligned}$$

If there is $n^* \in \mathbb{N}$ such that $M_s(x_{n^*-1}, x_{n^*}) = d(x_{n^*-1}, x_{n^*})$, then (3.2) implies

$$\psi(d(x_{n^*-1}, x_{n^*})) \leq \delta_{n^*}\psi(d(x_{n^*-1}, x_{n^*})) < \psi(d(x_{n^*-1}, x_{n^*})),$$

which is a contradiction. So for each $n \in \mathbb{N}$, we have

$$\psi(d(x_{n-1}, x_n)) \leq \delta_n \psi(d(x_n, x_{n+1})).$$

Repeating this process, we obtain

$$\begin{aligned}
\psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\
&\leq \delta_n \psi(d(x_{n-1}, x_n)) \\
&\leq (\delta_n \delta_{n-1})\psi(d(x_{n-2}, x_{n-1})) \\
&\vdots \\
&\leq (\delta_n \delta_{n-1} \ldots \delta_1)\psi(d(x_0, x_1)).
\end{aligned}$$

Thus $\delta_1 \delta_2 \ldots \delta_n \leq \delta_1^n$ and so $\delta_1^n \to 0$. This implies that

$$\lim_{n\to\infty} (\delta_1 \delta_2 \ldots \delta_n) = 0$$

and hence

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0.$$

Since ψ is an altering distance function, we obtain that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Next, we will prove that $\{x_n\}$ is a b-Cauchy sequence. For $m, n \in \mathbb{N}$, $m > n$ and property of ψ, we have

$$\begin{aligned}
&\psi(d(x_n, x_m)) \\
&\quad \leq \psi(sd(x_n, x_{n+1}) + + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{m-n} d(x_{m-1}, x_m)) \\
&\quad \leq \psi(s^{m-n} d(x_n, x_{n+1}) + s^{m-n} d(x_{n+1}, x_{n+2}) + \ldots + s^{m-n} d(x_{m-1}, x_m)) \\
&\quad = \psi(s^{m-n}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m))) \\
&\quad = s^{m-n}\psi(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)) \\
&\quad \leq s^{m-n}\Big[\psi(d(x_n, x_{n+1})) + \psi(d(x_{n+1}, x_{n+2})) + \ldots + \psi(d(x_{m-1}, x_m))\Big] \\
&\quad \leq s^{m-n}\Big[(\delta_n \delta_{n-1} \ldots \delta_1) + (\delta_{n+1}\delta_n \ldots \delta_1) + \ldots + (\delta_{m-1}\delta_{m-2} \ldots \delta_1)\Big] \\
&\qquad \times\ \psi(d(x_0, x_1)) \\
&\quad = s^{m-n} \sum_{k=n}^{m-1} (\delta_k \delta_{k-1} \ldots \delta_1)\psi(M_s(x_0, x_1)).
\end{aligned}$$

Assume that $\beta_k = (\delta_k \delta_{k-1} \dots \delta_1)$. From

$$\lim_{k\to\infty} \frac{\beta_{k+1}}{\beta_k} = \lim_{k\to\infty} \delta_{k+1} = 0,$$

we get $\sum_{k=1}^{\infty} \beta_k < \infty$. It follows that

$$\sum_{k=n}^{m-1} (\delta_k \delta_{k-1} \dots \delta_1) \to 0 \quad \text{as} \quad m, n \to \infty.$$

Therefore, $\{x_n\}$ is a b-Cauchy sequence. Since (X, d) is a complete b-metric space, the sequence $\{x_n\}$ in X b-converges to some point $z \in X$, that is, $\lim_{n\to\infty} d(x_n, z) = 0$ and so $\lim_{n\to\infty} x_n = z$.

Observe that

$$\psi(d(x_{n+1}, Tz)) \leq \left(\frac{d(x_n, Tz) + d(z, Tx_n)}{s^2 d(x_n, Tx_n) + s^2 d(z, Tz) + s^2} \right) \psi(M_s(x_n, z)), \quad (3.3)$$

where

$$M_s(x_n, z) = \max \left\{ d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{d(x_n, Tz) + d(z, Tx_n)}{2s} \right\}$$

for all $n \in \mathbb{N}$. Taking limit supremum as $n \to \infty$ in (3.3) and using Lemma 2.9, we have

$$\begin{aligned} \limsup_{n\to\infty} \psi(d(x_{n+1}, Tz)) &\leq \left(\frac{sd(z, Tz)}{s^2 d(z, Tz) + s^2} \right) \psi(d(z, Tz)) \\ &= \left(\frac{d(z, Tz)}{sd(z, Tz) + s} \right) \psi(d(z, Tz)). \end{aligned} \quad (3.4)$$

By triangle inequality and property of ψ, we get

$$\begin{aligned} \psi(d(z, Tz)) &\leq \psi(s[d(z, x_{n+1}) + d(x_{n+1}, Tz)]) \\ &\leq s[\psi(d(z, x_{n+1})) + \psi(d(x_{n+1}, Tz))]. \end{aligned}$$

By taking limit supremum in above inequality and using (3.4), we obtain

$$\begin{aligned} \psi(d(z, Tz)) &\leq s \left(\frac{d(z, Tz)}{sd(z, Tz) + s} \right) \psi(d(z, Tz)) \\ &= \left(\frac{d(z, Tz)}{d(z, Tz) + 1} \right) \psi(d(z, Tz)) \end{aligned}$$

and so $\psi(d(z, Tz)) = 0$. Since ψ is an altering distance function, we have

$$d(z, Tz) = 0$$

and hence z is a fixed point of T.

If there exist two distinct fixed points $x, y \in X$, then

$$\begin{aligned}\psi(d(x,y)) &= \psi(d(Tx,Ty)) \\ &\leq \left(\frac{d(x,Ty)+d(y,Tx)}{s^2 d(x,Tx)+s^2 d(y,Ty)+s^2}\right)\psi(M_s(x,y)) \\ &= \left(\frac{d(x,Ty)+d(y,Tx)}{s^2}\right)\psi(M_s(x,y)) \\ &= \frac{2}{s^2}d(x,y)\psi(M_s(x,y)) \\ &= \frac{2}{s^2}d(x,y)\psi(d(x,y)).\end{aligned}$$

Therefore, $d(x,y) \geq \frac{s^2}{2}$. This completes the proof. □

Example 3.3. Let $X = \{0,1,2,3,4\}$ and $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ (x+y)^2, & \text{if } x \neq y. \end{cases}$$

Then (X,d) is a b-metric space with coefficient $s = \frac{49}{25}$. Define the mappings $T : X \to X$ and $\psi : [0,\infty) \to [0,\infty)$ by

$$Tx = \begin{cases} 0, & x = 0,1,2,3; \\ 4, & x = 4, \end{cases}$$

for all $x \in X$ and

$$\psi(t) = 2t$$

for all $t \in [0,\infty)$. It is easy to see that ψ is altering distance function, homogeneous and subadditive function.

Now, we claim that all the conditions in Theorem 3.2 hold in this situation. It suffices to show that T is a ψ_s-rational type contraction mapping in 3 cases.

Case I: If $x = y$ for all $x, y \in X$, then the argument holds.

Case II: For $x, y \in \{0,1,2,3\}$, the claim is obvious.

Case III: For $x \in \{0,1,2,3\}$ and $y=4$. Then, we have

$$\begin{aligned}\psi(d(Tx,Ty)) &= \psi(d(0,4))\\ &= 32\\ &\leq \left(\frac{(x+4)^2+16}{(x+4)^2}\right)(x+4)^2\\ &\leq \left(\frac{(x+4)^2+16}{\left(\frac{49}{25}\right)^2(x^2+1)}\right)(x+4)^2\\ &\leq \left(\frac{d(x,Ty)+d(y,Tx)}{s^2d(x,Tx)+s^2d(y,Ty)+s^2}\right)\psi(d(x,y))\\ &\leq \left(\frac{d(x,Ty)+d(y,Tx)}{s^2d(x,Tx)+s^2d(y,Ty)+s^2}\right)\psi(M_s(x,y)).\end{aligned}$$

From 3 case above, we can see that T is a ψ_s-rational type contraction mapping. Hence T has at least one fixed point $z \in X$, that is, 0, 4 are fixed points of T. Note that

$$d(0,4)=(0+4)^2=16>\frac{s^2}{2}.$$

Moreover, we can see that this example is not hold for the results of Khojasteh et al. [14] in the case of usual metric. Indeed, if $x=3$ and $y=4$, then we have

$$d(Tx,Ty)=|0-4|=4>\frac{5}{4}=\left(\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1}\right)d(x,y).$$

Corollary 3.4. *Let (X,d) be a complete b-metric space with coefficient $s>1$ and self mapping T on X be a mapping such that*

$$d(Tx,Ty)\leq\left(\frac{d(x,Ty)+d(y,Tx)}{s^2d(x,Tx)+s^2d(y,Ty)+s^2}\right)M_s(x,y)$$

for all $x,y \in X$. Then T has at least one fixed point $z \in X$ and if x,y are two distinct fixed points of T, then $d(x,y) \geq \frac{s^2}{2}$.

Moreover, for each $x_0 \in X$, the Picard sequence $\{x_n\}$ in X which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point z.

Proof. Setting ψ is an identity function in Theorem 3.2, we get this result. □

Corollary 3.5. *Let (X,d) be a complete metric space and self mapping T on X be a mapping such that*

$$\psi(d(Tx,Ty))\leq\left(\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1}\right)\psi(M(x,y))$$

for all $x, y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is altering distance function, homogeneous and subadditive function. Then T has at least one fixed point $z \in X$ and if x, y are two distinct fixed points of T, then $d(x,y) \geq \frac{1}{2}$.

Moreover, for each $x_0 \in X$, the Picard sequence $\{x_n\}$ in X which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point z.

Proof. Setting $s = 1$ in Theorem 3.2, we get this result. □

Corollary 3.6. ([14]) *Let (X,d) be a complete metric space and self mapping T on X be a mapping such that*

$$d(Tx,Ty) \leq \left(\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1}\right) M(x,y)$$

for all $x, y \in X$. Then T has at least one fixed point $z \in X$ and if x, y are two distinct fixed points of T, then $d(x,y) \geq \frac{1}{2}$.

Moreover, for each $x_0 \in X$, the Picard sequence $\{x_n\}$ in X which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point z.

Proof. Setting $s = 1$ and ψ is an identity function in Theorem 3.2, we get this result. □

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