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AN INERTIAL TSENG ALGORITHM FOR SOLVING QUASIMONOTONE VARIATIONAL INEQUALITY AND FIXED POINT PROBLEM IN HILBERT SPACES

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Abstract. In this paper, we propose an inertial method for solving a common solution to fixed point and Variational Inequality Problem in Hilbert spaces. Under some standard and suitable assumptions on the control parameters, we prove that the sequence generated by the proposed algorithm converges strongly to an element in the solution set of Variational Inequality Problem associated with a quasimonotone operator which is also solution to a fixed point problem for a demimetric mapping. Finally, we give some numerical experiments for supporting our main results and also compare with some earlier announced methods in the literature.

1. Introduction

In this paper, we are concerned with solving variational inequality problems (shortly,VIP) in real Hilbert spaces. The notion of VIP was introduced

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independently by Stampacchia [40] and Fichera [18, 17] for modeling problems arising from mechanics and for solving the Signorini problem. The theory of VIP has been studied widely in several branches of pure and applied areas, such as economics, transportation, engineering, network, finance and many more, see for example [25, 26].

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset in H. The VIP is formulated as: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset in a real Hilbert space H and $A : H \to H$ is a single-valued mapping. Let VIP(C,A) denote the solution set of the variational inequality problem (1.1).

The problem of finding an iterative solution to the VIP is an interesting area of research in optimization theory. One of the simplest method for approximating a solution to the VIP is the gradient method (see, Dafermo [13]) in which only one projection onto the feasible set C is performed. It is known that the convergence of this method requires that the operator A be strongly monotone or inverse strongly monotone [21]. To avoid the strong monotonicity assumption on A, authors have used the extragradient method (EGM) proposed by Korpelevich [28] (also by Antipin [5] independently) which was initially used for saddle point problems. After that, it was extended to VIPs in both Euclidean spaces and Hilbert spaces. To be more precise, the EGM for a monotone and L-Lipschitz continuous operator $A : H \to H$ is given as follows:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda_n A(y_n)), \ \forall n \geq 1, \end{cases} \tag{1.2}$$

where $\lambda_n \in (0, \frac{1}{L}), P_C : H \to C$ denotes the metric projection from H onto C. Whenever the solution set $VI(C, A)$ is nonempty then the sequence $\{x_n\}$ generated by the algorithm (1.2) converges weakly to an element in $VI(C, A)$. In recent years, the EGM has received great attention by many authors, who improved it in various ways, see, e.g., [7, 8, 9, 10, 11, 22, 23, 24, 27, 30, 31, 34, 48] and the references therein.

In the execution of the EGM, there is the need to compute the projection onto the feasible set C twice per iteration which could be computationally expensive most especially when the structure of C is not simple. For this reason, authors have provided several modifications to the EGM (for example, see [9, 36, 44, 46]) and the references therein. The notable modifications is the

Tseng extragradient method (TEGM) (see [46]) proposed in Hilbert space H given below:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda_n A(x_n)), \\ x_{n+1} = y_n - \lambda_n (A(y_n) - A(x_n)), \ \forall n \geq 1. \end{cases} \tag{1.3}$$

which only needs to calculate the metric projection once in each iteration. The weak convergence theorem of TEGM in Hilbert space was obtained under some mild assumptions. Since then, the TEGM has attracted the interest of many authors, see, for instance, [4, 12, 33, 38, 43, 45, 47] and the references therein. We note that the TEGM preserves one of the drawback of the EGM which is the dependence of the methods on the Lipschitz constant of the associated cost operator. To solve this problem, researchers in this direction have introduced several self adaptive techniques [6, 16, 32]

On other hand, we consider the fixed point problem (shortly, FPP). A point $x \in H$ is called a fixed point of T if

$$x = Tx, \tag{1.4}$$

where $T : H \to H$ is a nonlinear mapping. The set of solutions of problem (1.4) is denoted by $F(T)$. In real life, many mathematical models have been formulated as the problem of finding a fixed point of a nonlinear mapping. Currently, many mathematicians are interested in finding a common solution to the FPP (1.4) and the VIP (1.1). Thus in this article, we are interested in finding a point $x \in C$ such that

$$x \in F(T) \cap VI(C, A). \tag{1.5}$$

This problem is of particular interest due to the fact that several problems in applied sciences and engineering have constraint which can be expressed as a problem of finding a solution to (1.5).

In an attempt to solve (1.5), Takahashi and Toyoda [42] introduced an iterative scheme for finding a solution to (1.5) for an inverse strongly monotone mapping and then obtained a weak convergence theorem. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and $A : C \to H$ be an inverse strongly monotone and $T : C \to C$ is nonexpansive mapping, such that $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A(x_n)), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C(x_n - \lambda_n A x_n), \ \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, $\{x_n\}$ converges weakly to $z \in F(T) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(T)\cap VI(C,A)} x_n$.

Very recently, Yin *et al.* [49] introduced an iterative scheme for approximating a solution to (1.5) and obtain the following weak convergence theorem. Let C be a nonempty convex closed subset of Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive operator. Let the *nth* iterate x_n be given by;

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \\ z_n = (1 - \alpha_n)x_n + \alpha_n T(y_n). \end{cases} \tag{1.7}$$

Let the nth stepsize λ_n be known,

$$\begin{cases} w_n = P_C(z_n - \lambda_n A(z_n)), \\ x_{n+1} = (1 - \zeta_n)z_n + \zeta_n \lambda_n (A(z_n) - A(w_n)), \end{cases} \tag{1.8}$$

where $\lambda_1 > 0, \{\zeta_n\}, \{\alpha_n\}$ and $\{\alpha_n\}$ are sequences in $(0, 1)$ satisfying some standard assumption. The step size λ_n is updated as

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|w_n - z_n\|}{\|A(w_n) - A(z_n)\|}\|\right\}, & \text{if } A(w_n) \neq A(z_n), \\ \lambda_n, & \text{otherwise,} \end{cases} \tag{1.9}$$

where $\mu \in (0, 1)$. They proved that the sequence $\{x_n\}$ generated by (1.7)-(1.8) converges weakly to some point in (1.5).

A technique used to speed up the convergence of iterative procedures in fixed point and optimization theory is the Inertial Method. For this reason, many mathematicians have proposed inertial based algorithm for solving fixed point and optimization problems. The Inertial Method adds the technical term $\theta_n(x_n - x_{n-1})$, where θ_n is an inertial parameter which controls the inertial force $x_n - x_{n-1}$ (see [3, 35]). For more on inertial based algorithms, we refer the readers to see the following [2, 15, 19] and the references therein.

In this work, inspired and motivated by the results of Yin *et al.* [49], Takahashi and Toyoda [42], we proposed an inertial Tseng extragradient algorithm for solving a common solution to a fixed point and variational inequality problem in a real Hilbert space. Our proposed method is self adaptive and thus the knowledge of the Lipschitz constant of the cost operator is not required. Using this method we obtain and prove a strong convergent result for approximating a common solution to a fixed point and variational inequality problem associated with demi-metric and quasimonotone operators respectively. Finally, we report some numerical experiments to show the convergence of our algorithm.

This paper is organized as follows: In Section 2, we recall some definitions and preliminary results for further use. Section 3 deals with analyzing the convergence of the proposed algorithms. In Section 4, some numerical experiments are performed to illustrate the convergence of the propose algorithms and compare them with previously known algorithms.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Lemma 2.1. ([41]) *Let $x, y \in H$ and $t \in [0,1]$. Then the following properties hold on H :*

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$;
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(iii) $\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$.

For all point H, there exists the unique nearest points in C, denoted $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \ \forall y \in C. \tag{2.1}$$

The mapping P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

A mapping $T : H \to H$ is said to be

(a) monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall x, y \in H;$$

(b) pseudomonotone, if

$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, y - x\rangle \geq 0, \ \forall x, y \in H;$$

(c) a contraction, if there exists a constant $k \in (0,1)$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \ \forall x, y \in H;$$

(d) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in H;$$

(e) quasinonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x \in H \text{ and } y \in F(T);$$

(f) L-Lipschitz continuous with L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in H;$$

(g) quasimonotone, if

$$\langle Tx, x-y\rangle > 0 \Rightarrow \langle Ty, x-y\rangle \geq 0,\ \forall x, y \in H;$$

(h) sequentially weakly continuous if for each $\{x_n\}$, we obtain $\{x_n\}$ converges weakly to x implies that $T(x_n)$ converges weakly to $T(x)$;

(i) a mapping $T : C \to H$ is called k-demimetric, if $F(T) \neq 0$ and there exists $k \in (-\infty, 1)$ such that

$$\langle x-y, x-T(x)\rangle \geq \frac{1-k}{2}\|x-T(x)\|^2,\ \forall x \in C, y \in F(T).$$

Lemma 2.2. ([20, 50]) *Let C be a nonempty, closed and convex subset of a Hilbert space H and $A : H \to H$ be a L-Lipschitzian and quasimonotone operator. Suppose that $y \in C$ and for some $p \in C$, we have $\langle Ay, p-y\rangle \geq 0$. Then at least one of the following hold*

$$\langle Ap, p-y\rangle \geq 0 \quad or \quad \langle Ay, q-y\rangle \leq 0,\ \forall q \in C. \tag{2.2}$$

Lemma 2.3. ([1]) *Let C be a nonempty closed subset of a real Hilbert space H. Let $k \in (-\infty, 1)$ and T be a k-demimetric mapping of C onto H such that $F(T) \neq \emptyset$. Let $\beta \in (0, 1)$ and define $S = (1-\beta)I + \beta T$ where I is the identity mapping. Then there holds*

(i) *$F(S) = F(T)$ if $\beta \neq 0$;*
(ii) *S is a quasinonexpansive mapping for $\beta \in (0, 1-k)$;*
(iii) *$F(T)$ is closed and convex.*

Lemma 2.4. ([37]) *Let $\{a_n\} \subset \mathbb{R}_+, \{b_n\} \subset \mathbb{R}$ and $\{\epsilon_n\} \subset (0, 1)$ such that $\sum_{n=1}^{\infty} \epsilon_n = \infty$ and*

$$a_{n+1} \leq (1-\epsilon_n)a_n + \epsilon_n b_n,\ \forall n \in \mathbb{N}.$$

If $\limsup_{i\to\infty} b_{n_i} \leq 0$ for every subsequence $\{a_{n_i}\}$ of $\{a_n\}$ satisfying

$$\liminf_{i\to\infty}(a_{n_i+1} - a_{n_i}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3. Main result

In this section, we state and prove our main result of the sequel. First, we make the following assumptions:

Assumption 3.1. Suppose

(A1) C is a nonempty, closed and convex subset of a real Hilbert space H.

(A2) $f : C \to C$ be a contraction mapping with constant $\kappa \in (0,1)$.
(A3) $T : C \to C$ is a η-demimetric mapping such that $F(T) \neq \emptyset$.
(A4) $A : H \to H$ is a quasimonotone, sequentially weakly and Lipschitz continuous operator. We note however that the execution of our method does not depend on the knowledge of the Lipschitz constant of A.
(A5) The solution set of the VIP (1.1) denoted by $\Gamma := \{x \in F(T) \cap VI(C, A)\}$ is nonempty.

Assumption 3.2. We also require the following assumptions:

(C1) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(C2) $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.
(C3) $\{\alpha_n\} \in (0,1)$ such that $\beta_n \in (0, 1-\eta)$.
(C4) $\{\delta_n\} \in (0,1)$.

We introduce an inertial extrapolation method for finding common solution to the fixed point problem and variational inequality problem:

Algorithm 3.3. Inertial Tseng-type method for FPP and VIP.

Initialization: Given $\theta > 0, \lambda_0 > 0$ and $0 < \mu < 1$. Choose a nonnegative real sequence $\{\delta_n\}$ such that $\sum_{n=1}^{\infty} \delta_n < +\infty$. Let $x_0,\ x_1 \in H$ be arbitrary starting points. Define

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}. \end{cases} \tag{3.1}$$

Iterative Steps: Calculate x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A(w_n)), \\ z_n = y_n - \lambda_n(A(y_n) - A(w_n)), \\ t_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)t_n. \end{cases} \tag{3.2}$$

Update

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|A(w_n) - A(y_n)\|}, \lambda_n + \delta_n\right\}, & \text{if } A(w_n) \neq A(y_n), \\ \lambda_n + \delta_n, & \text{otherwise}. \end{cases} \tag{3.3}$$

Remark 3.4. The stepsize in (3.3) is similar to the ones in [29, 44] if $\delta_n = 0$. Also, stepsize employed in the (3.3) is to increase from iteration to iteration and so (3.3) reduces the dependence on the initial step size λ_0. Since the sequence $\{\delta_n\}$ is summable, we have that $\lim_{n\to\infty} \delta_n = 0$. Hence, the stepsize λ_n may be non-increasing when n is large.

Lemma 3.5. *Let $\{x_n\}$ be the sequence defined iteratively by Algorithm* 3.3 *and $p \in \Gamma$. Then, the following holds:*

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2. \tag{3.4}$$

Proof. Let $p \in \Gamma$. From Lemma 2.1(i) and $z_n = y_n - \lambda_n(A(y_n) - A(w_n))$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - \lambda_n(A(y_n) - A(w_n)) - p\|^2 \\
&= \|y_n - p\|^2 + \lambda_n^2\|A(y_n) - A(w_n)\|^2 - 2\lambda_n\langle y_n - p, A(y_n) - A(w_n)\rangle \\
&= \|w_n - p\|^2 + \|w_n - y_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle \\
&\quad + \lambda_n^2\|A(y_n) - A(w_n)\|^2 - 2\lambda_n\langle y_n - p, A(y_n) - A(w_n)\rangle \\
&= \|w_n - p\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - p, y_n - w_n\rangle \\
&\quad + \lambda_n^2\|A(y_n) - A(w_n)\|^2 - 2\lambda_n\langle y_n - p, A(y_n) - A(w_n)\rangle.
\end{aligned} \tag{3.5}$$

Since $y_n = P_C(w_n - \lambda_n A(w_n))$, we have

$$\langle y_n - w_n + \lambda_n A(w_n), y_n - p\rangle \le 0$$

equivalently

$$\langle y_n - w_n, y_n - p\rangle \le -\lambda_n\langle A(w_n), y_n - p\rangle. \tag{3.6}$$

From (3.5) and (3.6), it follows that

$$\begin{aligned}
\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|w_n - y_n\|^2 - 2\lambda_n\langle A(w_n), y_n - p\rangle \\
&\quad + \lambda_n^2\|A(y_n) - A(w_n)\|^2 - 2\lambda_n\langle y_n - p, A(y_n) - A(w_n)\rangle \\
&= \|w_n - p\|^2 - \|w_n - y_n\|^2 + \lambda_n^2\|A(y_n) - A(w_n)\|^2 \\
&\quad - 2\lambda_n\langle y_n - p, A(y_n)\rangle.
\end{aligned} \tag{3.7}$$

Since $p \in \Gamma$, we have $\langle A(p), y_n - p\rangle \ge 0$. Using Lemma 2.2, we obtain $\langle A(y_n), y_n - p\rangle \ge 0$. This, together with (3.7), implies

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \|w_n - y_n\|^2 + \lambda_n^2\|A(y_n) - A(w_n)\|^2, \tag{3.8}$$

combining (3.3) and (3.8), we obtain,

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2. \tag{3.9}$$

□

Lemma 3.6. *The sequence $\{x_n\}$ defined iteratively by Algorithm 3.3 is bounded. Consequently, the sequences, $\{t_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded.*

Proof. From $\lim_{n\to\infty}\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)=1-\mu^2>0$, there exists $n_0\in\mathbb{N}$ such that

$$1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}>0,\ \forall n\leq n_0.$$

Using (3.9) we have

$$\|z_n-p\|\leq\|w_n-p\|,\ \forall n\leq n_0. \tag{3.10}$$

By definition of w_n, we get

$$\begin{aligned}\|w_n-p\|&=\|x_n+\theta_n(x_n-x_{n-1})-p\|\\&\leq|x_n-p\|+\theta_n\|x_n-x_{n-1}\|\\&\leq\|x_n-p\|+\theta_n\cdot\frac{\alpha_n}{\alpha_n}\|x_n-x_{n-1}\|\\&\leq\|x_n-p\|+\alpha_n\cdot\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|.\end{aligned} \tag{3.11}$$

From (3.3), we get

$$\theta_n\|x_n-x_{n-1}\|\leq\epsilon_n$$

for all n, which together with

$$\lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n}=0,$$

implies that

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq\lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n}=0.$$

Thus there exists a constant $M_1>0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq M_1,\ \forall n\geq 1. \tag{3.12}$$

Therefore,

$$\|z_n-p\|\leq\|w_n-p\|\leq\|x_n-p\|+\alpha_nM_1,\ \forall n\geq n_0. \tag{3.13}$$

Set $W_n=(1-\beta_n)I+\beta_nT$. Since W_n is quasi-nonexpansive with $F(W_n)=F(T)$, due to Lemma 2.3 and t_n in Algorithm 3.3, we obtain

$$\begin{aligned}\|t_n-p\|&=\|W_nz_n-p\|\\&\leq\|z_n-p\|.\end{aligned} \tag{3.14}$$

Again, from Algorithm 3.3, we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n f(x_n)+(1-\alpha_n)t_n-p\| \\
&= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(t_n-p)\| \\
&\leq \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|t_n-p\| \\
&\leq \alpha_n\kappa\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|t_n-p\| \\
&\leq \alpha_n\kappa\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|z_n-p\| \\
&\leq \alpha_n\kappa\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|w_n-p\| \\
&\leq (1-\alpha_n(1-\kappa))\|x_n-p\|+(1-\kappa)\alpha_n\left(\frac{M_1+\|f(p)-p\|}{1-\kappa}\right) \\
&\leq \max\left\{\|x_n-p\|,\frac{M_1+\|f(p)-p\|}{1-\kappa}\right\} \\
&\quad\vdots \\
&\leq \max\left\{\|x_1-p\|,\frac{M_1+\|f(p)-p\|}{1-\kappa}\right\}. \qquad (3.15)
\end{aligned}$$

This implies that $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\},\{z_n\}$ and $\{t_n\}$ are also bounded. □

Theorem 3.7. *Assume that Conditions* 3.1 *and* 3.2 *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.3 *converges strongly to an element* $p\in\Gamma$, *where* $p=P_\Gamma f(p)$.

Proof. As before, fix $p\in\Gamma$. We can see that the operator $P_\Gamma f$ is a contraction. By Banach contraction principle, there exists a unique point $p\in\Gamma$ such that $p=P_\Gamma f(p)$. It follows from the characterization of P_Γ that

$$\langle f(p)-p, q-p\rangle \leq 0,\ \forall q\in\Gamma. \qquad (3.16)$$

Observe from 3.3 and Lemma 2.1 (iii), that

$$\begin{aligned}
\|t_n-p\|^2 &= \|(1-\beta_n)z_n+\beta_n Tz_n-p\|^2 \\
&= \|(1-\beta_n)(z_n-p)+\beta_n(Tz_n-p)\|^2 \\
&= (1-\beta_n)\|z_n-p\|^2+\beta_n\|Tz_n-p\|^2-\beta_n(1-\beta_n)\|z_n-Tz_n\|^2 \\
&\leq (1-\beta_n)\|z_n-p\|^2+\beta_n(\|z_n-p\|^2+\eta\|z_n-Tz_n\|^2) \\
&\quad -\beta_n(1-\beta_n)\|z_n-Tz_n\|^2 \\
&= \|z_n-p\|^2-\beta_n(1-\eta-\beta_n)\|z_n-Tz_n\|^2,
\end{aligned}$$

from (3.9), we get

$$\|t_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 \\ - \beta_n(1 - \eta - \beta_n)\|z_n - Tz_n\|^2.$$

But,

$$\|w_n - p\|^2 = \|x_n - p\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1}\rangle.$$

Thus, we obtain

$$\begin{aligned}\|t_n - p\|^2 &\le \|x_n - p\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1}\rangle \\ &\quad - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 - \beta_n(1 - \eta - \beta_n)\|z_n - Tz_n\|^2 \\ &\le \|x_n - p\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - p\|\|x_n - x_{n-1}\| \\ &\quad - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 - \beta_n(1 - \eta - \beta_n)\|z_n - Tz_n\|^2.\end{aligned} \tag{3.17}$$

From Algorithm 3.3, Lemma 2.1 (i), (iii) and (3.17)

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|(1 - \alpha_n)(t_n - p) + \alpha_n(f(x_n) - f(p))\|^2 \\ &\quad + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\ &= (1 - \alpha_n)\|t_n - p\|^2 + \alpha_n \|f(x_n) - f(p)\|^2 \\ &\quad + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n \\ &\quad - p\|\|x_n - x_{n-1}\| - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 \\ &\quad - \beta_n(1 - \eta - \beta_n)\|z_n - Tz_n\|^2 + \alpha_n \kappa \|x_n - p\|^2 \\ &\quad + 2\alpha_n \langle f(p) - p, x_{n+1} - p\rangle \\ &= (1 - \alpha_n(1 - \kappa))\|x_n - p\|^2 + \alpha_n(1 - \kappa)b_n \\ &\quad - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - w_n\|^2 \\ &\quad - \beta_n(1 - \eta - \beta_n)\|z_n - Tz_n\|^2,\end{aligned} \tag{3.18}$$

where

$$b_n := \frac{1}{1-\kappa}\Big(2\langle f(p)-p, x_{n+1}-p\rangle + \frac{\theta_n^2}{\alpha_n}\|x_n - x_{n-1}\|^2 + 2\frac{\theta_n}{\alpha_n}\|x_n - p\|\|x_n - x_{n-1}\|\Big). \tag{3.19}$$

It follows that,

$$\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|y_n - w_n\|^2 + \beta_n(1-\eta-\beta_n)\|z_n - Tz_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n(1-\kappa)M', \tag{3.20}$$

where

$$M' = \sup\{b_n : n \in \mathbb{N}\}. \tag{3.21}$$

We now show that $\{x_n\}$ converges strongly to p. Set $a_n := \|x_n - p\|$ and $\epsilon_n := \alpha_n(1-\kappa)$. From (3.18), we have the following inequality;

$$a_{n+1} \leq (1-\epsilon_n)a_n + \epsilon_n b_n. \tag{3.22}$$

To apply Lemma 2.4, we have to show that $\limsup_{k\to\infty} b_{n_k} \leq 0$, whenever a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfies.

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0. \tag{3.23}$$

Now suppose that $\{a_{n_k}\} \subset \{a_n\}$ is a subsequence satisfying (3.23). Then, by (3.20) and Assumption 3.2(C1), we have

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\alpha_{n_k})\left(1-\mu^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\right)\|y_{n_k}-w_{n_k}\|^2+\beta_{n_k}(1-\eta-\beta_{n_k})\|z_{n_k}-Tz_{n_k}\|^2\\
&\leq \limsup_{k\to\infty}(a_{n_k} - a_{n_k+1}) + (1-\kappa)M' \lim_{k\to\infty}\alpha_{n_k}\\
&= -\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k})\\
&\leq 0,
\end{aligned} \tag{3.24}$$

which implies,

$$\lim_{k\to\infty}\|y_{n_k} - w_{n_k}\| = \lim_{k\to\infty}\|z_{n_k} - Tz_{n_k}\| = 0. \tag{3.25}$$

Observe that

$$\|w_{n_k} - x_{n_k}\| = \alpha_{n_k}.\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0 \quad \text{and} \quad k\to\infty. \tag{3.26}$$

Combining (3.25) and (3.26), we obtain

$$\lim_{k\to\infty}\|y_{n_k} - x_{n_k}\| = 0. \tag{3.27}$$

Observe from Algorithm 3.3, that

$$\|z_{n_k} - y_{n_k}\| = \lambda_{n_k}\|(A(y_{n_k}) - A(w_{n_k}))\| \\ \leq \mu\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\|y_{n_k} - w_{n_k}\|.$$

Thus, by (3.25), we have

$$\lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = 0, \tag{3.28}$$

combining this with (3.25) and (3.28), we obtain

$$\lim_{k\to\infty}\|z_{n_k} - w_{n_k}\| = 0 = \lim_{k\to\infty}\|z_{n_k} - x_{n_k}\|. \tag{3.29}$$

Using (3.2)(ci),(3.25), (3.26) and (3.29), we have;

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &\leq \alpha_{n_k}\|f(x_{n_k}) - t_{n_k}\| + \|t_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|f(x_{n_k}) - t_{n_k}\| + \|t_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|f(x_{n_k}) - t_{n_k}\| + \|t_{n_k} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|f(x_{n_k}) - t_{n_k}\| + \beta_{n_k}\|Tz_{n_k} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| \\ &\quad + \|w_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty.\end{aligned} \tag{3.30}$$

We next show that $\limsup_{k\to\infty} b_{n_k} \leq 0$. It suffices to show that,

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle \leq 0.$$

Let $\{x_{n_{k_j}}\}$ be a subsequence of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle = \limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle. \tag{3.31}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \bar{p} \in H$. By (3.27), we have that $\{y_{n_{k_j}}\}$ also converges weakly to $\bar{p}$.

We next show that $\bar{p} \in VI(C, A)$. From $y_{n_k} = P_C(w_{n_k} - \lambda_{n_k}A(w_{n_k}))$ and the characteristic of the metric projection, we have

$$\langle w_{n_k} - \lambda_{n_k}A(w_{n_k}) - y_{n_k}, x - y_{n_k}\rangle \leq 0,\ \forall x \in C \tag{3.32}$$

which implies that

$$\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \lambda_{n_k}\langle A(w_{n_k}), x - y_{n_k}\rangle. \tag{3.33}$$

Since $\lambda_{n_k} > 0$, we have

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle A(w_{n_k}), y_{n_k} - w_{n_k}\rangle \leq \langle A(w_{n_k}), x - w_{n_k}\rangle. \tag{3.34}$$

Using (3.25), we have

$$0 \leq \liminf_{k\to\infty}\langle A(w_{n_k}), x - w_{n_k}\rangle \\ \leq \limsup_{k\to\infty}\langle A(w_{n_k}), x - w_{n_k}\rangle. \tag{3.35}$$

Now, observe that

$$\langle A(y_{n_k}), x - y_{n_k}\rangle = \langle A(y_{n_k}) - A(w_{n_k}), x - w_{n_k}\rangle \\ + \langle A(w_{n_k}), x - w_{n_k}\rangle \\ + \langle A(y_{n_k}), w_{n_k} - y_{n_k}\rangle. \tag{3.36}$$

From (3.3) and (3.25), we obtain

$$\lim_{k\to\infty}\|A(w_{n_k}) - A(y_{n_k})\| = 0.$$

Using this (3.25) and (3.36), we have

$$0 \leq \lim_{k\to\infty}\inf\langle A(y_{n_k}), x - y_{n_k}\rangle \\ \leq \lim_{k\to\infty}\sup\langle A(y_{n_k}), x - y_{n_k}\rangle. \tag{3.37}$$

First, we consider the case in which

$$\limsup_{k\to\infty}\langle A(y_{n_k}), x - y_{n_k}\rangle > 0$$

for all $x \in C$. Then there exists a sequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that

$$\limsup_{j\to\infty}\langle A(y_{n_{k_j}}), x - y_{n_{k_j}}\rangle > 0$$

for all $x \in C$. It follows that we can find n_0 such that

$$\langle A(y_{n_{k_j}}), x - y_{n_{k_j}}\rangle > 0,\ \forall j > n_0. \tag{3.38}$$

Using the fact that A is quasimonotone, we obtain that

$$\langle Ax, x - y_{n_k}\rangle > 0,\ \forall j > n_0. \tag{3.39}$$

Passing to the limit in (3.39) with n_k replaced by n_{k_j}, we have

$$\lim_{j\to\infty}\langle Ax, x - y_{n_{k_j}}\rangle = \langle Ax, x - \bar{p}\rangle \\ > 0.$$

Hence, $\bar{p} \in VI(C, A)$.

Secondly, we consider the case in which $\limsup_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle = 0$ for all $x \in C$. Let $\{\delta_k\}$ be a non-increasing positive sequence defined by

$$\delta_k = \langle Ay_{n_k}, x - y_{n_k}\rangle + \frac{1}{\kappa + 1}. \tag{3.40}$$

It is easy to see that

$$\lim_{k\to\infty}\delta_k = \lim_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle + \lim_{k\to\infty}\frac{1}{\kappa+1}$$
$$= 0. \tag{3.41}$$

Therefore, we have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle + \delta_k \geq 0, \tag{3.42}$$

for each $k \geq 1$.

Since $\{y_{n_k}\} \subset C$, it implies that $\{Ay_{n_k}\} > 0$ and $\liminf_{k\to\infty}\|Ay_{n_k}\| = n_0$, we therefore deduce that,

$$\|Ay_{n_k}\| > \frac{n_0}{2}. \tag{3.43}$$

Furthermore, let $\{\epsilon_{n_k}\}$ be a sequence defined by

$$\epsilon_{n_k} = \frac{Ay_{n_k}}{\|Ay_{n_k}\|^2}.$$

Then

$$\langle Ay_{n_k}, \epsilon_{n_k}\rangle = 1. \tag{3.44}$$

Combining (3.42) and (3.44), we have

$$\langle Ay_{n_k}, x - \delta_k\epsilon_{n_k} - y_{n_k}\rangle > 0.$$

Since A is quasimonotone on H, we get that

$$\langle A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \geq 0. \tag{3.45}$$

Now, observe that

$$\begin{aligned}\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle &= \langle Ax - A(x + \delta_k\epsilon_{n_k}) + A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \\ &= \langle Ax - A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \\ &\quad + \langle A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle,\end{aligned} \tag{3.46}$$

using this (3.45) and applying the well-known Cauchy Schwartz inequality, we have

$$\begin{aligned}\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle &\geq \langle Ax - A(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \\ &\geq -\|Ax - A(x + \delta_k\epsilon_{n_k})\|\|x + \delta_k\epsilon_{n_k} - y_{n_k}\|.\end{aligned} \tag{3.47}$$

Since A is Lipschitz continuous, we have

$$\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + L\|\delta_k\epsilon_{n_k}\|\|x + \delta_k\epsilon_{n_k} - y_{n_k}\| \geq 0, \tag{3.48}$$

combining (3.43), (3.48) and using the definition of $\{\epsilon_{n_k}\}$, we have

$$\langle Ax, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + \frac{2L}{n_0}\delta_k\|x + \delta_k\epsilon_{n_k} - y_{n_k}\| \geq 0.$$

Thus,

$$\lim_{k\to\infty}\left[\langle Ax, x+\delta_k\epsilon_{n_k}-y_{n_k}\rangle+\frac{2L}{n_0}\delta_k\|x+\delta_k\epsilon_{n_k}-y_{n_k}\|\right]=\langle Ax, x-\bar{p}\rangle$$
$$>0.$$

Therefore, $\bar{p}\in VI(C,A)$.

Further, we obtain from (3.25), (3.29) and the demiclosedness of $I-T$ that $\bar{p}\in F(T)$. Hence $\bar{p}\in\Gamma$. Finally, from (3.16), (3.30) and (3.31), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(q)-q, x_{n_{k+1}}-q\rangle &\leq \limsup_{k\to\infty}\langle f(q)-q, x_{n_{k+1}}-x_{n_k}\rangle\\ &\quad+\limsup_{k\to\infty}\langle f(q)-q, x_{n_k}-q\rangle\\ &=\lim_{j\to\infty}\langle f(q)-q, x_{n_{k_i}}-q\rangle\\ &=\langle f(q)-q, \bar{q}-q\rangle\\ &\leq 0. \end{aligned}\tag{3.49}$$

By Lemma 2.4, we conclude that $\{x_n\}$ converges to $\bar{p}$. The proof is completed. □

4. Numerical example

In this section, we give some numerical illustrations of our main result in the sequel. The following examples show the competitive advantage of our method Algorithm 3.3 over Algorithm 3.1 and Algorithm 1 as announced in [44] and [49], respectively. We choose the parameters for this experiment as follows: $\alpha_n=\frac{1}{n+1}$, $\beta_n=\frac{n}{7(n+1)}$, $\theta=\frac{1}{3}$, $\mu=\frac{1}{2}$, $\delta_n=\frac{1}{n\sqrt{n}}$ and $\epsilon_n=\frac{1}{n^{1.1}}$.

All computations were done on MATLAB 2023a with a Dell Latitude E7450 personal computer.

Example 4.1. We start with the following problem given in Liu and Yang [29]. Let $H=\mathbb{R}$, $C=[-1,1]$ and

$$A(x)=\begin{cases}2x-1, & x>1,\\ x^2, & x\in[-1,1],\\ -2x-1, & x<-1.\end{cases}\tag{4.1}$$

Then A is quasimonotone, Lipschitz continuous and $VI(C,A)=\{-1,0\}$.

Now, let $T:C\to C$ be given by $T(x)=-2x$. Then $F(T)=\{0\}$ and thus $\Gamma=\{0\}$. For this experiment, we consider the following initial values:

Case Ia: $x_0=1, x_1=0.075$;

Case Ib: $x_0=0.95, x_1=0.064$;

The results are presented in Figure 1.

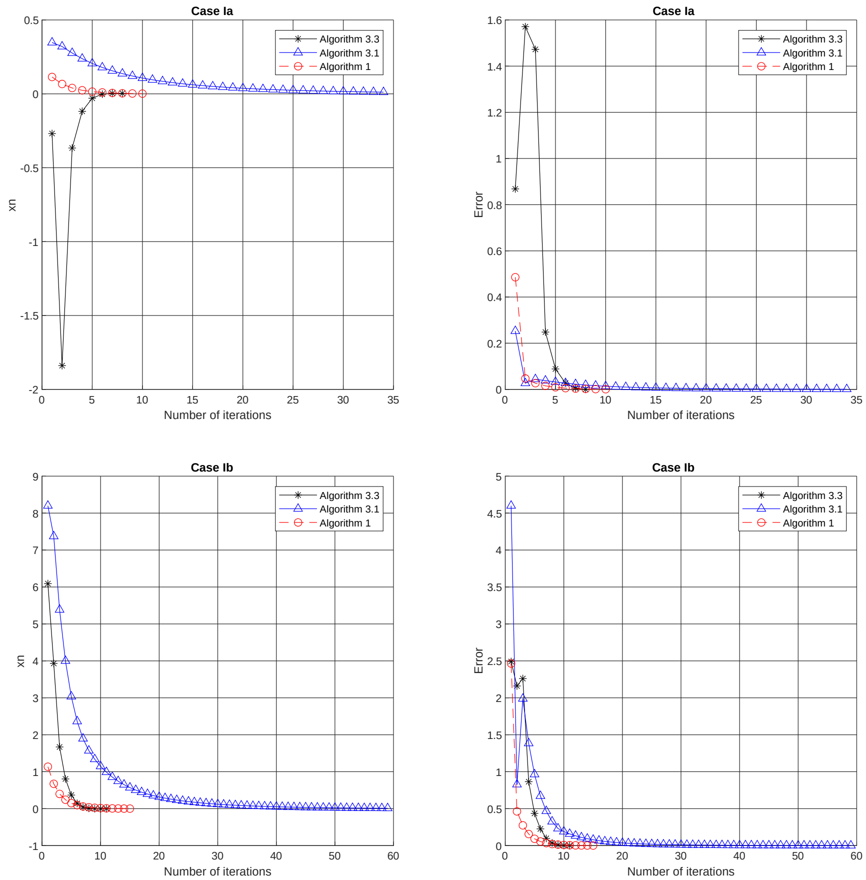

FIGURE 1. Left: Plot of $\{x_n\}$ against number of iteration; Right: Plot of error against number of iteration- Top: Case 1a; Bottom: Case 1b. **Numerical report for Example 4.1**.

Example 4.2. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $A(x_1, x_2) = (-x_1 \exp^{x_2}, x_2)$ and $C = \left\{x \in \mathbb{R}^2 : {x_1}^2 + {x_2}^2 \leq 1, 0 \leq x_1\right\}$. Then, $VI(C, A) = \{(1, 0)^T, (0, 0)^T\}$. It can easily be verified that all the conditions of Algorithms 3.3 are satisfied. Now, we choose different initial values as follows:

Case IIa: $x_0 = (0.3, 1.2)^T, x_1 = (0.6, 0.01)^T$;
Case IIb: $x_0 = (0.5, 0.6)^T, x_1 = (0.4, 0.2)^T$;
Case IIc: $x_0 = (1.2, 1.5)^T, x_1 = (1.0, 1.3)^T$;
Case IId: $x_0 = (-1.5, -1.1)^T, x_1 = (0.1, 0.6)^T$.
The results are presented in Figure 2.

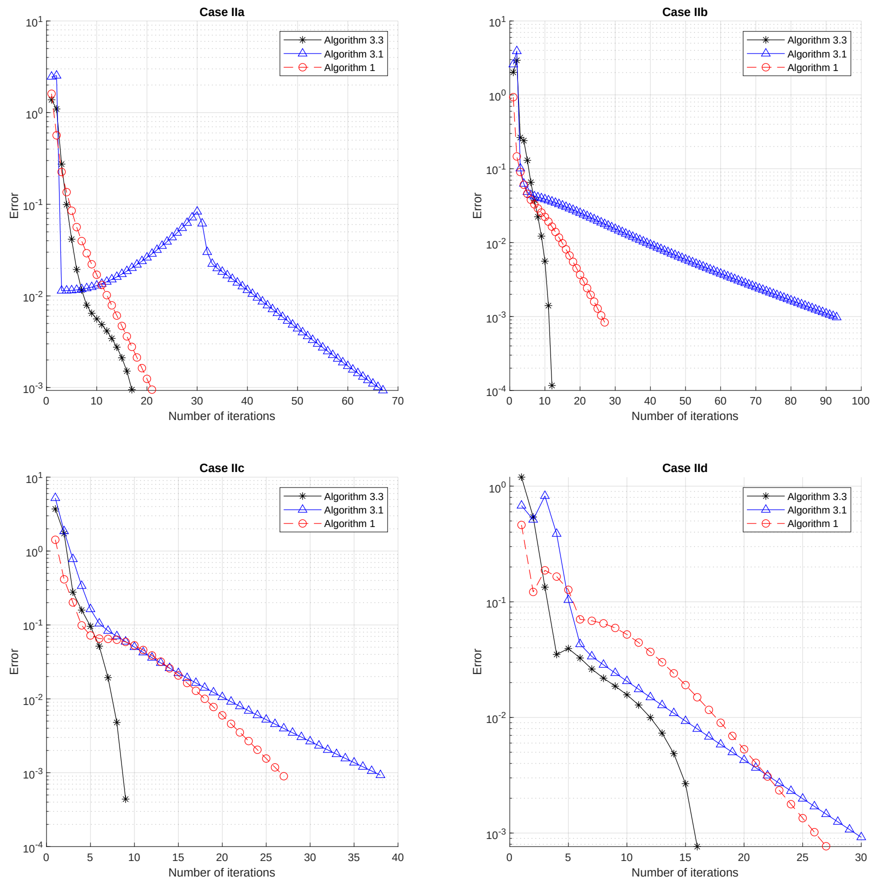

FIGURE 2. Plot of error against number of iteration - Top left: Case IIa; Top Right: Case IIb; Bottom left: Case IIc; Bottom right: Case IId. **Numerical report for Example 4.2**.

Example 4.3. We consider the next example in an infinite dimensional Hilbert space. Let

$$H = \left\{ x = (x_1, x_2, ..., x_i, ...) : \sum_{i=1}^{\infty} |x_i|^2 < +\infty \right\}.$$

Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha > \frac{\beta}{2} > 0$. Take $C = C_\alpha = \{x \in H : \|x\| \leq \alpha\}$ and $A = A_\beta(x) = (\beta - \|x\|)x$. Then A is quasimonotone and Lipschitz continuous (see Salahuddin [39]). Furthermore, we have that $VI(C, A) = \{0\}$.

We take $\beta = 3$ and $\alpha = 2$. Define the mapping $T : C \to C$ by

$$T(x) = \begin{cases} x, & \text{if } x < 0, \\ -3x, & 0 \leq x < \infty. \end{cases}$$

Choose different initial values as follows:

Case IIIa: $x_0 = (\frac{1}{10}, \cdots, \cdots), x_1 = (\frac{2}{5}, \cdots, \cdots)$,

Case IIIb: $x_0 = (\frac{1}{5}, \cdots, \cdots), x_1 = (\frac{1}{2}, \cdots, \cdots)$.

The results of this experiment are presented in Figure 3.

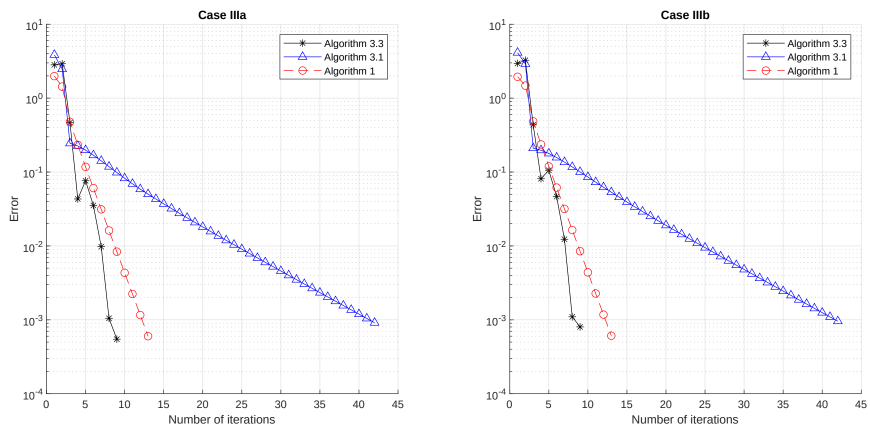

FIGURE 3. Plot of error against number of iteration - Left: Case IIIa; Right: Case IIIb (1). **Numerical report for Example 4.3**.

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