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A NOVEL FIXED POINT ITERATION PROCEDURE FOR APPROXIMATING THE SOLUTION OF IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this research, we propose a new efficient iterative method for fixed point problems of generalized α-nonexpansive mappings. We show the weak and strong convergence analysis of the proposed method under some mild assumptions on the control parameters. We consider the application of the new method to some real world problems such as convex minimization problems, image restoration problems and impulsive fractional differential equations. We carryout a numerical experiment to show the computational advantage of our method over some well known existing methods.

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1. Introduction

Over the last ten decades or so, the theory of fixed point has been vital and powerful tool in the study of nonlinear phenomena. The concept of fixed point theory has been applied to solve problems in various fields such as chemistry, physics, biology, engineering, economics, geometry, computer science, astronomy, control theory, optimization theory and games theory. Let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{B}$. Then, the mapping $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ is said to admit a fixed point $p^* \in \mathcal{K}$, if $\mathcal{S}p^* = p^*$ and $\mathcal{S}$ is called a contraction if there exists a constant $\delta \in [0, 1)$ such that $\|\mathcal{S}p - \mathcal{S}q\| \leq \delta\|p - q\|$, for all $p, q \in \mathcal{K}$. If $\delta = 1$, then $\mathcal{S}$ is called nonexpansive. The mappings $\mathcal{S}$ is called quasi-nonexpansive if $\|\mathcal{S}p - q\| \leq \|p - q\|$, for all $p \in \mathcal{K}$ and $q \in F(\mathcal{S}) = \{p^* \in \mathcal{K} : \mathcal{K}p^* = p^*\}$, where $F(S)$ is the set of all fixed points of S. One of the well known fixed point theorems was given by Brouwer in 1912. After this, several fixed point theorems have been given by many researchers. In 1922, the first metric fixed point theorem was considered by Stefan Banach and the theorem is known as Banach contraction principle. The Banach contraction principle has attracted the attention of several researchers as a results of its connection in the determination of the existence and uniqueness of solutions of many problems. The metric fixed point theory is very useful for topology and nonlinear functional analysis.

The Banach contraction principle includes the Picard iteration method for approximating the solutions of contraction mappings if such solutions exists. However, it is known that the Banach contraction principle is limited to contraction mappings since the Picard iteration method fails to convergence to the fixed point of nonexpansive mappings and other larger classless of mappings even when the fixed points of such mappings exist. In recent years, many fixed point theorems with more efficient iterative methods have been introduced and studied, see for example [1, 3, 12, 14, 16, 17, 19, 20, 28, 29, 40].

Fixed point theory of nonexpansive mappings has found applications in several applied sciences and engineering problems such as game theory, compressed sensing, mathematical economics, approximation theory and mathematics of fractals. It is well known that convex feasibility, monotone inclusion, image/signal restoration, variational inequality, equilibrium and convex optimization problems can all be transformed into a problem of finding the fixed point of some appropriate nonlinear operator, see, for instance, [2, 26]. Many iterative methods have been constructed for approximating the fixed points of nonlinear operators in different spaces (see, for example, [18, 21, 22, 23, 30]).

Many Modifications and generalizations of the class of nonexpansive mappings have been considered by many authors in the literature, see for instance [6, 31, 32, 33, 39]. One of these modifications is the generalized α-nonexpansive mappings which was proposed by Pant and Shukla [33]. They established convergence results of this mapping. This class of mappings is well known to include the class of mappings satisfying condition (C) which was introduced by Suzuki [39].

Let $\{d_k\}, \{e_k\}$ and $\{f_k\}$ be control sequences in $(0,1)$. The following iterative algorithms are called Mann [16], Ishikawa [12], Noor [17], S [3], Picard-S [11] and M [44] iterative algorithms:

$$\begin{cases} p_1 \in \mathcal{K}, \\ p_{k+1} = (1-d_k)p_k + d_k\mathcal{S}p_k, \end{cases} \qquad k \in \mathbb{N}. \tag{1.1}$$

$$\begin{cases} p_1 \in \mathcal{K}, \\ q_k = (1-e_k)p_k + e_k\mathcal{S}p_k, \\ p_{k+1} = (1-d_k)p_k + d_k\mathcal{S}q_k, \end{cases} \qquad k \in \mathbb{N}. \tag{1.2}$$

$$\begin{cases} p_1 \in \mathcal{K}, \\ w_k = (1-f_k)p_k + f_k\mathcal{S}p_k \\ q_k = (1-e_k)p_k + e_k\mathcal{S}w_k, \\ p_{k+1} = (1-d_k)p_k + d_k\mathcal{S}q_k, \end{cases} \qquad k \in \mathbb{N}. \tag{1.3}$$

$$\begin{cases} p_1 \in \mathcal{K}, \\ q_k = (1-e_k)p_k + e_k\mathcal{S}p_k, \\ p_{k+1} = (1-d_k)\mathcal{S}p_k + d_k\mathcal{S}q_m, \end{cases} \qquad k \in \mathbb{N}. \tag{1.4}$$

$$\begin{cases} p_1 \in \mathcal{K}, \\ w_k = (1-e_k)p_k + e_k\mathcal{S}p_k, \\ q_k = (1-d_k)\mathcal{S}p_k + d_k\mathcal{S}w_m, \\ p_{k+1} = \mathcal{S}q_m \end{cases} \qquad k \in \mathbb{N}. \tag{1.5}$$

$$\begin{cases} p_1 \in \mathcal{K}, \\ w_k = (1-d_k)p_k + d_k\mathcal{S}p_k, \\ q_k = \mathcal{S}w_k, \\ p_{k+1} = \mathcal{S}q_k, \end{cases} \qquad k \in \mathbb{N}. \tag{1.6}$$

The above iterative methods have been used to approximate the fixed points of certain nonlinear operators and several authors have shown that they converge faster than many other methods. It is therefore natural to ask the following question:

Is it possible to construct a more efficient iterative method that outperforms the above methods such that it can be used to solve some real-world problems?

In this article, one of our aims is to give affirmative answer to the above question by introducing the following iterative method called Picard-S**:

$$\begin{cases} p_1 \in \mathcal{K}, \\ w_k = (1-e_k)p_k + e_k\mathcal{S}p_k, \\ q_k = (1-d_k)\mathcal{S}(\mathcal{S}p_k) + d_k\mathcal{S}(\mathcal{S}w_k), \\ p_{k+1} = \mathcal{S}q_k, \end{cases} \quad k \in \mathbb{N}, \tag{1.7}$$

where $\{e_k\}$ and $\{d_k\}$ are sequences in (0,1).

Motivated and inspired by the above results, in this article, we prove the weak and strong convergence theorems of the proposed method (1.7) for the approximation of fixed points of generalized α-nonexpansive mappings in uniformly convex Banach space. The convergence results are obtained under some standard assumptions on the control parameters. We apply our method to solve some real world problems such as convex minimization problems, image restoration problems and impulsive fractional deferential equations. Lastly, we carryout a numerical experiment to show the computational advantage of our new method over some well known existing methods.

2. Preliminaries

In this section, we recall some definitions and lemmas that will be used in obtaining our main results.

Definition 2.1. Let $\mathcal{B}$ be a Banach space. If for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for $p, q \in \mathcal{B}$ such that $\|p\| \leq 1$, $\|q\| \leq 1$ and $\|p-q\| > \epsilon$, implies $\left\|\frac{p+q}{2}\right\| < 1-\delta$. Then $\mathcal{B}$ is said to be a uniformly convex Banach space (UCBS).

Definition 2.2. A Banach $\mathcal{B}$ is said to fulfill the Opial's property if for any sequence $\{p_k\} \subset \mathcal{B}$ which converges weakly to $p \in \mathcal{B}$ implies

$$\limsup_{k\to\infty} \|p_k - p\| < \limsup_{k\to\infty} \|p_k - q\|, \ \ \forall\, q \in \mathcal{B} \text{ with } q \neq p.$$

Definition 2.3. Let $\{p_k\}$ be a bounded sequence in a Banach space $\mathcal{B}$ and let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{B}$. For $p \in \mathcal{B}$, we take

$$r(p, \{p_k\}) = \limsup_{k\to\infty} \|p_k - p\|.$$

The asymptotic radius of $\{p_k\}$ relative to $\mathcal{K}$ is given as:

$$r(\mathcal{K}, \{p_k\}) = \inf\{r(p, \{p_k\}) : g \in \mathcal{K}\}.$$

The asymptotic center of $\{p_k\}$ relative to $\mathcal{K}$ is defined by

$$A(\mathcal{K}, \{p_k\}) = \{p \in \mathcal{K} : r(p, \{p_k\}) = r(\mathcal{K}, \{p_k\})\}.$$

It is well known that in a UCBS, the set $A(\mathcal{K}, \{p_k\})$ is a singleton.

Definition 2.4. Let $\mathcal{B}$ be a Banach space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{B}$. Then, the self-mapping $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ is said to demiclosed with respect to $p \in \mathcal{B}$, if for each sequence $\{p_k\}$ which is weakly convergent to $p \in \mathcal{K}$ and $\{\mathcal{S}p_k\}$ converges strongly to q implies that $\mathcal{S}p = q$.

Lemma 2.5. ([36]) *Let $\mathcal{B}$ be a UCBS and $\{r_k\}$ be any sequence fulfilling $0 < h \leq r_k \leq q < 1$ for all $k \geq 1$. Suppose $\{p_k\}$ and $\{q_k\}$ are any sequences in $\mathcal{B}$ with*

$$\limsup_{k\to\infty} \|p_k\| \leq z, \ \limsup_{k\to\infty} \|q_k\| \leq z$$

and

$$\limsup_{k\to\infty} \|r_k p_k + (1 - r_k)q_k\| = z$$

hold for some $z \geq 0$. Then $\lim\limits_{k\to\infty} \|p_k - q_k\| = 0$.

Definition 2.6. ([33]) $\mathcal{S} : \mathcal{B} \to \mathcal{B}$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ such that for $\frac{1}{2}\|p - \mathcal{S}p\| \leq \|p - q\|$,

$$\|\mathcal{S}p - \mathcal{S}q\| \leq \alpha\|\mathcal{S}p - h\| + \alpha\|\mathcal{S}q - p\| + (1 - \alpha)\|p - q\| \tag{2.1}$$

for all $p, q \in \mathcal{B}$.

Lemma 2.7. ([33]) *Let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{B}$. Assume that $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ is a generalized α–nonexpansive mapping. Then for all $p, q \in \mathcal{K}$, the following condition holds:*

$$\|p - \mathcal{S}q\| \leq \left(\frac{3 + \alpha}{1 - \alpha}\right) \|p - \mathcal{S}p\| + \|p - q\|.$$

Definition 2.8. ([37]) The condition (I) is said to be satisfied by the mapping $\mathcal{S} : \mathcal{K} \to \mathcal{K}$, if a nondecreasing function $h : [0, \infty) \to [0, \infty)$ exists with $h(0) = 0$ and for all $c > 0$ then $h(c) > 0$ with $\|p - \mathcal{S}p\| \geq h(d(p, F(\mathcal{S}))$ for all $p \in \mathcal{K}$, where $d(p, F(\mathcal{S})) = \inf_{p^* \in F(\mathcal{S})} \|p - p^*\|$.

3. Convergence analysis

In this section, we establish the weak and strong convergence results of the proposed method (1.7) for approximation of the fixed points of the class of mapping satisfying (2.1) in the frame work of UCBS.

Theorem 3.1. *Let $\mathcal{B}$ be a UCBS and $\mathcal{K}$ be a nonempty, closed and convex subset of $\mathcal{B}$. If $\mathcal{S}$ is the mapping satisfying* (2.1) *with $F(\mathcal{S}) \neq \emptyset$ and $\{p_k\}$ is the sequence generated by the Picard–S^{**} iterative method* (1.7)*. Then $\lim_{k\to\infty} \|p_k - p^*\|$ exists for each $p^* \in F(\mathcal{S})$.*

Proof. Let $p^* \in F(\mathcal{S})$ and $p \in \mathcal{K}$. Since $\mathcal{S}$ is a α-nonexpansive mapping with $F(\mathcal{S}) \neq \emptyset$, we obtain

$$\|\mathcal{S}p - \mathcal{S}p^*\| \leq \|p - p^*\|. \tag{3.1}$$

By (1.7), we have

$$\begin{aligned} \|w_k - p^*\| &= \|(1-e_k)p_k + e_k\mathcal{S}p_k - p^*\| \\ &\leq (1-e_k)\|p_k - p^*\| + e_k\|\mathcal{S}p_k - p^*\| \\ &\leq (1-e_k)\|p_k - p^*\| + e_k\|p_k - p^*\| \\ &= \|p_k - g^*\|. \end{aligned} \tag{3.2}$$

Combining (3.2) and (1.7), we obtain

$$\begin{aligned} \|q_k - p^*\| &= \|(1-d_k)\mathcal{S}(\mathcal{S}p_k) + d_k\mathcal{S}(\mathcal{S}w_k) - p^*\| \\ &\leq (1-d_k)\|\mathcal{S}p_k - p^*\| + d_k\|\mathcal{S}w_k - p^*\| \\ &\leq (1-d_k)\|p_k - p^*\| + d_k\|w_k - p^*\| \\ &\leq (1-d_k)\|p_k - p^*\| + d_k\|p_k - p^*\| \\ &= \|p_k - p^*\|. \end{aligned} \tag{3.3}$$

Using (3.3) and (1.7), we get

$$\begin{aligned} \|p_{k+1} - p^*\| &\leq \|\mathcal{S}q_k - p^*\| \\ &\leq \|q_k - p^*\| \\ &\leq \|p_k - p^*\|. \end{aligned} \tag{3.4}$$

It follows that $\{\|p_k - p^*\|\}$ is a sequence that is non-increasing and bounded below. Consequently, $\lim_{k\to\infty} \|p_k - p^*\|$ exists for each $p^* \in F(\mathcal{S})$. □

Theorem 3.2. *Let $\mathcal{B}$, $\mathcal{K}$, $\mathcal{S}$ and $\{p_k\}$ be same as in Theorem* 3.1*. Then, $F(\mathcal{S}) \neq \emptyset$ if and only if $\{p_k\}$ is bounded and $\lim_{k\to\infty} \|p_k - \mathcal{S}p_k\| = 0$.*

Proof. By Theorem 3.1, it is shown that $\{p_k\}$ is bounded and $\lim_{k\to\infty} \|p_k - p^*\|$ exists for any $p^* \in F(S)$. Suppose

$$\lim_{k\to\infty} \|p_k - p^*\| = h. \tag{3.5}$$

From (3.5) and (3.2), we have

$$\limsup_{k\to\infty} \|w_k - p^*\| \le \limsup_{k\to\infty} \|p_k - p^*\| = h. \tag{3.6}$$

By (3.5) and (3.1), we obtain

$$\limsup_{k\to\infty} \|\mathcal{S}p_k - p^*\| \le \limsup_{k\to\infty} \|p_k - p^*\| = h. \tag{3.7}$$

Recalling (1.7), we have

$$\begin{aligned}
\|p_{k+1} - p^*\| &= \|\mathcal{S}q_k - p^*\| \\
&\le \|q_k - p^*\| \\
&= \|(1-d_k)\mathcal{S}(\mathcal{S}p_k) + d_k\mathcal{S}(\mathcal{S}w_k) - p^*\| \\
&\le (1-d_k)\|\mathcal{S}p_k - p^*\| + d_k\|\mathcal{S}w_k - p^*\| \\
&\le (1-d_k)\|p_k - p^*\| + d_k\|w_k - p^*\| \\
&= \|p_k - p^*\| - d_k\|p_k - p^*\| + d_k\|w_k - p^*\|.
\end{aligned} \tag{3.8}$$

From (3.9), we have

$$\frac{\|p_{k+1} - p^*\| - \|p_k - p^*\|}{d_k} \le \|w_k - p^*\| - \|p_k - p^*\|. \tag{3.9}$$

Since $\{d_k\} \in (0, 1)$, it follows that

$$\begin{aligned}
\|p_{k+1} - p^*\| - \|p_k - p^*\| &\le \frac{\|p_{k+1} - p^*\| - \|p_k - p^*\|}{d_k} \\
&\le \|w_k - p^*\| - \|p_k - p^*\|,
\end{aligned} \tag{3.10}$$

which implies that

$$\|p_{k+1} - p^*\| \le \|w_k - p^*\|. \tag{3.11}$$

Therefore, from (3.5), we have

$$h \le \liminf_{k\to\infty} \|w_k - p^*\|. \tag{3.12}$$

From (3.12) and (3.6), we have

$$\begin{aligned} h &= \lim_{k\to\infty} \|w_k - p^*\| \\ &\le \lim_{k\to\infty} \|(1-e_k)p_k + e_k\mathcal{S}p_k - p^*\| \\ &= \lim_{k\to\infty} \|(1-e_k)(p_k - p^*) + e_k(\mathcal{S}p_k - p^*)\| \\ &= \lim_{k\to\infty} ((1-e_k)\|p_k - p^*\| + e_k\|\mathcal{S}p_k - p^*\|) \\ &\le \lim_{k\to\infty} ((1-e_k)\|p_k - p^*\| + e_k\|p_k - p^*\|) \\ &\le h. \end{aligned} \tag{3.13}$$

Thus,

$$\lim_{k\to\infty} \|(1-e_k)(p_k - p^*) + e_k(\mathcal{S}p_k - p^*)\| = h. \tag{3.14}$$

Using (3.5), (3.7), (3.14) and Lemma 2.5, we obtain

$$\lim_{k\to\infty} \|p_k - \mathcal{S}p_k\| = 0. \tag{3.15}$$

On the other hand, suppose $\{p_k\}$ is bounded and $\lim_{k\to\infty} \|p_k - \mathcal{S}p_k\| = 0$. Let $p^* \in A(\mathcal{K}, \{p_k\})$. From Lemma 2.7, we get

$$\begin{aligned} r(\mathcal{S}p^*, \{p_k\}) &= \limsup_{k\to\infty} \|p_k - \mathcal{S}p^*\| \\ &\le \left(\frac{3+\alpha}{1-\alpha}\right) \limsup_{k\to\infty} \|p_k - \mathcal{S}p_k\| + \limsup_{k\to\infty} \|\mathcal{S}p_k - \mathcal{S}p^*\| \\ &= \limsup_{k\to\infty} \|p_k - p^*\| \\ &= r(p^*, \{p_k\}). \end{aligned}$$

It means that $\mathcal{S}p^* \in A(\mathcal{K}, \{p_k\})$. Since $\mathcal{B}$ is a UCBS, it implies that $A(\mathcal{K}, \{p_k\})$ is a singleton set and therefore, we have that $\mathcal{S}p^* = p^*$. Thus, $F(\mathcal{S}) \neq \emptyset$. □

Theorem 3.3. *Let $\mathcal{B}$, $\mathcal{K}$, $\mathcal{S}$ and $\{p_k\}$ be same as in Theorem* 3.1 *such that $F(\mathcal{S}) \neq \emptyset$. Assume that $\mathcal{B}$ fulfills the Opial's property., Then the Picard–S^{**} iterative scheme $\{p_k\}$ weakly converges to an element in $F(\mathcal{S})$.*

Proof. For $F(\mathcal{S}) \neq \emptyset$, it is shown in Theorems 3.1 and 3.2 that $\lim_{k\to\infty} \|p_k - p^*\|$ exists and $\lim_{k\to\infty} \|p_k - \mathcal{S}p_k\| = 0$. In what follows, we will show the impossibility of $\{p_k\}$ to posses two weak sub-sequential limits in $F(\mathcal{S})$. Let c and d be two weak sub-sequential limits of $\{p_{k_i}\}$ and $\{p_{k_j}\}$, respectively. Thanks to Theorem 3.2, we have that $(I-\mathcal{S})$ is demiclosed at 0, it follows that $(I-\mathcal{S})c =$

0. Therefore, $\mathcal{S}c = c$. Using similar approach, we can prove that $\mathcal{S}d = d$. Now, we prove uniqueness. Suppose $c \neq d$. Then by Opial's condition

$$\begin{aligned}\lim_{k\to\infty} \|p_k - c\| &= \lim_{i\to\infty} \|p_{k_i} - c\| \\ &< \lim_{i\to\infty} \|p_{k_i} - d\| = \lim_{k\to\infty} \|p_k - d\| \\ &= \lim_{j\to\infty} \|p_{k_j} - d\| \\ &< \lim_{j\to\infty} \|p_{k_j} - c\| = \lim_{k\to\infty} \|p_k - c\|,\end{aligned}$$

which is a contradiction, therefor $c = d$. Thus, $\{p_k\}$ weakly converges to $c \in F(\mathcal{S})$. □

Now, we present some strong convergence theorems.

Theorem 3.4. *Let $\mathcal{B}$, $\mathcal{K}$, $\mathcal{S}$ and $\{p_k\}$ be same as in Theorem* 3.1 *such that $F(\mathcal{S}) \neq \emptyset$. Then, $\{g_m\}$ converges strongly to an element in $F(\mathcal{S})$ if and only if $\liminf_{k\to\infty} d(p_k, F(\mathcal{S})) = 0$, where $d(p_k, F(\mathcal{S})) = \inf\{\|p_k - p^*\| : p^* \in F(\mathcal{S})\}$.*

Proof. The necessity case is trivial. Thus, we consider only the sufficient case. Assume that $\liminf_{k\to\infty} d(p_k, F(\mathcal{S})) = 0$ and $p^* \in F(\mathcal{S})$. Then, by Theorem 3.1, we have that $\lim_{k\to\infty} \|p_k - p^*\|$ exists, for each $p^* \in F(\mathcal{S})$. It is now enough to prove that $\{p_k\}$ is a Cauchy sequence in $\mathcal{K}$. Due to $\lim_{k\to\infty} d(p_k, F(\mathcal{S})) = 0$, then for $\xi > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $k \geq m_0$

$$d(p_k, F(\mathcal{S})) < \frac{\xi}{2}$$

and

$$\inf\{\|p_k - p^*\| : p^* \in F(\mathcal{S})\} < \frac{\xi}{2}.$$

In particular, $\inf\{\|p_{m_0} - p^*\| : p^* \in F(S)\} < \frac{\xi}{2}$. Therefore, there exists $p^* \in F(\mathcal{S})$ such that

$$\|p_{m_0} - p^*\| < \frac{\xi}{2}.$$

If $m, k \geq m_0$, we have

$$\begin{aligned}\|p_{k+l} - p_k\| &\leq \|p_{k+l} - p^*\| + \|p_k - p^*\| \\ &\leq \|p_{m_0} - p^*\| + \|p_{m_0} - p^*\| \\ &= 2\|p_{m_0} - p^*\| < \xi.\end{aligned}$$

This means that the sequence $\{p_k\}$ is Cauchy in $\mathcal{K}$. Since $\mathcal{K}$ is closed, it follows that a point $t \in \mathcal{K}$ with $\lim_{k\to\infty} p_k = t$. So that $\lim_{k\to\infty} d(p_k, F(\mathcal{S})) = 0$ implies that $d(t, F(\mathcal{S})) = 0$, that is, $t \in F(\mathcal{S})$. □

Theorem 3.5. *Let $\mathcal{B}$, $\mathcal{K}$, $\mathcal{S}$ and $\{p_k\}$ be same as in Theorem* 3.1 *such that $F(\mathcal{S}) \neq \emptyset$. Assume $\mathcal{K}$ is a nonempty convex compact subset of $\mathcal{B}$. Then $\{p_k\}$ strongly converges to an element in $F(\mathcal{S})$.*

Proof. Thanks to Theorem 3.2, it is shown $\lim_{k\to\infty} \|p_k - \mathcal{S}p_k\| = 0$. By the compactness of $\mathcal{K}$, it follows that $\{p_k\}$ has a strong convergent subsequence $\{p_{k_i}\}$ with a strong limit c. By Lemma 2.7, we get

$$\|p_{k_i} - \mathcal{S}c\| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \|p_{k_i} - \mathcal{S}p_{k_i}\| + \|p_{k_i} - c\|.$$

Letting $i \to \infty$, we get $p_{k_i} \to \mathcal{S}c$. Thus, $\mathcal{S}c = c$, that is, $c \in F(S)$. From Theorem 3.1, we know that $\lim_{k\to\infty} \|p_k - c\|$ exists. In what follows, we have that c is a strong limit for $\{p_k\}$. □

Theorem 3.6. *Let $\mathcal{B}$, $\mathcal{K}$, $\mathcal{S}$ and $\{p_k\}$ be same as in Theorem* 3.1 *such that $F(S) \neq \emptyset$. If $\mathcal{S}$ satisfies the condition I, then $\{p_k\}$ converges strongly to an element in $F(S)$.*

Proof. It shown in Theorem 3.2 that

$$\lim_{k\to\infty} \|p_k - \mathcal{S}p_k\| = 0. \tag{3.16}$$

By (3.16) and Definition 2.8, we have

$$0 \leq \lim_{k\to\infty} f(d(p_k, F(S))) \leq \lim_{k\to\infty} \|p_k - \mathcal{S}p_k\| = 0.$$

Hence we have $f(d(p_k, F(S))) = 0$.

Since the function $h : [0, \infty) \to [0, \infty)$ is nondecreasing such that $h(0) = 0$ and $h(g) > 0$, for all $g > 0$, we have

$$\lim_{k\to\infty} d(p_k, F(S)) = 0.$$

Using Theorem 3.4, the remainder of the proof is obtained. □

4. Application to convex minimization problem

In this section, we apply our method to solve convex minimization problem. The compressive sensing (CS) has recently been applied to image and signal processing by many authors. A notable concept of image/signal processing problem is the image restoration which is one of the most celebrated classical inverse problems. Image processing problem has widely been studied in various applications such astronomical imaging, image deblurring, remote sensing, digital photography, radar imaging, microscopic imaging and many more. The image recovery problem can be expressed in one dimensional vector by a model defined as follows:

$$Fp = q + h, \tag{4.1}$$

where h is additive noise, $q \in \mathbb{R}^{m+1}$ is the observed image, $p \in \mathbb{R}^{k+1}$ is the original image and $F \in \mathbb{R}^{k\times m}$. One of the well-known methods for solving the problem (4.1) is the least absolute shrinking and selection operator (LASSO), which was introduced and studied by Tibshirani [42] as follows:

$$\min_p \|Fp - q\|_2^2 + w\|p\|_1, \tag{4.2}$$

where $\|\cdot\|_1$ is the l_1-norm defined by $\|p\|_1 = \sum_{i=1}^k |p_i|$, $\|\cdot\|_2$ is the l_2 norm defined by $\|p\|_2 = \sum_{i=1}^k |p_i|^2$ and $w > 0$ is known as the regularization parameter. It is well known that the LASSO can be used to solve regression problem [9], image restoration problem [42], etc. The expression (4.2) can be rewritten in a general form by estimating the minimizer of the sum of two functions as follows:

$$\min_p A(p) = f(p) + g(p), \tag{4.3}$$

where f is a smooth convex loss function with gradient having Lipschitz constant L and g is a convex smooth (or possible non-smooth) function. By applying Theorem 6.3 of Fermats rule [8], the solution of the problem (4.3) can characterized as follows: p^* is a minimizer of $(f+g)$ if and only if $0 \in \partial g(p^*) + \triangledown f(p^*)$, where $\triangledown f(p^*)$ and $\partial g(p^*)$ denote the gradient and sub-differential of f and g, respectively. It is shown in Parikh and Boyd [34] that that problem (4.3) can be reformulated as a fixed point problem as follows: p^* is a minimizer of $(f+q)$ if and only if

$$p^* = prox_{ug}(I - u\triangledown f)(p^*) = J_{u\partial g}(I - u\triangledown f)(p^*), \tag{4.4}$$

where $prox_g(p) = argmin_{q\in\mathcal{B}}(g(q) + \frac{1}{2}\|p-q\|^2)$, $u > 0$, $prox_{ug}$ is the proximity operator of ug, I is an identity operator and $J_{\partial g} = (I + \partial g)^{-1}$. It easy to see that (4.4) can be reformulated as a fixed point problem as follows:

$$p^* = \mathcal{S}p^*, \tag{4.5}$$

where $S = prox_{ug}(I - u\triangledown f)$ and this is called the forward–backward operator. It is well known that S is a nonexpansive mapping for $u \in (0, \frac{2}{L})$ and a solution of (4.5) is a fixed point of $\mathcal{S}$ [13].

Now, we present our main result in section as follows:

Theorem 4.1. *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{B}$ and the sequence $\{p_k\}$ be defined as follows:*

$$\begin{cases} p_1 \in \mathcal{K}, \\ w_k = (1-e_k)p_k + e_k prox_{ug}(I - u\nabla f)p_k, \\ q_k = (1-d_k)prox_{ug}(I-u\nabla f)(prox_{ug}(I-u\nabla f)p_k) \\ +d_k prox_{ug}(I-u\nabla f)(prox_{ug}(I-u\nabla f)w_k), \\ p_{k+1} = prox_{ug}(I-u\nabla f)q_k, \end{cases} \tag{4.6}$$

where $k \in \mathbb{N}$, $\{e_k\}$, $\{d_k\}$ are sequences in (0,1), $u \in (0, \frac{2}{L})$ and $f, g : \mathbb{R}^k \to (-\infty, +\infty]$ are such that g is a convex function and f is smooth convex function with gradient having Lipschitz constant L. Then $\{p_k\}$ converges strongly to an element $p^* \in Argmin(f+g)$.

Proof. Let $\mathcal{S}$ be the forward-backward operator of f and g with respect to c, that is, $S = prox_{ug}(I - u\nabla f)$. Then $\mathcal{S}$ is a nonexpansive operator and $F(\mathcal{S}) = argmin(f+g)$, see Proposition 26.1 in [8]. Therefore, since every nonexpansive mapping is a generalized α-nonexpansive mapping, we obtain the required result directly from Theorem 3.4. □

5. Application to image restoration problem

In this section, the deblurring efficiency of our method for solving image restoration problem is compared with Ishikawa [12], Noor [17], S [3], Picard-S [11] and M [44] iteration methods. All algorithms are applied to solve the problem (4.2), where $f(q) = \|Fq - v\|_2^2$, $g(q) = w\|q\|_1$ and w is the regularization parameter, F is the blurring operator and v is the observed image. For this numerical test, two images, apples and Brain X-ray of sizes 365×365 are used as original images. The images went through standard deviation $\sigma = 5$ and Gaussian blur of size 92. The quality of the images are measured by peak signal-to-noise ratio (PSNR) [29], where PSNR(p_k) is defined by:

$$PSNR(p_k) = 20 \log 10 \left(\frac{354^2}{MSE} \right),$$

where $MSE = \frac{1}{Q}\|p_k - \bar{p}\|_2^2$, $\bar{p}$ is the original image and Q is the number of image samples. For this experiment, the regularization parameter is taken as $w = 8 \times 10^{-8}$, and the initial images were the blurred images. The Lipschitz constant L, is determined using maximum eigenvalues of the matrix $F^T F$. The control parameters c, $\{e_k\}, \{d_k\}$ and $\{f_k\}$ are choosen as follows: $c = \frac{1}{L}$ and $e_k = d_k = f_k = \frac{3}{4}$. The numerical results are shown in Figures 1–4 and Tables 1–2.

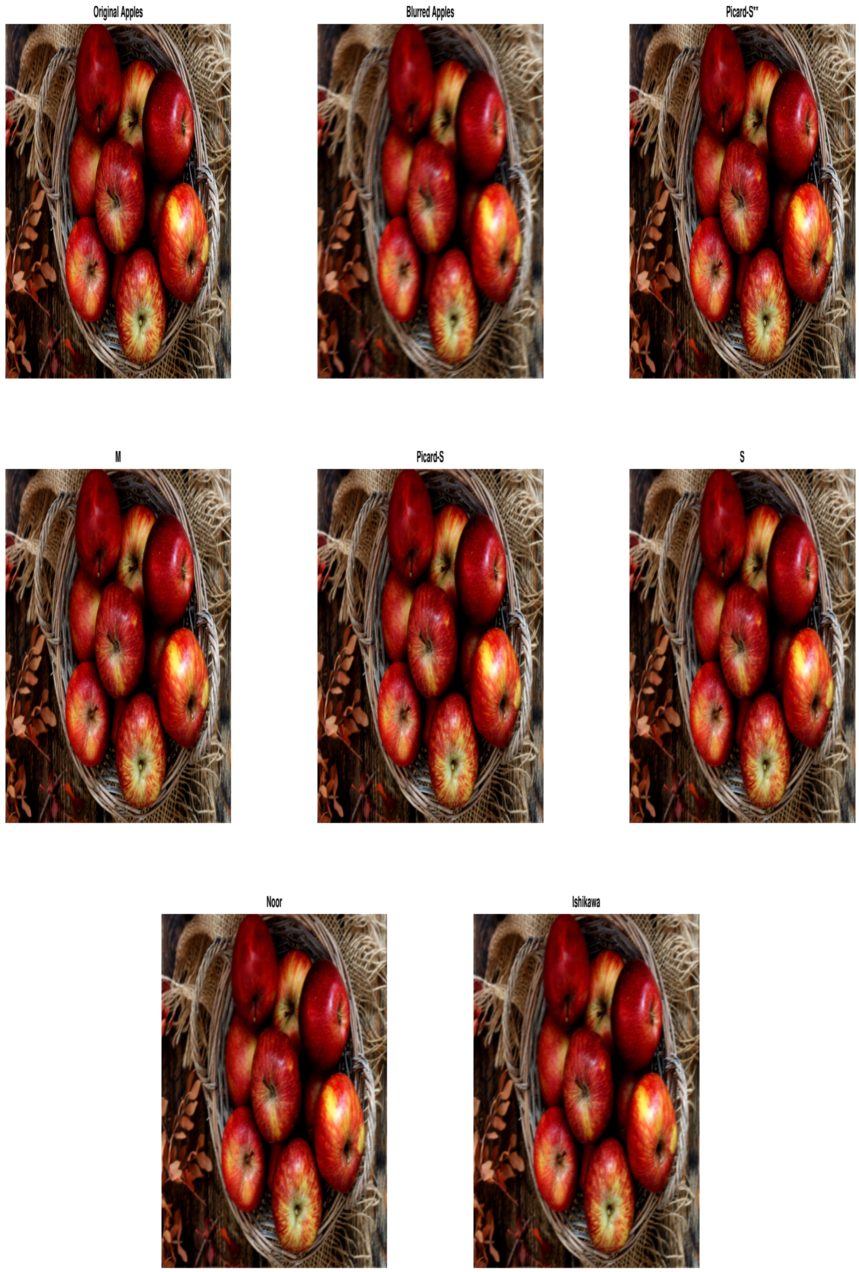

FIGURE 1. Apples image deblurring by various algorithms.

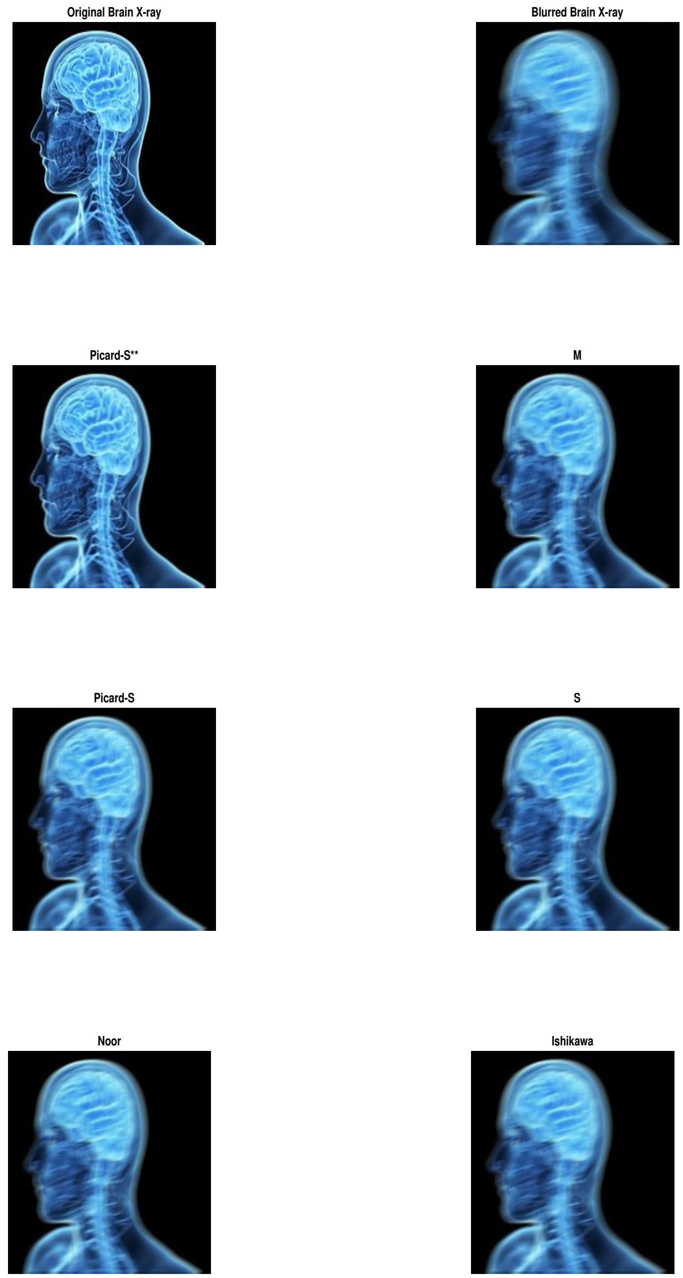

FIGURE 2. Brain X-ray image deblurring by various algorithms.

TABLE 1. Numerical behaviour of various methods using their PSNR values for Apples image.

Images	k	Picard-S**	M	Picard-S	S	Noor	Ishikawa
Apples.png		PSNR	PSNR	PSNR	SNR	PSNR	PSNR
(365 × 365)	300	37.3425	32.3464	29.2435	26.2347	24.4362	10.4567
	600	37.4536	33.4536	29.5346	27.6473	24.6983	15.8456
	8000	37.4867	34.6745	30.2367	28.6457	25.3563	16.7834
	1500	37.7543	34.9875	32.2356	30.6473	25.6457	17.5863

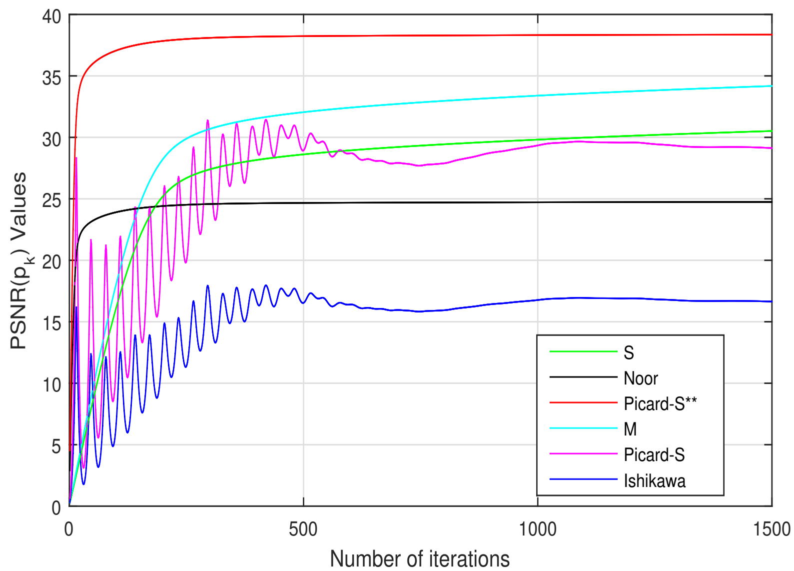

FIGURE 3. Graph corresponding to Table 1

.

TABLE 2. Numerical behaviour of various methods using their PSNR values for Brain X-ray image.

Images	k	Picard-S**	M	Picard-S	S	Noor	Ishikawa
Brain X-ray.png		PSNR	PSNR	PSNR	PSNR	PSNR	PSNR
(365 × 365)	300	32.4356	24.6754	23.2321	22.3432	21.4535	13.3542
	600	32.5768	25.6575	25.4353	23.5463	22.2343	13.7637
	800	33.3435	27.7658	25.6575	24.5464	23.4536	13.8757
	1500	33.7683	28.3246	26.4564	24.8764	23.6546	13.9840

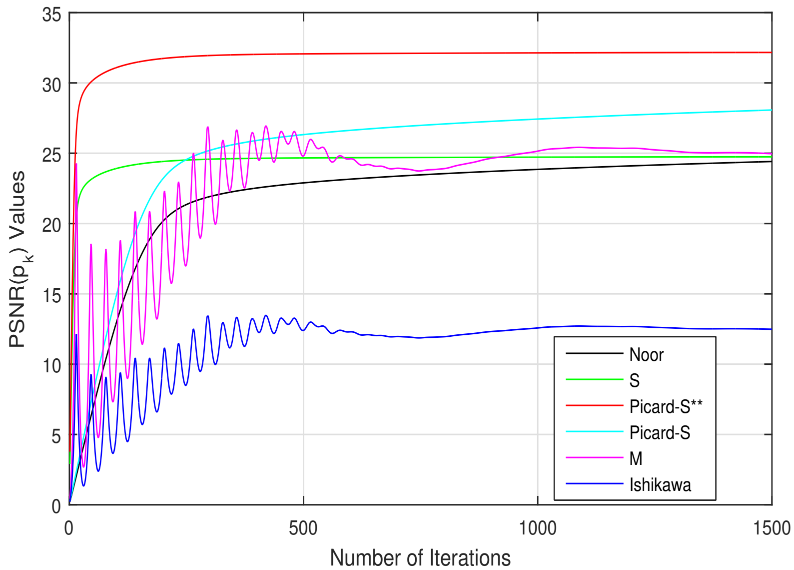

FIGURE 4. Graph corresponding to Table 2

.

Remark 5.1. It is well known that better restoration is achieved with higher PSNR value. From Figure 1–4 and Tables 1–2, observe that our new method has a better restoration efficiency than the compared methods.

6. Application to impulsive fractional differential equations

It is well known that fractional calculus is a valuable tool in modeling many phenomena in various fields of engineering and applied sciences such as acoustic control, astrophysics, chemical engineering, chaotic dynamics, porous media, electro chemistry, economics, optics, polymer physics and medicine [10]. Dynamics of several evolutionary phenomena emanating from diverse fields such as control theory, population dynamics, biology, physics and medicine undergo abrupt changes at some time like harvesting, earthquake, shocks, and so forth, these perturbations can be approximated as instantaneous change of states or impulses. These phenomena are modeled by impulsive differential equations.

In this research, we approximate the solution of the following impulsive fractional differential equations (IFDEs) ([4, 5, 7]) that incorporates the Caputo fractional derivative with a nonlocal condition in Banach space $PC(E,\mathcal{B})$:

$$\begin{cases} {}^c\mathcal{D}^q p(s) = F(s,p(s)),\ s\in E/\{s_1,s_2,\cdots,s_m\},\ \ E=[0,T], \\ \triangle p(s_j) = I_j(p(s_j)),\ j=1,2,\cdots,m, \\ p(0)+\eta(p)=p_0, \end{cases} \tag{6.1}$$

where ${}^c\mathcal{D}^q$ is the Caputo fractional derivative of order $0<q\le 1$, $F: E\times\mathcal{B}\to\mathcal{B}$, $p_0\in\mathcal{B}$ and $\eta: PC(E,\mathcal{B})\to\mathcal{B}$ is a continuous map. The map $I_j:\mathcal{B}\to\mathcal{B}$ is continuous and s_j fulfills, $0=s_0<s_1<s_2<\cdots<s_m<s_{m+1}=T$.

Now, let $PC(E,\mathcal{B}) = \{p: E\to\mathcal{B} : p\in C((I_j,I_{j+1}],\mathcal{B}), j=0,\cdots,m+1$ and there exist, $p(I_j^-)$ and $p(I_j^+), j=1,\cdots,m$ such that $p(I_j^-)=p(I_j^+)\}$. Then $PC(E,\mathcal{B})$ is a Banach space with the induced norm $\|p\|_{PC}=\sup\{\|p(s)\|: s\in E\}$.

The following definition and lemma will be useful in this section:

Definition 6.1. The Caputo fractional order derivation of a continuous function F in the closed interval $[a,b]$ is defined by

$$({}^c\mathcal{D}^q_{a+}F)(s) = \frac{1}{\Gamma(k-q)}\int_a^s (s-t)^{k-q-1}F^{(k)}(t)dt, \tag{6.2}$$

where $k=[q]+1$.

Definition 6.2. A function $p\in PC(E,\mathcal{B})$ is called a solution of the problem (6.1) if it satisfies the equation ${}^c\mathcal{D}^q = F(s,p(s))$ a.e. on E such that $\triangle p(s_j) = I_j(p(s_j))$, $j=1,2,\cdots,m$, and $p(0)+\eta(p)=p_0$.

Lemma 6.3. ([10]) *The fractional integral equation*

$$\begin{aligned} p(s) &= p_0-\eta(p)-\frac{1}{\Gamma(q)}\sum_{0<s_j<s}\int_{s_{j-1}}^{s_j}(s_j-t)^{q-1}F(t,p(t))dt \\ &\quad+\frac{1}{\Gamma(q)}\int_0^s (s-t)^{q-1}F(t,p(t))dt \\ &\quad+\sum_{0<s_j<s} I_j(p(s_j)),\ j=1,\cdots,m \end{aligned} \tag{6.3}$$

has a solution $p\in PC(E,\mathcal{B})$, for $s\in(s_j,s_{j+1}), j=1,2,\cdots,m$ if and only if p is the solution to the problem (6.1).

It is shown in [10] that under some standard assumptions, the problem (6.1) has a unique solution. In this research, we will approximate the solution of the problem (6.1) via our new method (1.7) under the following assumptions:

(a) A constant $L_F>0$ exists such that

$$\|F(s,p)-F(s,q)\|_{PC}\le L_F\|p-q\|_{PC},\ \forall p,q\in\mathcal{B}, s\in E.$$

(b) There exists $\delta \in [0, \frac{1}{m})$ and $\gamma_\eta > 0$ such that

$$\|I_j(p) - I_j(q)\| \leq \delta\|p - q\|, \ \forall p, q \in \mathcal{B}, \ j = 1, 2, \cdots, m,$$
$$\|\eta(p) - \eta(q)\|_{PC} \leq \gamma_\eta\|p - q\|_{PC}, \ \forall p, q \in PC(E, \mathcal{B}).$$

(c) $\gamma_k + m\delta + \frac{(m+1)L_F T^q}{\Gamma(q+1)} < 1$.

Now, we present our main result in this section in the next theorem.

Theorem 6.4. *Let the operator $\mathcal{S} : PC(E, \mathcal{B}) \to PC(E, \mathcal{B})$ be defined by*

$$(Sp)(s) = p_0 - \eta(p) - \frac{1}{\Gamma(q)} \sum_{0<s_j<s} \int_{s_{j-1}}^{s_j} (s_j - t)^{q-1} F(t, p(t))dt$$
$$+ \frac{1}{\Gamma(q)} \int_0^s (s - t)^{q-1} F(t, p(t))dt + \sum_{0<s_j<s} I_j(p(s_j)), \ j = 1, \cdots, m. \tag{6.4}$$

Suppose assumptions (a)–(c) *hold. Then the sequence involving operator $\mathcal{S}$ and generated by algorithm* (1.7) *converges to a unique solution of the problem* (6.1)*, say $p \in PC(E, \mathcal{B})$.*

Proof. It suffices to show that the operator $\mathcal{S}$ is nonexpansive. Indeed, for $p, q \in PC(E, \mathcal{B})$ and $s \in (s_j, s_{j+1}), j = 1, 2, \cdots, m$, we have

$$\begin{aligned}
\|(Sp)(s) - (Sq)(s)\|_{PC} &\leq \|\eta(p) - \eta(q)\|_{PC} + \sum_{0<s_j<s} \|I_j(p(s_j)) - I_j(q(s_j))\| \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<s_j<s} \int_{s_{j-1}}^{s_j} (s_j - t)^{q-1} \|F(t, p(t)) - F(t, q(t))\|_{PC} dt \\
&\quad + \frac{1}{\Gamma(q)} \int_0^s (s - t)^{q-1} \|F(t, p(t)) - F(t, q(t))\|_{PC} dt \\
&\leq \gamma_\eta\|p - q\|_{PC} + \sum_{0<s_j<s} \delta\|p - q\| \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<s_j<s} \int_{s_{j-1}}^{s_j} (s_j - t)^{q-1} L_F\|p - q\|_{PC} dt \\
&\quad + \frac{1}{\Gamma(q)} \int_0^s (s - t)^{q-1} L_F\|p - q\|_{PC} dt \\
&\leq \left[\gamma_k + m\delta + \frac{(m+1)L_F T^q}{\Gamma(q+1)}\right] \|p - q\|_{PC} \\
&\leq \|p - q\|_{PC}.
\end{aligned}$$

Therefore, $\mathcal{S}$ is a nonexpansive operator. Hence, since every nonexpansive mapping is a generalized α-nonexpansive mapping, then algorithm (1.7) converges to the solution of the problem (6.1). $\square$

7. Numerical experiment

In this section, we present an example of a generalized α-nonexpnsive which does not satisfy condition (C). We show the computational advantage of our suggested method by comparing it with some well known existing methods.

Example 7.1. Let $\mathbb{M} = \mathbb{R}$ with the usual norm and $\mathcal{K} = [7, 10]$ and the mapping $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ be defined by

$$\mathcal{S}p = \begin{cases} \frac{p+56}{8}, & \text{if } \ p < 10, \\ 7, & \text{if } \ p = 10, \end{cases}$$

for all $p \in \mathcal{K}$.

(i) If $p = 9$ and $q = 10$, we get

$$\begin{aligned} \frac{1}{2}\|p - \mathcal{S}q\| &= \frac{7}{8} \\ &< 1 \\ &= \|p - q\|. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\mathcal{S}p - \mathcal{S}q\| &= \frac{9}{8} \\ &> 1 \\ &= \|p - q\|. \end{aligned}$$

Thus, $\mathcal{S}$ does not satisfy condition (C).

(ii) Next, we illustrate that $\mathcal{S}$ is a generalized α-nonexpansive mapping for $\alpha = \frac{1}{3}$. To do this, the following cases will be considered:
Case 1: When $p, q < 10$, we have

$$\begin{aligned} \|\mathcal{S}p - \mathcal{S}q\| &= \alpha\|p - \mathcal{S}q\| + \alpha\|q - \mathcal{S}p\| + (1 - 2\alpha)\|p - q\| \\ &= \frac{1}{3}\left|p - \left(\frac{q+56}{8}\right)\right| + \frac{1}{3}\left|q - \left(\frac{g+56}{8}\right)\right| + \frac{1}{3}|p - q| \\ &\geq \frac{1}{8}\,|p - q| \\ &= \|\mathcal{S}p - \mathcal{S}q\|. \end{aligned}$$

Case 2: When $p < 10$ and $q = 10$, we have

$$\begin{aligned}
\|\mathcal{S}p - \mathcal{S}q\| &= \alpha\|p - \mathcal{S}q\| + \alpha\|p - \mathcal{S}q\| + (1-2\alpha)\|p - q\| \\
&= \frac{1}{3}\,|p-7| + \frac{1}{3}\left|10 - \left(\frac{p+56}{8}\right)\right| + \frac{1}{3}|p-10| \\
&= \frac{1}{3}\,|p-7| + \frac{1}{3}\left|\frac{24-p}{8}\right| + \frac{1}{3}|p-10| \\
&\geq \frac{1}{3}\left|\frac{24-p}{8}\right| + 1 \\
&\geq \frac{1}{8}\,|p| \\
&= \|\mathcal{S}p - \mathcal{S}q\|.
\end{aligned}$$

Case 3: When $g = 10$ and $h < 10$, we obtain

$$\begin{aligned}
\|\mathcal{S}p - \mathcal{S}q\| &= \alpha\|p - \mathcal{S}q\| + \alpha\|q - \mathcal{S}p\| + (1-2\alpha)\|p - q\| \\
&= \frac{1}{3}\left|10 - \left(\frac{q+56}{8}\right)\right| + \frac{1}{3}\,|q-7| + \frac{1}{3}|10-q| \\
&= \frac{1}{3}\left|\frac{24-q}{8}\right| + \frac{1}{3}\,|q-7| + \frac{1}{3}|10-q| \\
&\geq \frac{1}{3}\left|\frac{24-q}{8}\right| + 1 \\
&\geq \frac{1}{8}\,|q| \\
&= \|\mathcal{S}p - \mathcal{S}q\|.
\end{aligned}$$

Case 4: When $p = q = 10$, we have

$$\begin{aligned}
\|\mathcal{S}p - \mathcal{S}q\| &= \alpha\|p - \mathcal{S}q\| + \alpha\|q - \mathcal{S}p\| + (1-2\alpha)\|p - q\| \\
&\geq 0 \\
&= \|\mathcal{S}p - \mathcal{S}q\|.
\end{aligned}$$

Thus, for all cases, we have that $\mathcal{S}$ is a generalized α-nonexpansive mapping.

Firstly, we take the control sequences $d_k = e_k = f_k = \frac{k}{k+1}$ and starting point $p_1 = 7$. The numerical results are presented in the following Table 3 and Figure 5. Observe that our new methods approaches 8 which is the fixed point of $\mathcal{S}$ faster than the compared methods.

Table 3. Comparison of the convergence behaviour of various iterative methods.

p_k	Ishikawa	Noor	S	Picard-S	M	Picard-S**
p_1	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
p_2	7.63194444	7.63599537	7.92361111	7.99680397	7.99999375	7.99880642
p_3	7.83966579	7.84250543	7.99345920	7.99998598	7.99999978	8.00000000
p_4	7.92555912	7.92727929	7.99941600	7.99999976	7.99999998	8.00000000
p_5	7.96425861	7.96525500	7.99994671	7.99999998	7.99999999	8.00000000
p_6	7.98248007	7.98304557	7.99999507	7.99999999	8.00000000	8.00000000
p_7	7.99129025	7.99160757	8.00000000	8.00000000	8.00000000	8.00000000
p_8	7.99562577	7.99580254	8.00000000	8.00000000	8.00000000	8.00000000
p_9	7.99778616	7.99788414	8.00000000	8.00000000	8.00000000	8.00000000
p_{10}	7.99887276	7.99892687	8.00000000	8.00000000	8.0000000000	8.00000000
p_{11}	7.99942324	7.99945303	8.00000000	8.00000000	8.0000000000	8.00000000
p_{12}	7.99970371	7.99972008	8.00000000	8.00000000	8.0000000000	8.00000000
p_{13}	7.99984728	7.99985626	8.00000000	8.00000000	8.0000000000	8.00000000
p_{14}	7.99992106	7.99992597	8.00000000	8.00000000	8.0000000000	8.00000000
p_{15}	8.00000000	8.0000000000	8.00000000	8.00000000	8.0000000000	8.00000000

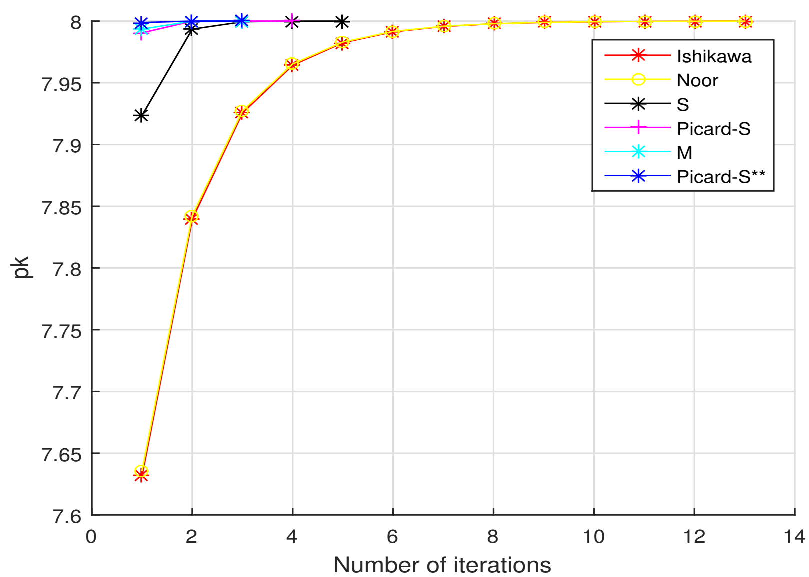

Figure 5. Graph corresponding to Table 3.

Secondly, we take the stopping criterion $E_k = \|p_{k+1} - p_k\| < 10^{-4}$ and consider the following cases of starting points:

Case a: $p_1 = 7$;

Case b: $p_1 = 7.5$;

Case c: $p_1 = 8$;

Case d: $p_1 = 9$.
The numerical results are presented in the following Table 4 and Figure 6.

TABLE 4. Numerical results of Example 7.1

Cases		Picard-S**	M	Picard-S	S	Noor	Ishikawa
Case a	CPU time (sec.)	0.0025	0.0039	0.0045	0.0064	0.0124	0.0126
	No of Iter.	2	4	5	6	13	14
Case b	CPU time (sec.)	0.0030	0.0047	0.056	0.0069	0.2354	0.2657
	No of Iter.	2	4	5	6	3	14
Case c	CPU time (sec.)	0.0034	0.0053	0.0066	0.0082	0.3234	0.3674
	No of Iter.	3	5	6	7	14	15
Case d	CPU time (sec.)	0.0040	0.0059	0.0074	0.0087	0.3467	0.3738
	No of Iter.	3	5	6	7	14	15

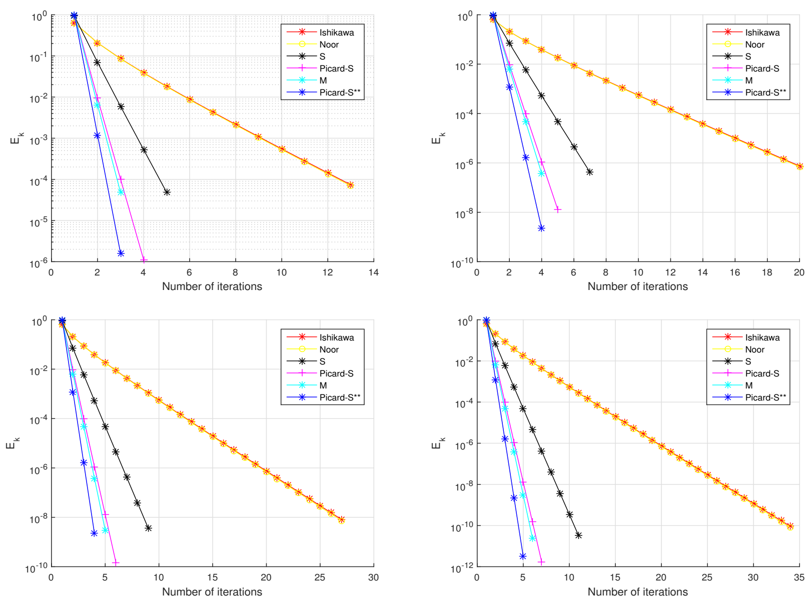

FIGURE 6. The behavior of E_k: Top Left: **Case a**; Top Right: **Case b**; Bottom Left: **Case c**; Bottom Right: **Case d**.

8. CONCLUSION

In this work, we have constructed a new iterative algorithm for fixed point approximation of a generalized nonexpansive-type mappings in the setting of uniformly convex Banach spaces. We proved the convergence results of this method with simple and standard assumptions on the control sequences of the method. We showed numerically that our method is more efficient than many existing methods for solving real world problems. Our results compliment, unify and and improve several well known results in the literature.

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