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CHEBYSHEV-KURCHATOV-TYPE METHODS FOR SOLVING EQUATIONS WITH NON-DIFFERENTIABLE OPERATORS

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Abstract. We introduce a new derivative free family of difference iterative methods for solving nonlinear equations with non-differentiable operators in a Banach space setting. This family is obtained as a combination of Chebyshev's and Kurchatov's method, and is shown to be faster than the corresponding one using the Chebyshev's or Kurchatov's methods. Numerical examples validate our semilocal convergence results.

1. Introduction

In this study we are concerned with problem of approximating a locally unique solution x^* of nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a continuous nonlinear operator defined on a non-empty convex subset D of a Banach space X with values in a Banach space Y. Many problems from computational sciences and other disciplines can be brought in a form similar to equation (1.1) using mathematical modelling [8, 10, 13, 19, 21, 22]. The solution of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. In applied sciences

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the practice of Numerical Analysis for finding solutions x^* of equation (1.1) is essentially connected to variants of Newton's method.

The Secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n) \ \ (n \geq 0) \ \ (x_{-1}, x_0 \in D) \tag{1.2}$$

is undoubtedly the most popular difference, iterative procedure for generating a sequence $\{x_n\}$ converging to x^*. Here, $[x, y; F] \in L(X, Y)$ the space of bounded linear operators from X into Y, denotes a divided difference of order one on the points x and y with $x \neq y$ if the following holds

$$[x, y; F](x - y) = F(x) - F(y). \tag{1.3}$$

The local as well as semilocal convergence of Secant method has been studied by many authors (cf. [1]-[28] and the references therein). The convergence order of the Secant method is $1.618\cdots$.

A less studied (linear interpolation based) difference method suggested by Kurchatov [20] is given by

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1} F(x_n), \\ A_n &= [2x_n - x_{n-1}, x_{n-1}; F], \ \ (n \geq 0), \ \ (x_{-1}, x_0 \in D). \end{aligned} \tag{1.4}$$

As in the case of the Secant method (1.3), Kurchatov's method (1.4) uses two previous iterations, but its convergence is quadratic. However, the semilocal convergence of Kurchatov's method shown in [20] requires the very restrictive boundedness condition on the norm of F'''.

Later the local as well as semilocal convergence of Kurchatov's method was given in [5], [6],[20],[27], where different techniques were used and divided difference of order only one. Under such conditions Potra [24] studied a method using three previous iterations, but the convergence is lower and equals to $1.839\cdots$.

The basic assumption in all studies on Kurchatov's method (1.4) is that the the divided difference of first order for the operator F is Lipschitz or Hölder continuous on some ball around the starting point x_0. But this assumption forces the operator F to be differentiable ([4], [25]).

In this paper, we have two main aims. Firstly, we consider a multiparametric family of iterative methods that do not use derivatives and generalize method (1.2). Secondly, we obtain a semilocal convergence result for non-differentiable operators. To do this, we change the conditions normally imposed on divided differences. Thus, we relax the requirements that the first order divided difference operator F is Lipschitz or Hölder continuous, and just assume the following condition:

$$\|[x, y; F] - [v, w; F]\| \leq \omega(\|x - v\|, \|y - w\|); \quad x, y, v, w \in D,$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function in its two arguments. It is clear that this condition generalizes the conditions previously indicated. In fact, when $\omega(u_1, u_2) = k(u_1 + u_2)$, we obtain the Lipschitz continuous case and, when $\omega(u_1, u_2) = k(u_1^p + u_2^p)$, we obtain the the (k, p)-Hölder continuous case. Moreover, in general, this condition does not involve the differentiability of the operator F.

The family of iterative methods that we consider in this paper arises from the Chebyshev method, a well known third-order convergence method that has been studied, for instance, in [11] or [1]. Firstly, Hernández [16] and later Ezquerro and Hernández [14]-[17], modify this method by avoiding the computation of the second derivative of F and reducing the number of evaluations of the first derivative of F. Actually, these authors have obtained a modification of the Chebyshev method with order of convergence at least three, which only need to evaluate the first derivative of F. This family of iterative methods is written as follows [14]:

$$\begin{cases} x_0 \in D, \\ y_k = x_k - F'(x_k)^{-1}\ F(x_k), \\ z_k = x_k + p\ (y_k - x_k),\ \ p \in (0, 1], \\ x_{k+1} = x_k - \dfrac{1}{p^2}\ F'(x_k)^{-1}\ ((p^2 + p - 1)\ F(x_k) + F(z_k)), \quad k \geq 0. \end{cases}$$

So, they obtain a uniparametric family of iterative methods which depends only on the first derivative of the operator F, which is evaluated only at one point. Then, to construct a family of iterative methods free of derivatives, as the classical Secant method, we consider an approximation of the first derivative of F from a divided difference of first order; that is, $F'(x_n) \simeq [x_{n-1}, x_n, F]$, where, $[x, y; F]$ is a divided difference of first order for the operator F at the points x, $y \in D$. So, we introduce a family of iterative methods that does not use derivatives. We recall these iterative methods as Chebyshev-Secant-type methods (CSTM) and they are written as follows:

$$\begin{cases} x_{-1}, x_0 \in D, \\ y_n = x_n - \overline{A_n}^{-1} F(x_n), \quad \overline{A_n} = [x_{n-1}, x_n; F], \\ z_n = x_n + a\ (y_n - x_n), \\ x_{n+1} = x_n - \overline{A_n}^{-1}\ (b\ F(x_n) + c\ F(z_n)), \quad n \geq 0, \end{cases} \tag{1.5}$$

where a, b, c are non-negative parameters to be chosen so that the sequence $\{x_n\}$ converges to x^*. Note that (1.7) reduces to (1.4) if $b = 1$ and $a = c = 0$. Even more, if $ac = 0$ and $b > 0$, the Chebyshev-Secant-type methods defined in (1.7) becomes the following Secant-type method:

$$x_{n+1} = x_n - b[x_{n-1}, x_n; F]^{-1} F(x_n); x_0,\ x_{-1}. \tag{1.6}$$

This kind of methods have been studied in [17], [18]. However, if $ac \neq 0$, the methods defined in (1.7) are essentially different from the Secant method (1.2) or Secant-type methods (1.6).

In the present paper we introduce the Chebyshev-Kurchatov-type method (CKTM) as follows.

$$\begin{cases} x_{-1}, x_0 \in D, \\ y_n = x_n - A_n^{-1} F(x_n), \quad A_n = [2x_n - x_{n-1}, x_{n-1}; F], \\ z_n = x_n + a\ (y_n - x_n), \\ x_{n+1} = x_n - A_n^{-1}\ (b\ F(x_n) + c\ F(z_n)), \quad n \geq 0. \end{cases} \tag{1.7}$$

Next, in Section 2, we provide a semilocal convergence result for the family of iterative methods (1.7) when they are applied to non-differentiable operators. Finally, in Section 3, we consider examples: where we compare (1.4) to (1.2).

Throughout the paper we denote $\overline{B(x,r)} = \{y \in X : \|y - x\| \leq r\}$ and $B(x,r) = \{y \in X : \|y - x\| < r\}$. In particular, we show that Kurchatov's method (CKTM) is faster than (CSTM).

2. Semilocal Convergence Analysis

We analyse the semilocal convergence of the new family of iterative methods given by (1.7). To do this, we use a technique based on proving a system of recurrence relations. Firstly, we suppose that there exists a first-order divided difference $[x, y; F] \in L(X, Y)$, for all $x, y \in D$. Let us suppose that

(C1) $(1 - b) = (1 - a)c$ and $a \in [0, 1]$,

(C2) $x_{-1}, x_0 \in D$ are such that $\|x_0 - x_{-1}\| \leq \alpha$,

(C3) the linear operator A_0 is invertible, $\|A_0^{-1}\| \leq \beta$ and $\|A_0^{-1} F(x_0)\| \leq \eta$,

(C4) $\|[x, y; F] - [u, v; F]\| \leq \omega(\|x - u\|, \|y - v\|)$, $x, y, u, v \in D$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous non-decreasing function in both arguments,

(C5) we denote $m = \max\{p, ac\beta\omega(2(1+p)\eta+\alpha, \alpha), ac\beta\omega(\alpha, a\eta+\alpha)\}$, where $p = ac\omega(\alpha, a\eta + \alpha)$. Suppose equation

$$(1 + \varphi(t))\eta = t\left(1 - \varphi(t)\left(1 + \frac{1}{ac}(1 + \varphi(t))\right)\right), \tag{2.1}$$

where $\varphi(t) = \frac{m}{1-\beta\omega(3t+\alpha,t+\alpha)}$, has at least one positive root; we denote the smallest positive root of this equation by R and $b\omega(3R+\alpha, R+\alpha) < 1$.

(C6) $(1+M)M\left(1+\frac{1}{ac}(1+M)\right)<1$, where $M=\frac{m}{1-\beta\omega(3R+\alpha,R+\alpha)}$,
(C7) $U(x_0,R)\subseteq D$.
(C8) $x,y\in D\Rightarrow 2y-x\in D$.
As noted in [5], [6], [8], [10] condition (C8) is automatically satisfied, if $X=D$. Otherwise, replace (C7) and (C8) by
(C7)' $U(x_0,3R)\subseteq D$.

Note that under (C7)', $\|2y-x-x_0\|\le 2\|y-x_0\|+\|x-x_0\|\le 2R+R=3R$ for all $x,y\in U(x_0,R)$.

Next, we present three auxiliary results for later usage.

Lemma 2.1. *If the method* (1.7) *is well defined, then the following assertions hold*

$$\begin{aligned} F(z_n) &= (1-a)F(x_n)+a(B_n-A_n)(y_n-z_n), \quad n\ge 0,\\ F(x_{n+1}) &= (A_{n+1}-A_n)(x_{n+1}-x_n)+ac(A_n-B_n)(y_n-x_n) \quad n\ge 0, \end{aligned}$$

where $A_n=[x_{n-1},x_n;F]$, $B_n=[x_n,z_n;F]$ *and* $(1-b)=(1-a)c$.

Lemma 2.2. *Suppose that* (C1)–(C7) *hold. Then, the following assertion holds*

$$\begin{aligned} \sum_{i=0}^{n} M^i\left(1+\frac{1}{ac}(1+M)\right)^i\eta &= (1+M)\,\frac{1-\left(M\left(1+\frac{1}{ac}(1+M)\right)\right)^{n+1}}{1-M\left(1+\frac{1}{ac}(1+M)\right)}\eta\\ &< \frac{(1+M)\,\eta}{1-M\left(1+\frac{1}{ac}(1+M)\right)}\\ &= R. \end{aligned}$$

Lemma 2.3. *Suppose that* (C1)–(C7) *hold. Then, for* $n\ge 1$, *we have the following recurrence relations:*

[I] *There exists* A_n^{-1} *and* $\|A_n^{-1}\|\le\frac{\beta}{1-\beta\omega(\alpha+R,R)}$,
[II] $\|y_n-x_n\|\le M\left(1+\frac{1}{ac}(1+M)\right)\|y_{n-1}-x_{n-1}\|$
$\le M^n\left(1+\frac{1}{ac}(1+M)\right)^n\|y_0-x_0\|$,
[III] $\|y_n-x_0\|\le(1+M)\sum_{i=0}^{n}M^i\left(1+\frac{1}{ac}(1+M)\right)^i\|y_0-x_0\|<R$,
[IV] $\|z_n-x_0\|\le(1+M)\sum_{i=0}^{n}M^i\left(1+\frac{1}{ac}(1+M)\right)^i\|y_0-x_0\|<R$,
[V] $\|x_{n+1}-x_n\|\le(1+M)\|y_n-x_n\|$
$\le(1+M)M^n\left(1+\frac{1}{ac}(1+M)\right)^n\|y_0-x_0\|$,
[VI] $\|x_{n+1}-x_0\|\le(1+M)\sum_{i=0}^{n}M^i\left(1+\frac{1}{ac}(1+M)\right)^i\|y_0-x_0\|<R$.

Proof. We have that $\|y_0 - x_0\| \leq \eta$, $\|z_0 - x_0\| \leq \eta$ and

$$\begin{aligned}\|x_1 - x_0\| &\leq \left(1 + ac\|A_0^{-1}\|\|A_0 - B_0\|\right)\|y_0 - x_0\| \\ &\leq (1+p)\|y_0 - x_0\| \leq (1+M)\eta \\ &< R.\end{aligned}$$

Next, we can prove that **[I]**–**[VI]** are true for $n = 1$ suppose that **[I]**–**[VI]** are true for $k = 1, 2, \ldots, n-1$. Then, we have in turn

[I]: Since $\|I - A_0^{-1}A_n\| \leq \beta\omega(\|x_{n-1} - x_{-1}\|, \|x_n - x_0\|) \leq \beta\omega(\alpha + R, R) < 1$, then, by Banach's lemma, it follows

$$\begin{aligned}\|A_n^{-1}\| &\leq \frac{1}{1 - b\omega(2\|x_n - x_0\| + \|x_{n-1} - x_{-1}\|, \|x_{n-1} - x_{-1}\|)} \\ &\leq \frac{\beta}{1 - \beta\omega(3R + \alpha, R + \alpha)}.\end{aligned} \tag{2.2}$$

[II]:

$$\begin{aligned}&\|y_n - x_n\| \\ &\leq \|A_n^{-1}\|\|F(x_n)\| \\ &\leq \|A_n^{-1}\|\omega(\|x_{n-1} - x_{n-2}\| + 2\|x_n - x_{n-1}\|, \|x_{n-1} - x_{n-2}\|)\|x_n - x_{n-1}\| \\ &\quad + ac\,\omega(\|x_{n-1} - x_{n-2}\|, \|z_{n-1} - x_{n-2}\|)\|y_{n-1} - x_{n-1}\| \\ &\leq M\left(1 + \frac{1}{ac}(1+M)\right)\|y_{n-1} - x_{n-1}\| \\ &\leq M^n\left(1 + \frac{1}{ac}(1+M)\right)^n\|y_0 - x_0\|,\end{aligned}$$

[III]:

$$\begin{aligned}\|y_n - x_0\| &\leq \|z_n - x_n\| + \|x_n - x_0\| \\ &\leq (1+M)\sum_{i=0}^{n} M^i\left(1 + \frac{1}{ac}(1+M)\right)^i\|y_0 - x_0\| \\ &< R,\end{aligned}$$

[IV]:

$$\begin{aligned}\|z_n - x_0\| &\leq \|z_n - x_n\| + \|x_n - x_0\| \\ &\leq (1+M)\sum_{i=0}^{n} M^i\left(1 + \frac{1}{ac}(1+M)\right)^i\|y_0 - x_0\| \\ &< R,\end{aligned}$$

[V]:

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \left(1+ac\|A_n^{-1}\|\|A_n-B_n\|\right)\|y_n-x_n\|\\ &\le \left(1+\frac{m}{1-\beta\omega(3R+\alpha,R+\alpha)}\right)\|y_n-x_n\|\\ &\le (1+M)\|y_n-x_n\|\\ &\le (1+M)M^n\left(1+\frac{1}{ac}(1+M)\right)^n\|y_0-x_0\|,\end{aligned}$$

[VI]:

$$\begin{aligned}\|x_{n+1}-x_0\| &\le \|x_{n+1}-x_n\|+\|x_n-x_0\|\\ &\le (1+M)\sum_{i=0}^{n}M^i\left(1+\frac{1}{ac}(1+M)\right)^i\|y_0-x_0\|\\ &< R.\end{aligned}$$

The proof of the Lemma is complete. □

Next, we present the main semilocal convergence theorem for iterative method (1.7) when they are applied to nondifferentiable operators that satisfy conditions (C1)–(C8).

Theorem 2.4. *Let X and Y be two Banach spaces and let $F : D \subseteq X \to Y$ be defined on on a non-empty open convex domain D. Suppose that there exists $[x, y; F] \in \mathcal{L}(X, Y)$, for all $x, y \in D$, and conditions* (C1)–(C8) *are satisfied. Then sequence* (1.7)*, starting from x_{-1} and x_0 is well defined, remains in $\overline{U}(x_0, R)$ for all $n \ge 0$ and converges to a unique solution $x^* \in \overline{U}(x_0, R)$ of $F(x) = 0$.*

Proof. We have that

$$\begin{aligned}&\|x_{n+k}-x_n\|\\ &\le \|x_{n+k}-x_{n+k-1}\|+\|x_{n+k-1}-x_{n+k-2}\|+\cdots+\|x_{n+1}-x_n\|\\ &\le (1+M)\left(\|y_{n+k-1}-x_{n+k-1}\|+\|y_{n+k-2}-x_{n+k-2}\|+\cdots+\|y_n-x_n\|\right.\\ &\le (1+M)\sum_{i=n}^{n+k-1}\left(M^i\left(1+\frac{1}{ac}(1+M)\right)^i\right)\|y_0-x_0\|\\ &< (1+M)M^n\left(1+\frac{1}{ac}(1+M)\right)^n\frac{1-\left(M\left(1+\frac{1}{ac}(1+M)\right)\right)^k}{1-M\left(1+\frac{1}{ac}(1+M)\right)}\,\eta.\end{aligned}$$

It follows from Lemma 2.3, that (1.7) is a complete sequence, and as such it converges to some $x^* \in \overline{U}(x_0, R)$ (since $\overline{U}(x_0, R)$ is a closed set).

Hence, for (II) there exists $\gamma > 0$ such that

$$\|F(x_n)\| \leq \gamma \|y_{n-1} - x_{n-1}\|$$

and $\|y_{n-1} - x_{n-1}\| \to 0$ as $n \to \infty$. That is we deduce $F(x^*) = 0$. To prove the uniqueness of the solution x^*, suppose that y^* is a solution of $F(x) = 0$ in $U(x_0, R)$. Define the operator $A = [x^*, y^*; F]$. Then, we have in turn that

$$\begin{aligned}\|A_0^{-1}A - I\| &\leq \|A_0^{-1}\|\|A - A_0\| \leq \|A_0^{-1}\|\|[y^*, x^*; F] - [x_{-1}, x_0; F]\| \\ &\leq \beta\omega(\|y^* - x_{-1}\|, \|x^* - x_0\|) \leq \beta\omega(\alpha + R, R) \\ &< 1.\end{aligned}$$

Hence, operator A^{-1} exists. It then follows from $A(y^* - x^*) = F(y^*) - F(x^*)$ that $x^* = y^*$. The proof of the Theorem is complete. □

Remark 2.5. Using the same information as in Theorem 2.4, we can provide at least as tight upper bounds on the distances involved, and at least as precise information on the location of the solution x^*. Indeed, let us assume

(C9) $\|[2x_0 - x_{-1}, x_{-1}; F] - [x, y; F]\| \leq \omega_0(\|2x_0 - x_{-1} - x\|, \|x_{-1} - y\|)$, $x, y \in D$, where $\omega_0 : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and non-decreasing function in both arguments,

(C10) there exists $R_1 \geq 3R$ such that

$$\beta\,\omega_0(R_1 + \alpha, R + \alpha) < 1,$$

and

(C11) $\overline{U}(x_0, 3R_1) \subseteq D$.

Condition (C9) always follows from (C4) (simply, set $\omega = \omega_0$). Hence, (C9) is not an additional (to (C4)) hypothesis. Note that

$$\omega_0 \leq \omega \tag{2.3}$$

holds in general, and $\dfrac{\omega}{\omega_0}$ can be arbitrarily large [7]–[9]. It follows from (2.3), and the proof of **[I]** in Lemma 2.3 that

$$\|A_n^{-1}\| \leq \frac{1}{1 - \beta\,\omega_0(\|x_{n-1} - x_{-1}\|, \|x_n - x_0\|)} \tag{2.4}$$

which is an at least as tight estimate as (2.2). In particular, if strict inequality holds in (2.3), then (2.4) can replace (I), which leads to tighter error bounds on the distances involved. Conditions (C9)–(C10) extend the uniqueness ball for the solution x^*. Indeed, assuming $y^* \in \overline{U}(x_0, R_1)$, the uniqueness proof of Theorem 2.4 now gives

$$\|A_0^{-1}\,A - I\| \leq \beta\,\omega_0(\alpha + R_1, R + \alpha) < 1.$$

We finish this section with a semilocal convergence theorem for the methods defined in 1.7 in the case $ac = 0$ or, equivalently, for Secant-type methods 1.6 with $b > 0$. We omit the proof because it follows directly from the proof of Theorem 2.1 in [17].

Theorem 2.6. *Let X and Y be two Banach spaces and let $F : D \subseteq X \to Y$ be a nonlinear operator defined on on a non-empty open convex domain D such that there exists $[x, y; F] \in \mathcal{L}(X, Y)$, for all $x, y \in D$. Let us suppose that*

(c1) *The linear operator A_0 is invertible, $\|A_0^{-1}\| \leq \beta$ and $\|A_0^{-1}F(x_0)\| \leq \eta$,*
(c2) *$x_{-1}, x_0 \in D$ are such that $\|x_0 - x_{-1}\| \leq \alpha$,*
(c3) *$\|[x, y; F] - [u, v; F]\| \leq \omega(\|x - u\|, \|y - v\|)$, $x, y, u, v \in D$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous non-decreasing function in both arguments,*
(c4) *we denote $m = \max\{\beta\omega(\alpha, \eta), \beta\omega(\eta, \eta)\}$, and suppose that the equation*

$$x\left(1 - \frac{bm}{1 - \beta\omega(x + \alpha, x)}\right) - \eta = 0,$$

has at least one positive root; we denote the smallest positive root of this equation by R,
(c5) *$\beta\omega(R + \alpha, R) < 1$ and* $\dfrac{bm}{1 - \beta\omega(\alpha + R, R)} < 1$,
(c6) *$U(x_0, R) \subseteq D$.*

Then sequence (1.6), starting from x_{-1} and x_0, converges to a unique solution x^* of $F(x) = 0$. Moreover, the solution x^* and the iterates x_n belong to $\overline{U(x_0, R)}$.

3. Applications

We provide two examples in this concluding Section. In the first example, we solve a system of two equations with two unknowns including a non-differentiable part. We suggest applications for solving nonlinear integral equations, where operator F is not necessarily differentiable in the concluding example of this Section.

Example 3.1. In this example we apply the above result to the following nonlinear system:

$$\begin{aligned} x_1^2 - x_2 + 1 + \tfrac{1}{9}|x_1 - 1| &= 0, \\ x_1 + x_2^2 - 7 + \tfrac{1}{9}|x_2| &= 0. \end{aligned} \tag{3.1}$$

Observe that system (3.1) is equivalent to $F(x) = 0$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F = (F_1, F_2)$, $x = (x_1, x_2)$, $F_1(x_1, x_2) = x_1^2 - x_2 + 1 + \frac{1}{9}|x_1 - 1|$ and $F_2(x_1, x_2) = x_1 + x_2^2 - 7 + \frac{1}{9}|x_2|$.

Moreover, for $u, v \in \mathbb{R}^2$, $[u, v; F] \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ and

$$[u, v; F]_{i1} = \frac{F_i(u_1, v_2) - F_i(v_1, v_2)}{u_1 - v_1},$$
$$[u, v; F]_{i2} = \frac{F_i(u_1, u_2) - F_i(u_1, v_2)}{u_2 - v_2}, \quad i = 1, 2,$$

so that

$$[u, v; F] = \begin{pmatrix} \frac{u_1^2 - v_1^2}{u_1 - v_1} & -1 \\ 1 & \frac{u_2^2 - v_2^2}{u_2 - v_2} \end{pmatrix} + \frac{1}{9} \begin{pmatrix} \frac{|u_1-1| - |v_1-1|}{u_1 - v_1} & -1 \\ 1 & \frac{|u_2| - |v_2|}{u_2 - v_2} \end{pmatrix}.$$

If we take the Chebyshev norm as vector norm and the matrix norm subordinated to this vector norm, we obtain

$$\|[x, y; F] - [u, v; F]\| \leq \|x - u\| + \|y - v\| + \frac{2}{9},$$

and consequently, from (C4), it follows $\omega(s, t) = s + t + 2/9$.

Now, we apply an iterative method of (1.7) for approximating a solution of (3.1). For example, we choose $a = b = c/2 = 1/2$. We start the corresponding iterative method with $x_{-1} = (0, 0)$ and $x_0 = (1, 1)$. The solution is

$$x^* = (1.159360850, 2.361824342).$$

The next two tables shows the result of 25^{th} to 35^{th} iterations of method (1.5) and (1.7) respectively.

Table 1. 25^{th} to 35^{th} Iterations of (CSTM) of Example 3.1.

n	x_n
25	(1.159360859, 2.361824364)
26	(1000991.18, -21674242.7900000000)
27	(1667673.863, -32560048.6399999999)
28	(-680216.541999999999, 16150460.95)
29	(-437579.255400000000, 13466090.79)
30	(-3684474.1379999999, 47500115.97)
31	(-811460.2386000000000, 14278327.17)
32	(-1224537.989000000000, 17372467.23)
33	(1435080.934, -31510306.4799999999)
34	(635016.9977, -21331918.4300000000)
35	(1870663.883, -43763005.8399999999)

TABLE 2. 25^{th} to 35^{th} Iterations of (CKTM) of Example 3.1.

n	x_n
25	(1.159360855, 2.361824353)
26	(1.159360853, 2.361824348)
27	(1.159360851, 2.361824345)
28	(1.159360851, 2.361824344)
29	(1.159360851, 2.361824343)
30	(1.15936085, 2.361824342)
31	(1.15936085, 2.361824342)
32	(1.15936085, 2.361824342)
33	(1.15936085, 2.361824342)
34	(1.15936085, 2.361824342)
35	(1.15936085, 2.361824342)

Example 3.2. Let $\mathcal{G}(x,t,x(t))$ be a continuous function of its arguments. Let operator F be given by

$$F(x(s)) = x(s) - \int_0^1 \mathcal{G}(x,t,x(t))\,dt.$$

Define divided difference A_n by

$$A_n(s,t) = \frac{\mathcal{G}(s,t,x_n(t)) - \mathcal{G}(s,t,x_{n-1}(t))}{x_n(t) - x_{n-1}(t)}.$$

We refer the reader to [8], [9], [17] for special choices of function $\mathcal{G}$, so that condition (C4) is satisfied.

References

[1] S. Amat, C. Bermúdez, S. Busquier and D. Mestiri, *A family of Halley-Chebyshev iterative schemes for non-Fréchet differentiable operators*, J. Comput. Appl. Math., **228** (2009), 486–493.

[2] I.K. Argyros, *The Secant method and fixed points of nonlinear operators*, Mh. Math., **106** (1988), 85–94.

[3] I.K. Argyros, *On an application of a variant of the closed graph theorem and the Secant method*, Tamkang J. Math., **24** (1993), 251–267.

[4] I.K. Argyros, *On the Secant method*, Publ. Math. Debrecen, **43(3-4)** (1993), 223–238.

[5] I.K. Argyros, *On a two-point Newton-like method of convergent order two*, Int. J. Comput. Math., **88(2)** (2005), 219–234.

[6] I.K. Argyros, *A Kantorovich-type analysis for a fast iterative method for solving nonlinear equations*, J. Math. Anal. Appl., **332** (2007), 97–108.

[7] I.K. Argyros, *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), 374–397.

[8] I.K. Argyros, Convergence and applications of Newton–type iterations, Springer–Verlag, 2008, New York.

[9] I.K. Argyros and S. Hilout, Efficient methods for solving equations and variational inequalities, Polimetrica Publisher, 2009.

[10] I.K. Argyros, S. Hilout and M.A. Tabatabai, Mathematical Modelling with Applications in Biosciences and Engineering, Nova Publishers, New York, 2011.

[11] I.K. Argyros, J.A. Ezquerro, J.M. Gutiérrez, M.A. Hernndez amd S. Hilout, *Chebyshev-Secant-type Methods for Non-differentiable Operators*, Milan J. Math.,(To appear).

[12] V. Candela and A. Marquina, *Recurrence Relations for Rational Cubic Methods II: The Chebyshev Method*, Computing, **45** (1990), 355–367.

[13] P. Deuflhard, *Newton methods for nonlinear problems, Affine invariance and adaptive algorithms*, Springer Series in Computational Mathematics, **35**, Springer-Verlag, Berlin, 2004.

[14] J.A. Ezquerro and M.A. Hernández, *An optimization of Chebyshev's method*, J. Complexity, **25** (2009), 343–361.

[15] M. Grau and M. Noguera, *A variant of Cauchy's method with accelerated fifth-order convergence*, Appl. Math. Lett., **17** (2004), 509–517.

[16] M.A. Hernández, *Chebyshev's approximation algorithms and applications*, Comput. Math. Appl., **41** (2001), 433–445.

[17] M.A. Hernández and M.J. Rubio, *The Secant method for nondifferentiable operators*, Appl. Math. Lett., **15** (2002), 395–399.

[18] M.A. Hernández and M.J. Rubio, *A uniparametric family of iterative processes for solving nondifferentiable equations*, J. Math. Anal. Appl., **275** (2002), 821–834.

[19] L.v. Kantorovich and G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.

[20] V.A. Kurchatov, *On a method of linear interpolation for the solution of functional equations*, Soviet Math. Dokl., **12(3)** (1971), 835–838.

[21] L.M. Ortega and W.C. Rheinboldt, Iterative solution of nonlinear equations in several variables, Academic press, New York, 1970.

[22] A.M. Ostrowski, Solution of equations in Euclidean and Banach spaces, Academic press, New York, 1973.

[23] F.A. Potra, *An aplication of the induction method of V. Pták to the study of Regula Falsi*, Aplikace Matematiky, **26** (1981), 111–120.

[24] F.A. Potra, *On an iterative algorithm of order 1.839$\cdots$. for solving nonlinear operator equations*, Numer. Funct. Anal. Optimiz., **7(1)** (1984-85), 75–106.

[25] F.A. Potra and V. Ptak, Nondiscrete induction and iterative methods, Pitman Publishing Limited, London, 1984.

[26] J.W. Schmidt, *Regula-Falsi verfahren mit konsistenter steigung und majoranten prinzip*, Periodica Mathematica Hungarica, **5** (1974), 187–193.

[27] S.M. Shakhno, *About the difference method with quadratic convergence for solving nonlinear operator equations*, PAMM, **4(1)** (2004), 650–651.

[28] J.F. Traub, Iterative methods for solution of equations, Prentic Hall, Englewood Cliffs, New Jersey, 1964.