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SOME FIXED POINT THEOREMS IN A GENERALIZED b_2-METRIC SPACE OF (ψ, φ)-WEAKLY CONTRACTIVE MAPPINGS

Pravin Singh[1], Shivani Singh[2] and Virath Singh[3]

[1]University of KwaZulu-Natal, Private Bag X54001, Durban, 4001, South Africa
e-mail: singhp@ukzn.ac.za

[2]University of South Africa, Department of Decision Sciences, PO Box 392, 0003, Pretoria
e-mail: singhs2@unisa.ac.za

[3]University of KwaZulu-Natal, Private Bag X54001, Durban, 4001, South Africa
e-mail: singhv@ukzn.ac.za

Abstract. The purpose of this paper is to introduce a class of distance altering functions that establish the existence and uniqueness of fixed points of ν-admissible mappings that are subject to a generalized (ψ, φ)-almost weakly contraction on a generalized b_2-metric space.

1. Introduction

The concept of a 2-metric was introduced by Gähler ([4]), as a generalization of the metric by using the concept of an area of a triangle in $\mathbb{R}^2$ as a basis for the formulation. The 2-metric spaces are not topologically equivalent to the metric spaces and so there is no easy relationship between results of these spaces ([2, 3]). In a recent paper, the authors Singh et al. introduce the concept of a generalized 2-metric ([1, 9, 10]).

Definition 1.1. ([4]) Let X be a nonempty set and $d : X \times X \times X \to [0, \infty)$ be a map satisfying the following properties:

(i) For $x, y, z \in X$ such that $d(x, y, z) = 0$ if at least two of the three points are the same.

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[0]Corresponding author: V. Singh(`singhv@ukzn.ac.za`).

(ii) For $x, y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(iii) symmetry property: for $x, y, z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) rectangle inequality:

$$d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t)$$

for $x, y, z, t \in X$.

Then d is a 2-metric and (X, d) is a 2-metric space.

Definition 1.2. ([6]) Let X be a nonempty set and $d : X \times X \times X \to [0, \infty)$ be a map satisfying the following properties:

(i) For $x, y, z \in X$ such that $d(x, y, z) = 0$ if at least two of the three points are the same.

(ii) For $x, y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(iii) symmetry property: for $x, y, z \in X$,

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$

(iv) s-rectangle inequality:there exists $s \geq 1$ such that

$$d(x,y,z) \leq s[d(x,y,t) + d(y,z,t) + d(z,x,t)]$$

for $x, y, z, t \in X$.

Then d is a b_2-metric and (X, d) is a b_2-metric space.

If $s = 1$, the b_2-metric reduces to the 2-metric.

Example 1.3. ([6]) Let $X = [0, \infty)$ and define $d(x, y, z) = [xy + yz + zx]^p$ where $p \geq 1$. it suffices to only verify property (iv) of Definition 1.2. For $x, y, z, t \in X$, we get by using the Jensen's inequality,

$$\begin{aligned} d(x,y,z) &= [xy + yz + zx]^p \\ &= 3^p \left(\frac{1}{3}xy + \frac{1}{3}yz + \frac{1}{3}zx\right)^p \\ &\leq 3^p \left(\tfrac{1}{3}[xy]^p + \tfrac{1}{3}[yz]^p + \tfrac{1}{3}[zx]^p\right) \\ &\leq 3^p \left(\tfrac{1}{3}[xy + yt + xt]^p + \tfrac{1}{3}[yz + zt + yt]^p + \tfrac{1}{3}[zx + xt + zt]^p\right) \\ &= 3^{p-1}[d(x,y,t) + d(y,z,t) + d(z,x,t)] \end{aligned}$$

It follows that d is a b_2-metric with $s \leq 3^{p-1}$.

2. Main result

Definition 2.1. Let X be a nonempty set and $d : X \times X \times X \to [0, \infty)$ be a map satisfying the following properties:

(i) If $x, y, z \in X$ such that $d(x, y, z) = 0$ if at least two of the three points are the same.

(ii) For $x, y \in X$ such that $x \neq y$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

(iii) symmetry property: for $x, y, z \in X$,

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x).$$

(iv) modified rectangle inequality: there exists $\alpha, \beta, \gamma \geq 1$ such that

$$d(x, y, z) \leq \alpha d(x, y, t) + \beta d(y, z, t) + \gamma d(z, x, t)]$$

for $x, y, z, t \in X$.

Then d is a generalized b_2-metric and (X, d) is a generalized b_2- metric space.

If $\alpha = \beta = \gamma$ then a generalized b_2-metric is a b_2-metric. Using the symmetry property, it can be shown that if d is a generalized b_2-metric then

$$d(x, y, z) \leq \left(\frac{\alpha+\beta+\gamma}{3}\right) [d(x, y, t) + d(y, z, t) + d(z, x, t)] .$$

It follows that d is a b_2-metric with $s = \frac{\alpha+\beta+\gamma}{3}$.

Example 2.2. Let $X = (0, 1)$ and define

$$d(x, y, z) = \begin{cases} 0, \text{if at least two of the three points are the same,} \\ \qquad e^{|x-y|+|y-z|+|z-x|}, \text{ otherwise.} \end{cases}$$

For $x, y, z \in X$ and using Jensen's inequality, we get

$$\begin{aligned} d(x, y, z) &= e^{|x-y|+|y-z|+|z-x|} \\ &= e^{\frac{1}{2}|x-y|+\frac{1}{3}|y-z|+\frac{1}{6}|z-x|} e^{\frac{1}{2}|x-y|+\frac{2}{3}|y-z|+\frac{5}{6}|z-x|} \\ &\leq e^2 e^{\frac{1}{2}|x-y|+\frac{1}{3}|y-z|+\frac{1}{6}|z-x|} \\ &\leq e^2 \left\{\frac{1}{2} e^{|x-y|} + \frac{1}{3} e^{|y-z|} + \frac{1}{6} e^{|z-x|}\right\} \\ &\leq e^2 \left\{\frac{1}{2} e^{|x-y|+|y-t|+|t-x|} + \frac{1}{3} e^{|z-y|+|y-t|+|t-z|} + \frac{1}{6} e^{|z-x|+|x-t|+|t-z|}\right\} \\ &= \alpha d(x, y, t) + \beta d(z, y, t) + \gamma d(z, x, t), \end{aligned}$$

where $\alpha = \frac{1}{2} e^2 \geq 1$, $\beta = \frac{1}{3} e^2 \geq 1$ and $\gamma = \frac{1}{6} e^2 \geq 1$. It follows that d is a generalized b_2-metric but not a b_2-metric.

Example 2.3. Let $X = [0, \infty)$ and define a mapping $d : X \times X \times X \to [1, \infty)$ by

$$d(x,y,z) = \begin{cases} 0, & \text{if at least two of the three points are the same,} \\ \left| |x-y|^{\xi} + |y-z|^{\xi} + |z-x|^{\xi} \right|^{\eta}, & \text{otherwise.} \end{cases}$$

for $x, y, z \in X$ and real number $\xi, \eta > 1$.

Properties (i)-(iii) of Definition 1.2 can be easily verified. We shall show property (iv) of Definition 1.2. For x, y, z, using Jensen's inequality, we get

$$\begin{aligned} d(x,y,z) &= \left| |x-y|^{\xi} + |y-z|^{\xi} + |z-x|^{\xi} \right|^{\eta} \\ &= 3^{\eta} \left| \frac{1}{3} |x-y|^{\xi} + \frac{1}{3} |y-z|^{\xi} + \frac{1}{3} |z-x|^{\xi} \right|^{\eta} \\ &\leq 3^{\eta-1} \left[|x-y|^{\xi\eta} + |y-z|^{\xi\eta} + |z-x|^{\xi\eta} \right] \\ &\leq 3^{\eta-1} \Big[\left| |x-y|^{\xi} + |y-t|^{\xi} + |t-x|^{\xi} \right|^{\eta} \\ &\quad + \left| |y-z|^{\xi} + |z-t|^{\xi} + |t-y|^{\xi} \right|^{\eta} \\ &\quad + \left| |z-x|^{\xi} + |x-t|^{\xi} + |t-z|^{\xi} \right|^{\eta} \Big] \\ &= 3^{\eta-1} \left[d(x,y,t) + d(y,z,t) + d(z,x,t) \right]. \end{aligned}$$

It follows that d is a b_2-metric but a special generalized b_2-metric with $\alpha = \beta = \gamma = 3^{\eta-1}$.

Definition 2.4. Let (X, d) be a generalized b_2-metric space. Let $x, y \in X$ and $\varepsilon > 0$. Then the subset

$$B_{\varepsilon}(x,y) = \{z \in X; d(x,y,z) < \varepsilon\}$$

of X is called a generalized b_2-ball centered at x, y with radius ε. A topology can be generated on X by taking the collection of all generalized b_2-balls as a subbasis, which we call the generalized b_2-metric topology and is denoted by τ. Thus (X, τ) is a generalized b_2-metric topological space. Members of τ are called b_2-open sets. From the property of the metric, it can easily be seen that $B_{\varepsilon}(x,y) = B_{\varepsilon}(y,x)$ for $\varepsilon > 0$.

Definition 2.5. ([6]) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a generalized b_2-metric space (X, d).

(1) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x \in X$, if for all $\xi \in X$,

$$\lim_{n\to\infty} d(x_n, x, \xi) = 0.$$

(2) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in X, if for all $\xi \in X$,

$$\lim_{n,m\to\infty} d(x_n, x_m, \xi) = 0.$$

In this paper we have amended the space of altering distance functions found in [5], to establish existence and uniqueness of fixed points for ν-admissible mappings subject to a generalized almost weakly (ψ, φ)-contraction type.

Definition 2.6. Let $\mathfrak{F}$ denote the class of all functions $\psi : [0, \infty) \to [0, \frac{1}{\beta})$, where $\beta > 1$, satisfying the following condition:

(i) the function ψ is continuous and non-decreasing,
(ii) if the function $\psi(t) = 0 \implies t = 0$.

Definition 2.7. ([8, 11]) Let (X, d) be a complete generalized b_2-metric space. Assume that $T : X \to X$ and $\nu : X \times X \times X :\to [0, \infty)$ are functions. The function T is an ν-admissible mapping if $\nu(x, y, \xi) \geq 1$ for $x, y, \xi \in X$ implies that $\nu(Tx, Ty, \xi) \geq 1$.

In [7], the author used a similar definition for a generalized (ψ, φ)-almost weakly contractive mapping.

Definition 2.8. Let (X, d) be a generalized b_2-metric space. A mapping $T : X \to X$ is a generalized (ψ, φ)-almost weakly contractive type mapping if there exists $\mu \geq 0$, $\beta > 1$ such that

$$\begin{aligned}
&\beta\psi(d(Tx, Ty, \xi)) \\
&\leq f(d(x, y, \xi))\psi\left(\max\left\{d(x, y, \xi), \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\}\right) \\
&\quad - \varphi\left(\max\left\{d(x, y, \xi), \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\}\right) \\
&\quad + \mu\psi\left(\min\left\{d(x, Tx, \xi), d(x, Ty, \xi), d(y, Ty, \xi), d(y, Tx, \xi)\right\}\right) \qquad (2.1)
\end{aligned}$$

for all $x, y, \xi \in X$ and $\psi, \varphi, f \in \mathfrak{F}$.

In [8], the authors have proved a similar result in a partially ordered b_2-metric space for mappings subject to an almost generalized (ψ, φ)-contraction.

Theorem 2.9. *Let (X, d) be a complete generalized b_2-metric space, $T : X \to X$ be a self-mapping and $\nu : X \times X \times X \to [0, \infty)$ be a function such that T is an ν-admissible mapping. Suppose that*

(i) *For all* $x, y, \xi \in X$ *and* $f, \psi, \varphi \in \mathfrak{F}$,

$$\begin{aligned}
&\beta\nu(x,Tx,\xi)\nu(y,Ty,\xi)\psi(d(Tx,Ty,\xi)) \\
&\le f(d(x,y,\xi))\psi\left(\max\left\{d(x,y,\xi), \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\}\right) \\
&\quad - \varphi\left(\max\left\{d(x,y,\xi), \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\}\right) \\
&\quad + \mu\psi\left(\min\left\{d(x,Tx,\xi), d(x,Ty,\xi), d(y,Ty,\xi), d(y,Tx,\xi)\right\}\right). \qquad (2.2)
\end{aligned}$$

(ii) *If* $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence such that* $x_n \to x$, $\nu(x_n, x_{n+1}, \xi) \ge 1$, *then* $\nu(x, Tx, \xi) \ge 1$.

If $\nu(x_0, Tx_0, \xi) \ge 1$ *for some* $x_0 \in X$, *then* T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$ such that $\nu(x_0, Tx_0, \xi) \ge 1$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by

$$x_n = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Since T is ν-admissible mapping and $\nu(x_0, Tx_0, \xi) \ge 1$, it follows that $\nu(x_1, Tx_1, \xi) = \nu(Tx_0, T^2x_0, \xi) \ge 1$. By continuing with the process, we get $\nu(x_n, Tx_n, \xi) \ge 1$ for all $n = 0, 1, 2, \cdots$. Then it follows that the product

$$\nu(x_n, Tx_n, \xi)\nu(x_{n-1}, Tx_{n-1}, \xi) \ge 1$$

for all $n = 1, 2, \cdots$.

We shall now show that the sequence $\{d(x_n, x_{n+1}, \xi)\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers. By (2.2), we get

$$\begin{aligned}
&\beta\psi\left(d(x_n, x_{n+1}, \xi)\right) \\
&= \beta\psi\left(d(Tx_{n-1}, Tx_n, \xi)\right) \\
&\le \beta\nu(x_{n-1}, Tx_{n-1}, \xi)\nu(x_n, Tx_n, \xi)\psi\left(d(Tx_{n-1}, Tx_n, \xi)\right) \\
&\le f(d(x_{n-1}, x_n, \xi))\psi\left(\max\left\{d(x_{n-1}, x_n, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(Tx_{n-1},Tx_n,\xi)}, \right.\right. \\
&\qquad\qquad \left.\left. \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(x_{n-1},x_n,\xi)}\right\}\right) \qquad (2.3) \\
&\quad - \varphi\left(\max\left\{d(x_{n-1}, x_n, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(Tx_{n-1},Tx_n,\xi)}, \right.\right. \\
&\qquad\qquad \left.\left. \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(x_{n-1},x_n,\xi)}\right\}\right) \\
&\quad + \mu\psi(\min\{d(x_{n-1}, Tx_n, \xi), d(x_n, Tx_n, \xi), d(x_{n-1}, Tx_{n-1}, \xi), d(x_n, Tx_{n-1}, \xi)\}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\max\left\{d(x_{n-1}, x_n, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(Tx_{n-1},Tx_n,\xi)}, \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_n,Tx_n,\xi)}{1+d(x_{n-1},x_n,\xi)}\right\} \\
&\le \max\left\{d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi)\right\} \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}&\min\{d(x_{n-1}, Tx_n, \xi), d(x_n, Tx_n, \xi), d(x_{n-1}, Tx_{n-1}, \xi), d(x_n, Tx_{n-1}, \xi)\}\\&= \min\{d(x_{n-1}, x_{n+1}, \xi), d(x_n, x_{n+1}, \xi), d(x_{n-1}, x_n, \xi), d(x_n, x_n, \xi)\}\\&= 0. \qquad (2.5)\end{aligned}$$

Using (2.4) and (2.5), inequality (2.3) reduces to

$$\begin{aligned}&\beta\psi d(x_n, x_{n+1}, \xi)\\&\leq f(d(x_{n-1}, x_n, \xi))\psi\left(\max\{d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi)\}\right)\\&\quad - \varphi\left(\max\{d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi)\}\right). \qquad (2.6)\end{aligned}$$

Inequality (2.6) further reduces, if we assume that

$$\max\{d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi)\} = d(x_{n-1}, x_n, \xi).$$

Thus, we get

$$\beta\psi((d(x_n, x_{n+1}, \xi)) \leq \frac{1}{\beta}\psi((d(x_{n-1}, x_n, \xi)). \qquad (2.7)$$

Since $\beta \geq 1$, we obtain

$$\psi((d(x_n, x_{n+1}, \xi)) \leq \frac{1}{\beta^2}\psi((d(x_{n-1}, x_n, \xi)) \leq \psi((d(x_{n-1}, x_n, \xi)). \qquad (2.8)$$

It follows that from the property of the altering function that $\{d(x_n, x_{n+1}, \xi)\}_{n\in\mathbb{N}}$ is decreasing that is bounded from below and thus converges.

Suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}, \xi) = r$, where $r > 0$ then taking limit as $n \to \infty$ in inequality (2.7), we get

$$\beta\psi(r) \leq \frac{1}{\beta}\psi(r), \qquad (2.9)$$

which leads to a contradiction unless $r = 0$, that is,,

$$\lim_{n\to\infty} d(x_n, x_{n+1}, \xi) = 0. \qquad (2.10)$$

In the case, we assume that

$$\max\{d(x_{n-1}, x_n, \xi), d(x_n, x_{n+1}, \xi)\} = d(x_{n+1}, x_n, \xi),$$

we get

$$\psi((d(x_n, x_{n+1}, \xi)) \leq \frac{1}{\beta^2}\psi((d(x_n, x_{n+1}, \xi)) < \psi((d(x_n, x_{n+1}, \xi)), \qquad (2.11)$$

which leads to a contradiction.

Next we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Using inequality (2.2), we get

$$\begin{aligned}
&\psi(d(x_n, x_m, \xi)) \\
&= \psi(d(Tx_{n-1}, Tx_{m-1}, \xi)) \\
&\leq \beta\nu(x_{n-1}, Tx_{n-1}, \xi)\nu(x_{m-1}, Tx_{m-1}, \xi)\psi(d(Tx_{n-1}, Tx_{m-1}, \xi)) \\
&\leq \beta(f(d(x_{n-1}, x_{m-1}, \xi)) \\
&\quad \times \psi\Big(\max\Big\{ d(x_{n-1}, x_{m-1}, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(Tx_{n-1},Tx_{m-1},\xi)}, \\
&\qquad \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(x_{n-1},x_{m-1},\xi)} \Big\}\Big) \\
&\quad - \beta\varphi\Big(\max\Big\{ d(x_{n-1}, x_{m-1}, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(Tx_{n-1},Tx_{m-1},\xi)}, \\
&\qquad \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(x_{n-1},x_{m-1},\xi)} \Big\}\Big) \\
&\quad + \mu\psi(\min\{d(x_{n-1}, Tx_{n-1}, \xi), d(x_{n-1}, Tx_{m-1}, \xi), d(x_{m-1}, Tx_{n-1}, \xi), \\
&\qquad d(x_{m-1}, Tx_{m-1}, \xi)\}).
\end{aligned} \tag{2.12}$$

Since $f(t) \leq \frac{1}{\beta}$, we obtain that

$$\begin{aligned}
\psi(d(x_n, x_m, \xi)) &= \psi(d(Tx_{n-1}, Tx_{m-1}, \xi)) \\
&\leq \beta\nu(x_{n-1}, Tx_{n-1}, \xi)\nu(x_{m-1}, Tx_{m-1}, \xi)\psi(d(Tx_{n-1}, Tx_{m-1}, \xi)) \\
&\leq \psi\Big(\max\Big\{ d(x_{n-1}, x_{m-1}, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(Tx_{n-1},Tx_{m-1},\xi)}, \\
&\qquad \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(x_{n-1},x_{m-1},\xi)} \Big\}\Big) \\
&\quad - \beta\varphi\Big(\max\Big\{ d(x_{n-1}, x_{m-1}, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(Tx_{n-1},Tx_{m-1},\xi)}, \\
&\qquad \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(x_{n-1},x_{m-1},\xi)} \Big\}\Big) \\
&\quad + \mu\psi(\min\{d(x_{n-1}, Tx_{n-1}, \xi), d(x_{n-1}, Tx_{m-1}, \xi), \\
&\qquad d(x_{m-1}, Tx_{n-1}, \xi), d(x_{m-1}, Tx_{m-1}, \xi)\}).
\end{aligned} \tag{2.13}$$

Taking $m, n \to \infty$ and using (2.10), we get,

$$\begin{aligned}
&\lim_{m,n\to\infty} \max\Big\{ d(x_{n-1}, x_{m-1}, \xi), \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(Tx_{n-1},Tx_{m-1},\xi)}, \\
&\qquad \tfrac{d(x_{n-1},Tx_{n-1},\xi)d(x_{m-1},Tx_{m-1},\xi)}{1+d(x_{n-1},x_{m-1},\xi)} \Big\} \\
&= \lim_{m,n\to\infty} \max\Big\{ d(x_{n-1}, x_{m-1}, \xi), \tfrac{d(x_{n-1},x_n,\xi)d(x_{m-1},x_m,\xi)}{1+d(x_n,x_m,\xi)}, \\
&\qquad \tfrac{d(x_{n-1},x_n,\xi)d(x_{m-1},x_m,\xi)}{1+d(x_{n-1},x_{m-1},\xi)} \Big\} \\
&= \lim_{m,n\to\infty} d(x_{n-1}, x_{m-1}, \xi)
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&\lim_{m,n\to\infty} \min\{d(x_{n-1},Tx_{n-1},\xi), d(x_{n-1},Tx_{m-1},\xi), d(x_{m-1},Tx_{n-1},\xi),\\
&\qquad\qquad d(x_{m-1},Tx_{m-1},\xi)\}\\
&= \lim_{m,n\to\infty} \min\{d(x_{n-1},x_n,\xi), d(x_{n-1},x_m,\xi), d(x_{m-1},x_n,\xi), d(x_{m-1},x_m,\xi)\}\\
&= 0. \qquad (2.15)
\end{aligned}$$

Taking $m,n\to\infty$ in (2.13), using (2.14) and (2.15), we get

$$\begin{aligned}
\lim_{m.n\to\infty} \psi(d(x_n,x_m,\xi)) &\le \psi(\lim_{m,n\to\infty} d(x_{n-1},x_{m-1},\xi))\\
&\quad - \beta\varphi(\lim_{m,n\to\infty} d(x_{n-1},x_{m-1},\xi)). \qquad (2.16)
\end{aligned}$$

Suppose that $\lim_{m,n\to\infty} d(x_n,x_m,\xi) = r$ with $r>0$. Then, since ψ is continuous, $\psi(d(x_n,x_m,\xi))\to\psi(r)$ as $n,m\to\infty$ and that $0\le\psi(r)<\frac{1}{\beta}$, we get

$$\psi(r) \le \psi(r) - \beta\varphi(r) \le \psi(r), \qquad (2.17)$$

which leads to a contradiction, unless $\psi(r)=0$, which implies that

$$\lim_{m,n\to\infty} d(x_n,x_m,\xi) = 0.$$

Thus $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since (X,d) is complete, there exists $x'\in X$ such that $\lim_{m,n\to\infty} d(x_n,x',\xi)=0$.

We now show that $x'\in X$ is a fixed point of T. Using (2.2) and from assumption (ii), $\nu(x',Tx',\xi)\ge 1$, we get

$$\begin{aligned}
&\psi(d(x',Tx',\xi))\\
&\le \lim_{n\to\infty} \beta\psi(d(Tx',\xi,x_{n+1}))\\
&\le \beta \lim_{n\to\infty} \nu(x_n,Tx_n,\xi)\nu(x',Tx',\xi)d(Tx',\xi,Tx_n)\\
&\le \lim_{n\to\infty} \Big[f(d(x_n,x',\xi))\psi\Big(\max\Big\{ d(x_n,x',\xi), \tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(Tx_n,Tx',\xi)},\\
&\qquad \tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(x_n,x',\xi)} \Big\}\Big)\\
&\quad - \varphi\Big(\max\Big\{ d(x_n,x',\xi), \tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(Tx_n,Tx',\xi)}, \tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(x_n,x',\xi)} \Big\}\Big)\\
&\quad + \mu\psi\Big(\min\Big\{ d(x_n,Tx',\xi), d(x',Tx_n,\xi), d(x_n,Tx_n,\xi), d(x',Tx',\xi)\Big\}\Big)\Big] \qquad (2.18)
\end{aligned}$$

$$\leq \lim_{n\to\infty}\Big[\frac{1}{\beta}\psi\Big(\max\Big\{d(x_n,x',\xi),\tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(Tx_n,Tx',\xi)},\tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(x_n,x',\xi)}\Big\}\Big)$$
$$-\varphi\Big(\max\Big\{d(x_n,x',\xi),\tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(Tx_n,Tx',\xi)},\tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(x_n,x',\xi)}\Big\}\Big)$$
$$+\mu\psi\Big(\min\big\{d(x_n,Tx',\xi),d(x',Tx_n,\xi),d(x_n,Tx_n,\xi),d(x',Tx',\xi)\big\}\Big)\Big].$$

Since

$$\lim_{n\to\infty}\max\Big\{d(x_n,x',\xi),\tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(Tx_n,Tx',\xi)},\tfrac{d(x_n,Tx_n,\xi)d(x',Tx',\xi)}{1+d(x_n,x',\xi)}\Big\}=0$$

and

$$\lim_{n\to\infty}\min\big\{d(x_n,Tx',\xi),d(x',Tx_n,\xi),d(x_n,Tx_n,\xi),d(x',Tx',\xi)\big\}=0.$$

We conclude from (2.18) that $\psi(d(x',Tx',\xi))\leq 0$ which implies that

$$d(x',Tx',\xi)=0$$

and since ξ is arbitrary, we get $Tx'=x'$.

To prove uniqueness of x', we assume that x'' is a fixed point of T such that $x'\neq x''$ and $\nu(x',Tx',\xi)\geq 1$, $\nu(x'',Tx'',\xi)\geq 1$. From inequality (2.2), we obtain

$$\beta\psi(d(x',x'',\xi))$$
$$\leq \beta\nu(x',Tx',\xi)\nu(x'',Tx'',\xi)\psi(d(Tx',Tx'',\xi))$$
$$\leq \frac{1}{\beta}\psi\Big(\max\Big\{d(x',x'',\xi),\tfrac{d(x',Tx',\xi)d(x'',Tx'',\xi)}{1+d(x',x'',\xi)},\tfrac{d(x',Tx',\xi)d(x'',Tx'',\xi)}{1+d(Tx',Tx'',\xi)}\Big)\Big)$$
$$-\varphi\Big(\max\Big\{d(x',x'',\xi),\tfrac{d(x',Tx',\xi)d(x'',Tx'',\xi)}{1+d(x',x'',\xi)},\tfrac{d(x',Tx',\xi)d(x'',Tx'',\xi)}{1+d(Tx',Tx'',\xi)}\Big)\Big)$$
$$+\mu\min\big\{d(x',Tx',\xi),d(x',Tx'',\xi),d(x'',Tx',\xi),d(x'',Tx'',\xi)\big\}. \quad (2.19)$$

It follows that

$$\left(\beta-\frac{1}{\beta}\right)\psi(d(x',x'',\xi))\leq -\beta\varphi\big(d(x',x'',\xi)\big)\leq 0 \quad (2.20)$$

is a contradiction unless $\psi(d(x',x'',\xi))=0$ which implies that $d(x',x'',\xi)=0$, and it follows that $x'=x''$. □

Example 2.10. Let $X=\left[0,\frac{1+\sqrt{17}}{8}\right]$ and define a generalized b_2-metric by

$$d(x,y,z)=\begin{cases}0, & \text{if at least two of the three points are the same,}\\ \frac{e^{|x-y|+|y-\xi|+|\xi-x|}}{\gamma}, & \text{otherwise,}\end{cases}$$

where $\gamma=\sup_{x,y,\xi\in X}e^{|x-y|+|y-\xi|+|\xi-x|}$.

Define $T:X\to X$ by

$$Tx=\sqrt{\frac{x+1}{4}}.$$

Since $0 \le x \le \frac{1+\sqrt{17}}{8}$, it follows that $\frac{1}{2} \le \sqrt{\frac{x+1}{4}} \le \frac{1}{2}\sqrt{\frac{1+\sqrt{17}}{8}} \le \frac{1+\sqrt{17}}{8}$.
If $x \ge y$ then $e^{\frac{1}{4}x-\frac{1}{4}y} \ge 1$. Define

$$\nu(x,y,\xi) = \begin{cases} e^{\frac{1}{4}x-\frac{1}{4}y}, & x \ge y, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that for $x \ge y$

$$\sqrt{\frac{x+1}{4}} \ge \sqrt{\frac{y+1}{4}},$$

which implies that

$$\nu(Tx,Ty,\xi) = e^{\frac{1}{4}\sqrt{\frac{x+1}{4}}-\frac{1}{4}\sqrt{\frac{y+1}{4}}} \ge 1.$$

Thus we conclude that T is a ν-admissible function. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to x = \frac{1+\sqrt{17}}{8}$ as $n \to \infty$ and $x_n \ge x_{n+1}$ for all $n \in \mathbb{N}$. Then by the definition of ν, we get $\nu(x_n, x_{n+1}, \xi) = e^{\frac{1}{4}x_n - \frac{1}{4}x_{n+1}} \ge 1$ and $\nu(x,Tx,\xi) = e^{\frac{x}{4}-\frac{1}{4}\sqrt{\frac{x+1}{4}}} = e^0 = 1$. Using the Mean value theorem for $[x,y]$, we get

$$\left|\sqrt{\frac{x+1}{4}} - \sqrt{\frac{y+1}{4}}\right| \le \frac{1}{4}\,|x-y|$$

and inequality $\left|\xi - \sqrt{\frac{y+1}{4}}\right| \le |\xi - y|$, we conclude that

$$e^{\left|\sqrt{\frac{x+1}{4}}-\sqrt{\frac{y+1}{4}}\right|+\left|\sqrt{\frac{y+1}{4}}-\xi\right|+\left|\xi-\sqrt{\frac{x+1}{4}}\right|} \le e^{|x-y|+|y-\xi|+|\xi-x|}.$$

For $x, y, \xi \in X$, we obtain that

$$\begin{aligned}
&\beta\nu(x,Tx,\xi)\nu(y,Ty,\xi)\psi(d(Tx,Ty,\xi)) \\
&= \beta e^{\frac{x}{4}-\frac{1}{4}\sqrt{\frac{x+1}{4}}} e^{\frac{y}{4}-\frac{1}{4}\sqrt{\frac{y+1}{4}}} \psi\left(\frac{e^{\left|\sqrt{\frac{x+1}{4}}-\sqrt{\frac{y+1}{4}}\right|+\left|\sqrt{\frac{y+1}{4}}-\xi\right|+\left|\xi-\sqrt{\frac{x+1}{4}}\right|}}{\gamma}\right) \\
&\le \beta\psi\left(\frac{e^{|x-y|+|y-z|+|z-x|}}{\gamma}\right) \\
&\le \beta\psi\left(\max\left\{d(x,y,\xi), \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \tfrac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\}\right),
\end{aligned}$$

since $\frac{x}{4} - \frac{1}{4}\sqrt{\frac{x+1}{4}} \leq 0$. Taking $f(t) = \frac{1}{\beta} < 1$ and define

$$\psi(t) = \begin{cases} \frac{t}{\beta^2}, & 0 \leq t \leq 1, \\ \frac{1}{\beta^2}, & t > 1, \end{cases} \tag{2.21}$$

then $\psi \in \mathfrak{F}$ and

$$\begin{aligned} &\beta\nu(x,Tx,\xi)\nu(y,Ty,\xi)\psi(d(Tx,Ty,\xi)) \\ &\leq \frac{1}{\beta}\max\left\{d(x,y,\xi), \frac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \frac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\} \\ &\leq \frac{1}{\beta}. \end{aligned}$$

Since $0 \leq d(x,y,\xi) \leq 1$ and $\frac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)} \leq 1$. It follows from Theorem 2.9, that T has a unique fixed point in X.

Corollary 2.11. *Let (X,d) be a complete generalized b_2-metric space, $T : X \to X$ be a self-mapping and $\nu : X \times X \times X \to [0,\infty)$ be a function such that T is an ν-admissible mapping. Suppose that*

$$\begin{aligned} &\beta\nu(x,Tx,\xi)\nu(y,Ty,\xi)d(Tx,Ty,\xi) \\ &\leq \frac{1}{\beta}\max\left\{d(x,y,\xi), \frac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \frac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\} \end{aligned} \tag{2.22}$$

for all $x,y,\xi \in X$. If there exists $x_0 \in X$ such that $\nu(x_0,Tx_0,\xi) \geq 1$, then T has a unique fixed point.

Proof. Follows from theorem 2.9, by setting $\psi(t) = t$, $\mu = 0$ and $\varphi(t) = 0$. □

Corollary 2.12. *Let (X,d) be a complete generalized b_2-metric space, $T : X \to X$ be a self-mapping and $\nu : X \times X \times X \to [0,\infty)$ be a function such that T is an ν-admissible mapping. Suppose that*

$$\begin{aligned} &\nu(x,Tx,\xi)\nu(y,Ty,\xi)d(Tx,Ty,\xi) \\ &\leq \frac{1}{\beta}\max\left\{d(x,y,\xi), \frac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(x,y,\xi)}, \frac{d(x,Tx,\xi)d(y,Ty,\xi)}{1+d(Tx,Ty,\xi)}\right\} \end{aligned} \tag{2.23}$$

for all $x,y,\xi \in X$. If there exists $x_0 \in X$ such that $\nu(x_0,Tx_0,\xi) \geq 1$, then T has a unique fixed point.

Proof. Follows from theorem 2.9, by setting $\psi(t) = \beta t$, $\mu = 0$ and $\varphi(t) = 0$. □

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