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MODIFICATION OF REGULAR FUNCTIONS ON TERNARY REAL NUMBERS IN THE VIEW OF QUATERNION

Ji Eun Kim

Department of Mathematics, Dongguk University,
Gyeongju-si 38066, Republic of Korea
e-mail: jeunkim@pusan.ac.kr

Abstract. In this paper, we represent regular functions on ternary theory in the view of quaternion. By expressing quaternions using ternary number theory, a new form of regular function, called E-regular, is defined. From the defined regular function, we investigate the properties of the appropriate hyper-conjugate harmonic functions and corresponding Cauchy-Riemann equations by pseudo-complex forms.

1. Introduction

The non-commutative four dimensional $\mathbb{R}^4$ of the hypercomplex numbers have been studied by Hamilton, called quaternions with real numbers. Quaternions are interested in the characteristics of functions dealt with in complex analysis from the perspective of a number system extended from complex numbers. In 1935, Fueter [2] has given a definition of regular quaternionic function in $\mathbb{R}^4$. After then Deavours [1] and Sudbery [12] have developed a quaternion analysis theory. K. Nôno [10] have represented quaternions to complex number forms. Koriyama and K. Nôno [6] have investigated hyperholomorphic function and holomorphic functions in Quaternionic analysis.

Various combinations of number systems can be attempted by transforming quaternions into ternaries. In particular, by transforming quaternions into

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ternary numbers, this paper examine how the form and properties of the regular function defined in ternary numbers are related to the properties of the regular function applied in quaternions. Kim and Shon [3, 4] have investigated quaternion variables modified with tri-complex form and properties of regular functions of that variables. Kim [5] has given the corresponding differentiability of functions of generalized quaternionic variables and their properties.

We consider the ternary representation of real quaternions and make pseudo-complex number form with modified basis $\hat{i}$. We define E-regular in $\mathbb{R}^3$ and relative properties of E-regular functions.

In this paper, we introduce definitions of E-regular functions on real ternary numbers and represent pseudo-complex number form which is a special form of the quaternion. In section 2, we give necessary variables, functions and operators. In section 3, we define dirac operators and Cauchy integral theorems and introduce properties and corollaries by Naser [9] and Nôno [10, 11]. Also, we find a harmonic conjugate function on $\mathbb{R}^3$. In section 4, we reserch two generalized Cauchy-Riemman systems introduced in [7, 8].

2. Preliminaries

In the skew quaternion field $\mathcal{T}$, let $\mathbb{T}$ be the set of all ternary numbers

$$\mathbb{T} = \{z \mid z = x_0 + x_1e_1 + x_2e_2,\ x_0, x_1, x_2 \in \mathbb{R}\} \subset \mathcal{T},$$

where $e_1^2 = e_2^2 = -1$ and let $e_1e_2 = \sqrt{-1}$.

The element is

$$\begin{aligned} z &= x_0 + x_1e_1 + x_2e_2 \\ &= x_0 + \frac{ae_1 + be_2}{\sqrt{a^2+b^2}} \left(\frac{ax_1 + bx_2}{\sqrt{a^2+b^2}} + \frac{bx_1 - ax_2}{\sqrt{a^2+b^2}} e_1e_2 \right), \end{aligned}$$

where a, b are real numbers except both zeros.

Let

$$\hat{i} = \frac{ae_1 + be_2}{\sqrt{a^2+b^2}}\ ,\ z_0 = \frac{ax_1 + bx_2}{\sqrt{a^2+b^2}} + \frac{bx_1 - ax_2}{\sqrt{a^2+b^2}} e_1e_2.$$

Then $\mathbb{T} \cong \mathbb{R} \times \mathbb{C}$ with

$$z = x_0 + \hat{i}z_0 \in \mathbb{R} \times \mathbb{C}\ ,\ \hat{i}^2 = -1$$

called a pseudo-complex number form and

$$\begin{aligned} z \pm w &= (x_0 + \hat{i}z_0) \pm (y_0 + \hat{i}w_0) \\ &= (x_0 \pm y_0) + \hat{i}(z_0 \pm w_0), \\ zw &= (x_0 + \hat{i}z_0)(y_0 + \hat{i}w_0) \\ &= (x_0y_0 - \overline{z_0}w_0) + \hat{i}(x_0w_0 + z_0y_0), \end{aligned}$$

where $z_0\hat{i} = \hat{i}\overline{z_0}$, $w = y_0 + \hat{i}w_0$ and

$$w_0 = \left(\frac{ay_1 + by_2}{\sqrt{a^2+b^2}} + \frac{by_1 - ay_2}{\sqrt{a^2+b^2}} e_1e_2\right).$$

Let $\tilde{z}$ be the conjugate of z

$$\tilde{z} = x_0 - \hat{i}z_0.$$

Then

$$\begin{aligned} |z|^2 := z\tilde{z} &= (x_0 + \hat{i}z_0)(x_0 - \hat{i}z_0) \\ &= x_0^2 + \overline{z_0}z_0 \\ &= x_0^2 + x_1^2 + x_2^2 \end{aligned}$$

and

$$z^{-1} = \frac{\tilde{z}}{|z|^2}.$$

Consider the following differential operators:

$$\begin{aligned} D &:= \frac{\partial}{\partial x_0} - \hat{i}\frac{\partial}{\partial \overline{z_0}} = \frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2}, \\ \widetilde{D} &= \frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial \overline{z_0}} = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2}, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial}{\partial z_0} &= \left(\frac{a}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_1} + \frac{b}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_2}\right) \\ &\quad - \left(\frac{b}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_1} - \frac{a}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_2}\right) e_1e_2, \\ \frac{\partial}{\partial \overline{z_0}} &= \left(\frac{a}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_1} + \frac{b}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_2}\right) \\ &\quad + \left(\frac{b}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_1} - \frac{a}{\sqrt{a^2+b^2}}\frac{\partial}{\partial x_2}\right) e_1e_2. \end{aligned}$$

Then the Laplacian operator is

$$\Delta := D\widetilde{D} = \widetilde{D}D = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z_0 \partial \overline{z_0}} = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Let Ω be a domain in $\mathbb{R}^3$. We consider a function f defined on Ω and with values in $\mathbb{R} \times \mathbb{C}$:

$$\begin{gathered} f = u_0 + u_1e_1 + u_2e_2 = u_0 + \hat{i}f_0, \\ z = (x_0, x_1, x_2) \in \Omega \mapsto f(z) = u_0(x_0, x_1, x_2) + \hat{i}f_0(x_0, x_1, x_2) \in \mathbb{R} \times \mathbb{C}, \end{gathered}$$

where u_0, u_1, u_2 are real valued functions and let

$$v_0 := \frac{au_1 + bu_2}{\sqrt{a^2+b^2}}\ ,\ v_1 := \frac{bu_1 - au_2}{\sqrt{a^2+b^2}}.$$

Then

$$f_0 = v_0 + v_1 e_1 e_2 \tag{2.1}$$

is an usual complex valued function.

We let G be a domain in $\mathbb{R}\times\mathbb{C}$. We consider a function h defined on G and with values in $\mathbb{C}\times\mathbb{C}$:

$$z = (x_0, z_0) \in G \mapsto h(z) = u(x_0, z_0) + \hat{i}v(x_0, z_0) \in \mathbb{C}\times\mathbb{C},$$

where u and v are complex valued functions.

3. Properties of a E-regular function

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^3$. Let a function f be defined on Ω and with values in $\mathbb{R}\times\mathbb{C}$. Then $\widetilde{D}f = 0$ if and only if*

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial f_0}{\partial z_0}\ ,\ \frac{\partial f_0}{\partial x_0} = -\frac{\partial u_0}{\partial \overline{z_0}}. \tag{3.1}$$

Similarly, $f\widetilde{D} = 0$ if and only if

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\ ,\ \frac{\partial f_0}{\partial x_0} = -\frac{\partial u_0}{\partial \overline{z_0}}. \tag{3.2}$$

Proof. By the definitions of the operators D and $\widetilde{D}$, we have

$$\begin{aligned}
Df &= \left(\frac{\partial}{\partial x_0} - \hat{i}\frac{\partial}{\partial \overline{z_0}}\right)(u_0 + \hat{i}f_0) = \left(\frac{\partial u_0}{\partial x_0} + \frac{\partial f_0}{\partial z_0}\right) + \hat{i}\left(\frac{\partial f_0}{\partial x_0} - \frac{\partial u_0}{\partial \overline{z_0}}\right),\\
\widetilde{D}f &= \left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial \overline{z_0}}\right)(u_0 + \hat{i}f_0) = \left(\frac{\partial u_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0}\right) + \hat{i}\left(\frac{\partial f_0}{\partial x_0} + \frac{\partial u_0}{\partial \overline{z_0}}\right),\\
fD &= (u_0 + \hat{i}f_0)\left(\frac{\partial}{\partial x_0} - \hat{i}\frac{\partial}{\partial \overline{z_0}}\right) = \left(\frac{\partial u_0}{\partial x_0} + \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right) + \hat{i}\left(\frac{\partial f_0}{\partial x_0} - \frac{\partial u_0}{\partial \overline{z_0}}\right),\\
f\widetilde{D} &= (u_0 + \hat{i}f_0)\left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial \overline{z_0}}\right) = \left(\frac{\partial u_0}{\partial x_0} - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\right) + \hat{i}\left(\frac{\partial f_0}{\partial x_0} + \frac{\partial u_0}{\partial \overline{z_0}}\right).
\end{aligned}$$

Thus, the results are obtained. □

Referred [10] and [11], we give the following definitions:

Definition 3.2. $f(z) = u_0(x_0, x_1, x_2) + \hat{i}f_0(x_0, x_1, x_2)$ is called E_1-regular function if

(1) u_0 and f_0 are continuous differential functions,
(2) $\widetilde{D}f = 0$ on Ω.

Definition 3.3. $h(z) = u(x_0, z_0) + \hat{i}v(x_0, z_0)$ is called $LE_2(RE_2)$-regular function if

(1) u and v are continuous differential functions,
(2) $\widetilde{D}h = 0$ $(h\widetilde{D} = 0)$ on Ω.

By comparing (3.1) and (3.2), we don't need to distinguish left and right calculation and just call E_1-regular function. However, $\widetilde{D}h = 0$ if and only if

$$\frac{\partial u}{\partial x_0} = \frac{\partial v}{\partial z_0}\ ,\ \frac{\partial v}{\partial x_0} = -\frac{\partial u}{\partial \overline{z_0}}. \tag{3.3}$$

Similarly, $h\widetilde{D} = 0$ if and only if

$$\frac{\partial u}{\partial x_0} = \frac{\partial \overline{v}}{\partial \overline{z_0}}\ ,\ \frac{\partial v}{\partial x_0} = -\frac{\partial \overline{u}}{\partial \overline{z_0}}. \tag{3.4}$$

Since u and v are complex-valued functions, we have to distinguish LE_2-regular which is satisfied $\widetilde{D}h = 0$ and RE_2-regular which is satisfied $h\widetilde{D} = 0$ in the above definition.

From the definitions of f_0 and z_0, we obtain the following equations:

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1} \tag{3.5}$$

and

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}. \tag{3.6}$$

Equation (3.5) is equivalent to

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial f_0}{\partial z_0}, \tag{3.7}$$

and Equation (3.6) is equivalent to

$$\frac{\partial u_0}{\partial \overline{z_0}} = -\frac{\partial f_0}{\partial x_0}. \tag{3.8}$$

Proposition 3.4. *Let Ω be an open set in $\mathbb{R}^3$ and f be an E-regular function on Ω. Then, we have*

$$Df := f' = \frac{\partial f}{\partial x_0} = -\hat{i}\frac{\partial f}{\partial \overline{z_0}}.$$

In particular, we obtain

$$D^n f = \frac{\partial^n}{\partial x_0^n} f$$

and

$$D^n f = (-1)^{\frac{n}{2}} \frac{\partial^n}{\partial z_0^{\frac{n}{2}} \partial \overline{z_0}^{\frac{n}{2}}} f, \quad \text{if } n \text{ is even},$$
$$\hat{i}(-1)^{\frac{n+1}{2}} \frac{\partial^n}{\partial z_0^{\frac{n-1}{2}} \partial \overline{z_0}^{\frac{n+1}{2}}} f, \quad \text{if } n \text{ is odd}.$$

Proof. By the definition of a E-regular function on Ω and [11],

$$Df = \left(\frac{\partial u_0}{\partial x_0} + \frac{\partial f_0}{\partial z_0}\right) + \hat{i}\left(\frac{\partial f_0}{\partial x_0} - \frac{\partial u_0}{\partial \overline{z_0}}\right) = \frac{\partial u_0}{\partial x_0} + \hat{i}\frac{\partial f_0}{\partial x_0} = \frac{\partial}{\partial x_0} f$$

and

$$D^2 f = \frac{\partial}{\partial x_0}\Big(\frac{\partial}{\partial x_0} f\Big) = \frac{\partial^2 f}{\partial x_0^2}.$$

By continuing calculations, we have

$$D^n f = \frac{\partial^n f}{\partial x_0^n}.$$

Similarly,

$$Df = \frac{\partial f_0}{\partial z_0} - \hat{i}\frac{\partial u_0}{\partial \overline{z_0}} = -\hat{i}^2 \frac{\partial f_0}{\partial z_0} - \hat{i}\frac{\partial u_0}{\partial \overline{z_0}}$$
$$= -\hat{i}\left(\frac{\partial}{\partial \overline{z_0}} \hat{i} f_0 + \frac{\partial}{\partial \overline{z_0}} u_0\right) = -\hat{i}\frac{\partial}{\partial \overline{z_0}} f$$

and by continuing the operator D calculations, we get

$$D^n f = (-1)^{\frac{n}{2}} \frac{\partial^n}{\partial z_0^{\frac{n}{2}} \partial \overline{z_0}^{\frac{n}{2}}} f, \quad \text{if } n \text{ is even},$$
$$\hat{i}(-1)^{\frac{n+1}{2}} \frac{\partial^n}{\partial z_0^{\frac{n-1}{2}} \partial \overline{z_0}^{\frac{n+1}{2}}} f, \quad \text{if } n \text{ is odd}.$$

□

Example 3.5. Let $f(z) = z^n = (x_0 + \hat{i} z_0)^n$. Then if f is an E-regular function on Ω in $\mathbb{R}^3$, then

$$D^n f(z) = \frac{\partial^n f(z)}{\partial x_0^n} = n!$$

and hence

$$f(z) = \frac{1}{n!} \frac{\partial^n f(z)}{\partial x_0^n} z^n.$$

Similarly,

$$f(z) = (-\hat{i})^n \frac{1}{n!} \frac{\partial^n f(z)}{\partial z_0^{r_1} \partial \overline{z_0}^{r_2}} z^n,$$

where r_1 and r_2 are integers with $r_1 \leq r_2$ and $r_1 + r_2 = n$.

Proposition 3.6. *Let Ω be an open set in $\mathbb{R}^3$ and f and g be E-regular functions on Ω. Then*

(1) *αf is E-regular on Ω, if α is any real constant,*
(2) *$f\alpha$ is E-regular on Ω, if α is any ternary constant,*
(3) *$f \pm g$ is E-regular on Ω.*

Proof. It is sufficient to show the second condition of the Definition 3.2.
(1) We have

$$\begin{aligned}\widetilde{D}(\alpha f) = &\left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial \overline{z_0}}\right)(a_0 u_0 - \overline{\alpha_0} f_0) + \hat{i}(a_0 f_0 + u_0 \alpha_0) \\ =&a_0\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0}\right) - \left(\overline{\alpha_0}\frac{\partial f_0}{\partial x_0} + \frac{\partial u_0}{\partial z_0}\alpha_0\right) \\ &+ \hat{i}\left\{a_0\left(\frac{\partial f_0}{\partial x_0} + \frac{\partial u_0}{\partial \overline{z_0}}\right) + \left(\frac{\partial u_0}{\partial x_0}\alpha_0 - \overline{\alpha_0}\frac{\partial f_0}{\partial \overline{z_0}}\right)\right\}.\end{aligned}$$

If α is any real constant, $\widetilde{D}(\alpha f) = 0$.

(2) Let α be a ternary constant, $\alpha = a_0 + \hat{i}\alpha_0$, where a_0 is real and

$$\alpha_0 = \frac{c_1 a_1 + c_2 a_2}{\sqrt{c_1^2 + c_2^2}} + \frac{c_2 a_1 - c_1 a_2}{\sqrt{c_1^2 + c_2^2}} e_1 e_2,$$

where c_1, c_2, a_1 and a_2 are real numbers. Then

$$\begin{aligned}\widetilde{D}(f\alpha) &= \left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial \overline{z_0}}\right)\{(u_0 a_0 - \overline{f_0}\alpha_0) + \hat{i}(u_0 \alpha_0 + f_0 a_0)\} \\ &= \left(\frac{\partial u_0}{\partial x_0}a_0 - \frac{\partial \overline{f_0}}{\partial x_0}\alpha_0 - \frac{\partial u_0}{\partial z_0}\alpha_0 - \frac{\partial f_0}{\partial z_0}a_0\right) \\ &\quad + \hat{i}\left(\frac{\partial u_0}{\partial x_0}\alpha_0 + \frac{\partial f_0}{\partial x_0}a_0 + \frac{\partial u_0}{\partial \overline{z_0}}a_0 - \frac{\partial \overline{f_0}}{\partial \overline{z_0}}\alpha_0\right) \\ &= 0.\end{aligned}$$

(3) Since f, g is E-regular functions on Ω,

$$\begin{aligned}\widetilde{D}(f \pm g) = &\left(\frac{\partial}{\partial x_0} + \hat{i}\frac{\partial}{\partial \overline{z_0}}\right)\{(u_0 \pm v_0) + \hat{i}(f_0 \pm g_0)\} \\ =&\left(\frac{\partial u_0}{\partial x_0} \pm \frac{\partial v_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0} \mp \frac{\partial g_0}{\partial z_0}\right) + \hat{i}\left(\frac{\partial u_0}{\partial \overline{z_0}} \pm \frac{\partial v_0}{\partial \overline{z_0}} + \frac{\partial f_0}{\partial x_0}a_0 \pm \frac{\partial g_0}{\partial x_0}\right) \\ =&0.\end{aligned}$$

□

Corollary 3.7. *Let Ω be an open set of $\mathbb{R}^3$ and $f,\ g$ be continuously differential on Ω. Then the following equations are satisfied:*

$$D(fg) = (Df)g + f\frac{\partial g}{\partial x_0} - \hat{i}f\frac{\partial g}{\partial z_0},$$
$$\widetilde{D}(fg) = (\widetilde{D}f)g + f\frac{\partial g}{\partial x_0} + \hat{i}f\frac{\partial g}{\partial z_0}.$$

Proof. By the definitions of D and $\widetilde{D}$, we have

$$\begin{aligned} D(fg) &= D(u_0v_0 + \hat{i}f_0v_0 + \hat{i}u_0g_0 - \overline{f_0}g_0) \\ &= \frac{\partial}{\partial x_0}(u_0v_0) + \frac{\partial}{\partial x_0}\hat{i}(f_0v_0) + \frac{\partial}{\partial x_0}\hat{i}(u_0g_0) - \frac{\partial}{\partial x_0}(\overline{f_0}g_0) \\ &\quad + \hat{i}\frac{\partial}{\partial \overline{z_0}}(u_0v_0) - \frac{\partial}{\partial z_0}(f_0v_0) + \hat{i}\frac{\partial}{\partial z_0}(u_0g_0) - \hat{i}\frac{\partial}{\partial \overline{z_0}}(\overline{f_0}g_0) \\ &= \left(\frac{\partial u_0}{\partial x_0} + \hat{i}\frac{\partial f_0}{\partial x_0} + \hat{i}\frac{\partial u_0}{\partial \overline{z_0}} - \frac{\partial f_0}{\partial z_0}\right)(v_0 + \hat{i}g_0) + f\frac{\partial g}{\partial x_0} - \hat{i}f\frac{\partial g}{\partial z_0} \\ &= (Df)g + f\frac{\partial g}{\partial x_0} - \hat{i}f\frac{\partial g}{\partial z_0}. \end{aligned}$$

Similarly, by expanding $\widetilde{D}(fg)$ and rearranging each term, we obtain the following equation.

$$\widetilde{D}(fg) = (\widetilde{D}f)g + f\frac{\partial g}{\partial x_0} + \hat{i}f\frac{\partial g}{\partial z_0}.$$

□

Proposition 3.8. *Let Ω be an orientable smooth surface with boundary in $\mathbb{R}^3$, and $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$, $\gamma(x, y) = (x_0, x_1, x_2)$ be a smooth such that $\gamma(U) = \Omega$ and $\gamma(bU) = b\Omega$. Let $f = (u_0, u_1, u_2)$ be an E-regular on Ω, where $u_i : \mathbb{R}^3 \to \mathbb{R}$ $(i = 0, 1, 2)$. Then*

$$\oint_{b\Omega} \overline{f} \cdot d\gamma = 0,$$

where $\overline{f} = (u_0, -u_1, -u_2)$ and $\cdot$ is usual inner product.

Proof. We define

$$G_1 = f \cdot \frac{\partial \gamma}{\partial x}\ ,\ G_2 = f \cdot \frac{\partial \gamma}{\partial y}.$$

Then

$$\oint_{b\Omega} f \cdot d\gamma = \oint_{b\Omega} u_0dx_0 + u_1dx_1 + u_2dx_2 = \int_{bU} G_1dx + G_2dy,$$

by Greens' Theorem,

$$\int_{bU} G_1dx + G_2dy = \int\int_{bU} \nabla \times \overline{f} \cdot \left(\frac{\partial \gamma}{\partial x} \times \frac{\partial \gamma}{\partial y}\right) dxdy,$$

where $\times$ is usual outer product. Since f is an E-regular on Ω,

$$\nabla \times \overline{f} = 0$$

and hence

$$\oint_{b\Omega} \overline{f} \cdot d\gamma = 0.$$

□

Referred [9] and [11], we define harmonic function on Ω in $\mathbb{T}$:

Definition 3.9. Let Ω be an open set in $\mathbb{T}$. A function $f = u_0 + \hat{i} f_0$ is said to be harmonic on Ω if the component u_0 and f_0 are each harmonic on Ω.

Proposition 3.10. *Let Ω be an open set in $\mathbb{T}$. If a function $f = u_0 + \hat{i} f_0$ is E-regular on Ω. Then f is harmonic on Ω.*

Proof. Since f is an E-regular function on Ω,

$$\begin{aligned} D\widetilde{D}f &= \left(\frac{\partial}{\partial x_0} - \hat{i}\frac{\partial}{\partial \overline{z_0}}\right)\left\{\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0}\right) + \hat{i}\left(\frac{\partial f_0}{\partial x_0} + \frac{\partial u_0}{\partial \overline{z_0}}\right)\right\} \\ &= \left(\frac{\partial^2 u_0}{\partial x_0^2} + \frac{\partial^2 u_0}{\partial z_0 \partial \overline{z_0}}\right) + \hat{i}\left(\frac{\partial^2 f_0}{\partial x_0^2} + \frac{\partial^2 f_0}{\partial z_0 \partial \overline{z_0}}\right) \\ &= \left(\frac{\partial}{\partial x_0}\frac{\partial u_0}{\partial x_0} - \frac{\partial}{\partial z_0}\frac{\partial f_0}{\partial x_0}\right) + \hat{i}\left(-\frac{\partial}{\partial x_0}\frac{\partial u_0}{\partial \overline{z_0}} + \frac{\partial}{\partial \overline{z_0}}\frac{\partial u_0}{\partial x_0}\right) \\ &= 0. \end{aligned}$$

We obtain a harmonic function on Ω. □

Proposition 3.11. *Let G be an open set in $\mathbb{T}$. If a function $u(x_0, z_0)$ is harmonic on G, then there exists $v(x_0, z_0)$ such that $h = u + \hat{i}v$ is $hL(Rh)$-E-regular.*

Proof. Since u is harmonic on G,

$$\frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial z_0 \partial \overline{z_0}} = 0$$

and then,

$$\frac{\partial}{\partial x_0}\frac{\partial u}{\partial x_0} = -\frac{\partial}{\partial z_0}\frac{\partial u}{\partial \overline{z_0}}.$$

Let

$$\frac{\partial v}{\partial z_0} = \frac{\partial u}{\partial x_0}\ ,\ \frac{\partial v}{\partial x_0} = -\frac{\partial u}{\partial \overline{z_0}}.$$

Then, we have v which satisfied that $h = u + \hat{i}v$ is $hL(Rh)$-E-regular on G. □

Theorem 3.12. *Let Ω be an open set in $\mathbb{R}^3$. If a function $u_0(x_0, x_1, x_2)$ is harmonic on Ω, then there exists $f_0(x_0, x_1, x_2)$ such that $f = u_0 + \hat{i} f_0$ is E-regular.*

Proof. Since u_0 is harmonic on Ω,

$$\frac{\partial^2 u_0}{\partial x_0^2} + \frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial^2 u_0}{\partial x_2^2} = 0.$$

We put

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2},$$
$$\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}\ ,\ \frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}.$$

Then from u_0, we have u_1 and u_2, satisfied Cauchy-Riemann equations for E-regular on Ω, such that

$$u_0 + e_1 u_1 + e_2 u_2 = u_0 + \hat{i} f_0,$$

where f_0 is represented by (2.1). Hence we obtain f_0 by u_0. □

The following proposition is Cauchy integral formula for E-regular functions which is based on [7] and [8].

Example 3.13. Let $f(z)$ be a function such that

$$f(z) = \frac{x_0 - \hat{i} z_0}{(x_0^2 + z_0 \overline{z_0})^{\frac{3}{2}}}.$$

A function $f(z)$ is composed of

$$u_0(x_0, z_0) = \frac{x_0}{(x_0^2 + z_0 \overline{z_0})^{\frac{3}{2}}} \quad \text{and} \quad f_0(x_0, z_0) = -\frac{z_0}{(x_0^2 + z_0 \overline{z_0})^{\frac{3}{2}}}.$$

Since u_0 and f_0 are continuous differential functions and the function $f(z)$ satisfies

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial f_0}{\partial z_0} \quad \text{and} \quad \frac{\partial u_0}{\partial \overline{z_0}} = -\frac{\partial f_0}{\partial x_0},$$

f is E-regular on $\mathbb{R}^3$.

Example 3.14. Let $f(z)$ be a function such that

$$f(z) = \frac{1}{|\zeta - z|^2},$$

where ζ is a real number. Then Df is expressed

$$\left(\frac{\partial u_0}{\partial x_0} + \frac{\partial f_0}{\partial z_0}\right) + \hat{i}\left(\frac{\partial f_0}{\partial x_0} - \frac{\partial u_0}{\partial \overline{z_0}}\right)$$

and satisfies Equations (3.7) and (3.8). Thus, the function f is E-regular on $\mathbb{R}^3$.

4. Generalizations of Cauchy-Riemann equations

We consider two generalizations of Cauchy-Riemann equations to the 3-dimensional Euclidean space. The first one, obtained by the Dirac operator, is given by

$$\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0, \tag{4.1}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_2}, \tag{4.2}$$

$$\frac{\partial u_1}{\partial x_0} = -\frac{\partial u_0}{\partial x_1}\ ,\ \frac{\partial u_2}{\partial x_0} = -\frac{\partial u_0}{\partial x_2}. \tag{4.3}$$

The second one is given by

$$x_2\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right) + u_2 = 0, \tag{4.4}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_2}, \tag{4.5}$$

$$\frac{\partial u_1}{\partial x_0} = -\frac{\partial u_0}{\partial x_1}\ ,\ \frac{\partial u_2}{\partial x_0} = -\frac{\partial u_0}{\partial x_2}. \tag{4.6}$$

The second system can be considered as a non-Euclidean version of the former one.

Lemma 4.1. *In $\mathbb{T}$, we have that the first one, called the system (R), is*

$$\begin{cases} \dfrac{\partial u_0}{\partial x_0} = \dfrac{\partial f_0}{\partial z_0}, \\ \dfrac{\partial f_0}{\partial x_0} = -\dfrac{\partial u_0}{\partial \overline{z_0}}, \end{cases} \tag{4.7}$$

and the second one, called the system (H), is

$$\begin{cases} z_0\left(\dfrac{\partial u_0}{\partial x_0} - \dfrac{\partial f_0}{\partial z_0}\right) + f_0 = 0, \\ \dfrac{\partial f_0}{\partial x_0} = -\dfrac{\partial u_0}{\partial \overline{z_0}}. \end{cases} \tag{4.8}$$

In particular, a function $f = u_0 + \hat{i} f_0$ from an open set Ω in $\mathbb{T}$ into $\mathbb{T}$ satisfying (R) is called E-regular or monogenic and satisfying (H) is H-solution.

Proof. From the definition of operator $\frac{\partial}{\partial z_0}$, $\frac{\partial}{\partial \overline{z_0}}$ and complex-valued function f_0, we obtain the replacing systems. That is, we can replace (4.1) and (4.2) to

$$\frac{\partial u_0}{\partial x_0} = \frac{\partial f_0}{\partial z_0}.$$

Also, we can replace (4.3) to

$$\frac{\partial f_0}{\partial x_0} = -\frac{\partial u_0}{\partial \overline{z_0}}.$$

Similarly, we can replace (4.4) to

$$z_0 \left(\frac{\partial u_0}{\partial x_0} - \frac{\partial f_0}{\partial z_0} \right) + f_0 = 0.$$

□

Let $z = x_0 + \hat{i} z_0$ in $\mathbb{T}$. By induction, we have

$$z^m = \sum_{\substack{k=0 \\ k:\text{even}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k}{2}} z_0^{\frac{k}{2}} \overline{z_0}^{\frac{k}{2}} + \hat{i} \sum_{\substack{k=1 \\ k:\text{odd}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k-1}{2}} z_0^{\frac{k+1}{2}} \overline{z_0}^{\frac{k-1}{2}},$$

where $k = 0, 1, 2, \cdots$. More simply,

$$z^m = \sum_{k=0}^{m} \binom{m}{k} c(k) x_0^{m-k} z_0^{\left[\frac{k+1}{2}\right]} \overline{z_0}^{\left[\frac{k}{2}\right]},$$

where

$$c(k) = \begin{cases} (-1)^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ (-1)^{\frac{k-1}{2}} \hat{i} & \text{if } k \text{ is odd.} \end{cases}$$

Also, we have

$$\begin{aligned} &\frac{\partial}{\partial z_0} z_0 \\ =& \left\{ \frac{\partial}{\partial x_1} \left(\frac{a}{\sqrt{a^2+b^2}} - \frac{b}{\sqrt{a^2+b^2}} \sqrt{-1} \right) + \frac{\partial}{\partial x_1} \left(\frac{b}{\sqrt{a^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}} \sqrt{-1} \right) \right\} \\ &\times \left\{ \left(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}} \sqrt{-1} \right) x_1 + \left(\frac{b}{\sqrt{a^2+b^2}} - \frac{a}{\sqrt{a^2+b^2}} \sqrt{-1} \right) x_2 \right\} \\ =& 1 + 1 = 2. \end{aligned}$$

Similarly, we obtain

$$\frac{\partial}{\partial \overline{z_0}} \overline{z_0} = 2 \ , \ \frac{\partial}{\partial z_0} \overline{z_0} = \frac{\partial}{\partial \overline{z_0}} z_0 = 0,$$

$$\frac{\partial^n}{\partial z_0^n} z_0^m = 2^n \frac{m!}{(m-n)!} z_0^{m-n} \ , \ \frac{\partial^n}{\partial \overline{z_0}^n} \overline{z_0}^m = 2^n \frac{m!}{(m-n)!} \overline{z_0}^{m-n},$$

where a and b are real numbers both nonzero and m is a positive integer.

Theorem 4.2. *In the system (H), the function z^m is an H-solution, but it is not an E-regular function.*

Proof. We have

$$\begin{aligned}
\frac{\partial u_0}{\partial x_0} &= \frac{\partial}{\partial x_0} \sum_{\substack{k=0\\ k:\text{even}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k}{2}} z_0^{\frac{k}{2}} \overline{z_0}^{\frac{k}{2}} \\
&= \sum_{\substack{k=0\\ k:\text{even}}}^{m} \binom{m}{k} (m-k) x_0^{m-k-1} (-1)^{\frac{k}{2}} z_0^{\frac{k}{2}} \overline{z_0}^{\frac{k}{2}}, \\
\frac{\partial f_0}{\partial z_0} &= \frac{\partial}{\partial z_0} \sum_{\substack{k=1\\ k:\text{odd}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k-1}{2}} z_0^{\frac{k+1}{2}} \overline{z_0}^{\frac{k-1}{2}} \\
&= \sum_{\substack{k=1\\ k:\text{odd}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k-1}{2}} (k+1) z_0^{\frac{k-1}{2}} \overline{z_0}^{\frac{k-1}{2}}, \\
\frac{\partial f_0}{\partial x_0} &= \frac{\partial}{\partial x_0} \sum_{\substack{k=1\\ k:\text{odd}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k-1}{2}} z_0^{\frac{k+1}{2}} \overline{z_0}^{\frac{k-1}{2}} \\
&= \sum_{\substack{k=1\\ k:\text{odd}}}^{m} \binom{m}{k} (m-k) x_0^{m-k-1} (-1)^{\frac{k-1}{2}} z_0^{\frac{k+1}{2}} \overline{z_0}^{\frac{k-1}{2}}, \\
\frac{\partial u_0}{\partial \overline{z_0}} &= \frac{\partial}{\partial \overline{z_0}} \sum_{\substack{k=0\\ k:\text{even}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k}{2}} z_0^{\frac{k}{2}} \overline{z_0}^{\frac{k}{2}} \\
&= \sum_{\substack{k=2\\ k:\text{even}}}^{m} \binom{m}{k} x_0^{m-k} (-1)^{\frac{k}{2}} z_0^{\frac{k}{2}} k \overline{z_0}^{\frac{k}{2}-1}.
\end{aligned}$$

Substituting the above terms into (H)-system satisfies the definition of (H)-solution, we obtain the theorem. □

Theorem 4.3. *The function $D^n z^{m+n}$, $m \geq n$, is an H-solution for any positive integer n.*

Proof. Generalizing D^n by the definition of D, we obtain the following formula:

$$D^n = \sum_{\substack{k=0 \\ k:\text{even}}}^{n} \binom{n}{k} \frac{\partial^{n-k}}{\partial x_0^{n-k}} (-1)^{\frac{k}{2}} \frac{\partial^k}{\partial z_0^{\frac{k}{2}} \overline{z_0}^{\frac{k}{2}}}$$
$$+ \hat{i} \sum_{\substack{k=1 \\ k:\text{odd}}}^{n} \binom{n}{k} \frac{\partial^{n-k}}{\partial x_0^{n-k}} (-1)^{\frac{k+1}{2}} \frac{\partial^k}{\partial z_0^{\frac{k-1}{2}} \overline{z_0}^{\frac{k+1}{2}}}.$$

To further simplify this and apply

$$D^n = \sum_{k=0}^{n} \binom{n}{k} c^*(k) \frac{\partial^{n-k}}{\partial x_0^{n-k}} \frac{\partial^k}{\partial z_0^{\left[\frac{k}{2}\right]} \overline{z_0}^{\left[\frac{k+1}{2}\right]}},$$

where

$$c^*(k) = \begin{cases} (-1)^{\frac{k}{2}} & \text{if } k \text{ is even}, \\ (-1)^{\frac{k+1}{2}} \hat{i} & \text{if } k \text{ is odd}, \end{cases}$$

to z^{m+n}, we get the following formula:

$$D^n z^{m+n} = \frac{(m+n)!}{m!} \sum_{j=0}^{m} \binom{m}{j} c(j) x_0^{m-j} z_0^{\left[\frac{j+1}{2}\right]} \overline{z_0}^{\left[\frac{j}{2}\right]}, \tag{4.9}$$

where

$$c(j) = \begin{cases} (-1)^{\frac{j}{2}} & \text{if } j \text{ is even}, \\ (-1)^{\left[\frac{j}{2}\right]} \hat{i} & \text{if } j \text{ is odd}. \end{cases}$$

Since the equation (4.9) is satisfied Equation (4.8), the result is obtained. □

5. Conclusion

In order to express quaternions as ternary numbers, we propose a pseudo-complex form by defining a pseudo-basis of $\hat{i}$. The form of the function suitable for the new number system is specified and a regular function named E-Regular function is defined. Considering the properties of regular functions in the existing quaternion form, we define the form of a ternary regular function. This preserves the properties of regular functions in quaternion form. Hyper-conjugate harmonic functions are defined from regular functions that deal with the number system based on ternary numbers, and their relationship with regular functions is specified. In addition, by defining Cauchy-Riemann equations suitable for the ternary number form and investigating its properties, it is possible to expect the use of regular functions in the ternary number system.

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