Nonlinear Functional Analysis and Applications
Vol. 17, No. 1 (2012), pp. 131-141

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

LOCAL CONVERGENCE OF A THREE-POINT METHOD FOR SOLVING LEAST PROBLEMS

Ioannis K. Argyros[1] and Hongmin Ren[2]

[1]Department of Mathematical Sciences, Cameron University
Lawton, OK 73505, U.S.A
e-mail: iargyros@cameron.edu

[2]College of Information and Electronics, Hangzhou Radio and TV University
Hangzhou 310012, Zhejiang, P.R.China
e-mail: rhm65@126.com

Abstract. We propose a three-point method for approximating a solution of nonlinear least squares problems. The radius of convergence is determined as well as usable error estimates. Numerical examples are also provided.

1. Introduction

In this study, we are concerned with the problem of approximating a solution $x^\star$ of the nonlinear least square problem

$$\min_{x\in\mathbb{R}^i} \frac{1}{2}F(x)^T F(x), \tag{1.1}$$

where, $F : D \subseteq \mathbb{R}^i \to \mathbb{R}^j$ is a twice continuously Fréchet-differentiable operator $((j \geq i)$ are natural numbers), and D is an open convex subset of $\mathbb{R}^i$.

Nonlinear least square problem can be used to formulate many problems arising in applied sciences [4-12, 16-20]. This problem can be solved by Gauss-Newton-type methods [4, 7, 9, 12, 16-20]. However these methods use the derivative $F'(x)$ of operator F. The computation of the derivative $F'(x)$ is in general very expensive. That is why we propose the alternative iterative method

$$x_{n+1} = x_n - [A_n^T A_n]^{-1} A_n^T F(x_n) \quad (n \geq 0), \tag{1.2}$$

[0]Received August 12, 2011. Revised December 24, 2011.
[0]2000 Mathematics Subject Classification: 65K10, 65G99, 65H10, 65B05, 49M15.
[0]Keywords: Nonlinear least squares problem, three-point iterative method, divided difference, radius of convergence, error estimates.

where, A_n is a linear operator, which is a divided difference or a linear combination of divided differences. We shall only consider the special case, when

$$A_n = [x_n, x_{n-1}; F] + [x_{n-2}, x_n; F] - [x_{n-2}, x_{n-1}; F] \quad (n \geq 0), \tag{1.3}$$

where, $[., ., F]$ is a divided difference of order one [4], [12]. Note that case (1.3) as well as cases

$$A_n = [x_n, x_{n-1}; F] \quad (n \geq 0), \tag{1.4}$$

$$A_n = [2x_n - x_{n-1}, x_{n-1}; F] \quad (n \geq 0), \tag{1.5}$$

were studied in [18]. However the proof in [18] (Theorem 2) using (1.3) is not completed (see Remark 2.4).

Here, we provide a new local convergence analysis for iterative method (1.2) under the special choice of A_n given in (1.3). Numerical examples further validating the theoretical results are also provided in this study. Note also that a local as well as a semilocal convergence analysis of more general method

$$y_{n+1} = y_n - A_n^{-1} F(y_n) \quad (n \geq 0), \tag{1.6}$$

where, $F : D \subseteq B_1 \to B_2$, and B_1, B_2 are Banach spaces and A_n is given by (1.3) can be found in [1]-[5], [8], [12], [14], [15], [20].

Throughout the paper, $\|.\|$ denotes the Euclidean norm.

2. Local convergence analysis of (TPM)

We can show the main local convergence result for (TPM).

Theorem 2.1. *Let $F : \mathbb{R}^i \to \mathbb{R}^j$ be a twice continuously differentiable mapping defined on an open convex subset D of $\mathbb{R}^i$ with values in $\mathbb{R}^j$.*
Assume:
problem (1.1) has a solution $x^\star \in D$, and the inverse operator $L = (A_\star^T A_\star)^{-1} = [F'(x^\star)^T F'(x^\star)]^{-1}$ exists, and

$$\|L\| \leq B; \tag{2.1}$$

F has divided difference of order one and two, and for all $x, y, z, v \in D$:

$$\|[x, y; F] - [x, z; F]\| \leq M\|y - z\|, \tag{2.2}$$

$$\|[x, y; F] - [x^\star, x^\star; F]\| \leq M_0(\|x - x^\star\| + \|y - x^\star\|), \tag{2.3}$$

and

$$\|[v, x, y; F] - [z, x, y; F]\| \leq N\|v - z\|; \tag{2.4}$$

the following hold

$$\|F(x^\star)\| \leq \eta, \tag{2.5}$$

$$\|F'(x^\star)\| \leq \alpha, \tag{2.6}$$

$$4BM_0\eta < 1, \tag{2.7}$$

and

$$U(x^\star, r_\star) = \{x : \|x - x^\star\| < r_\star\} \subseteq D, \tag{2.8}$$

where, $r_\star$ is the unique positive root of polynomial q given by

$$\begin{aligned} q(r) &= B[(\alpha + 2(M_0 + M)r)(M + 4Nr)r + 2(M_0 + M)\eta] \\ &\quad +4(M_0 + M)B(\alpha + (M_0 + M)r)r - 1. \end{aligned} \tag{2.9}$$

Then, for $x_{-2}, x_{-1}, x_0 \in U(x^\star, r_\star)$, iterative process $\{x_n\}$ generated by (TPM) for A_n given by (1.3) *is well defined, remains in $U(x^\star, r_\star)$ for all $n \geq 0$, and converges to $x^\star$.*

Moreover, the following error estimates hold for all $n \geq 0$:

$$e_{n+1} = \|x_{n+1} - x^\star\| \leq \overline{g}_n \overline{\delta}_n \leq g_n \delta_n, \tag{2.10}$$

where,

$$\overline{g}_n = \frac{B}{1 - B[2\alpha + M_0(e_n + e_{n-1}) + M\|x_n - x_{n-1}\|][M_0(e_n + e_{n-1}) + M\|x_n - x_{n-1}\|]}, \tag{2.11}$$

$$g_n = \frac{B}{1 - B[2\alpha + (M_0 + M)(e_n + e_{n-1})](M_0 + M)(e_n + e_{n-1})}, \tag{2.12}$$

$$\begin{aligned} \overline{\delta}_n &= \big(\alpha + (M_0 + M)(e_n + e_{n-1})\big)(Me_n + N\|x_n - x_{n-2}\|\|x_n - x_{n-1}\|)e_n \\ &\quad +(M_0 + M)(e_n + e_{n-1})\eta, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \delta_n &= \big(\alpha + (M_0 + M)(e_n + e_{n-1})\big)\big(Me_n + N(e_n + e_{n-2})(e_n + e_{n-1})\big)e_n \\ &\quad +(M_0 + M)(e_n + e_{n-1})\eta. \end{aligned} \tag{2.14}$$

Proof. Polynomial q has a positive root denoted by $r_\star$, by the intermediate value theorem applied on $[0, r]$, for sufficiently large $r > 0$, and (2.7). Moreover, $q'(r) \geq 0$, $r \geq 0$. That is the graph of function q crosses the positive $x - axis$ only once.

By hypothesis, $x_{-2}, x_{-1}, x_0 \in U(x^\star, r_\star)$. Using (2.1), (2.3), (2.6), and the definition of $r_\star$, we obtain in turn:

$$\begin{aligned} \|I - LA_0^T A_0\| &= \|L[A_0^T(A_0 - A_\star) + (A_0 - A_\star)^T A_\star]\| \\ &\leq B(\|A_0^T\|\|A_0 - A_\star\| + \|(A_0 - A_\star)^T\|\|A_\star\|) \\ &= B(\|A_0\| + \|A_\star\|)\|A_0 - A_\star\| \\ &\leq B(\|A_\star\| + \|A_0 - A_\star\| + \|A_\star\|)\|A_0 - A_\star\| \\ &\leq B\big(2\alpha + M_0(\|x_0 - x^\star\| + \|x_{-1} - x^\star\|) + M\|x_0 - x_{-1}\|\big) \\ &\quad \times\big(M_0(\|x_0 - x^\star\| + \|x_{-1} - x^\star\|) + M\|x_0 - x_{-1}\|\big) \\ &\leq 4(M_0 + M)B(\alpha + (M_0 + M)r_\star)r_\star \\ &= 1 - B[(\alpha + 2(M_0 + M)r)(M + 4Nr)r + 2(M_0 + M)\eta] \\ &< 1. \end{aligned} \tag{2.15}$$

Let us define functions g, and p by

$$g(r) = \frac{B}{1-4B(M_0+M)(\alpha+(M_0+M)r)r}, \tag{2.16}$$

and

$$p(r) = g(r)[(\alpha + 2(M_0 + M)r)(M + 4Nr)r + 2(M_0 + M)\eta]. \tag{2.17}$$

It then follows from (2.15), and the Banach lemma on invertible operators [4], [12] that $(A_0^T A_0)^{-1}$ exists, and

$$\|(A_0^T A_0)^{-1}\| \leq g_0 \leq g(r_\star). \tag{2.18}$$

Consequently, the iterate x_1 is well defined.

Next, we will show $x_1 \in U(x^\star, r_\star)$. Using the fact $A_\star^T F(x^\star) = 0$, (TPM), (2.1), (2.2), (2.4), (2.5), (2.18), the definition of $r_\star$, and the approximation

$$\begin{aligned}
[x_k, x^\star; F] - A_k &= [x_k, x^\star; F] - [x_k, x_k; F] + [x_k, x_k; F] - [x_k, x_{k-1}; F] \\
&\quad -([x_{k-2}, x_k; F] - [x_{k-2}, x_{k-1}; F]) \\
&= ([x_k, x^\star; F] - [x_k, x_k; F]) + ([x_k, x_k, x_{k-1}; F] \\
&\quad -[x_{k-2}, x_k, x_{k-1}; F])(x_k - x_{k-1})
\end{aligned} \tag{2.19}$$

for $k = 0$, we get

$$\begin{aligned}
\|x_1 - x^\star\| &= \| - (A_0^T A_0)^{-1}\big(A_0^T F(x_0) - A_0^T A_0(x_0 - x^\star)\big)\| \\
&= \|(A_0^T A_0)^{-1} A_0^T \big(F(x_0) - F(x^\star) + F(x^\star) - A_0(x_0 - x^\star)\big)\| \\
&\leq \|(A_0^T A_0)^{-1}\|\|(A_0^T - A_\star^T + A_\star^T)\big(([x_0, x^\star; F] - A_0)(x_0 - x^\star) \\
&\quad +F(x^\star)\big)\| \\
&\leq g_0\|(A_0 - A_\star)^T\big(([x_0, x^\star; F] - A_0)(x_0 - x^\star) + F(x^\star)\big) \\
&\quad +A_\star^T\big(([x_0, x^\star; F] - A_0)(x_0 - x^\star)\big)\| \\
&\leq g_0\big((\|A_0 - A_\star\| + \|A_\star\|)\|A_0 - [x_0, x^\star; F]\|\|x_0 - x^\star\| \\
&\quad +\|A_0 - A_\star\|\|F(x^\star)\|\big) \\
&\leq g_0\big[\big(\alpha + M_0(\|x_0 - x^\star\| + \|x_{-1} - x^\star\|) + M\|x_0 - x_{-1}\|\big) \\
&\quad \times(M\|x_0 - x^\star\| + N\|x_0 - x_{-2}\|\|x_0 - x_{-1}\|)\|x_0 - x^\star\| \\
&\quad +\big(M_0(\|x_0 - x^\star\| + \|x_{-1} - x^\star\|) + M\|x_0 - x_{-1}\|\big)\eta\big] \\
&< g(r_\star)[(\alpha + 2(M_0 + M)r_\star)(M + 4Nr_\star)r_\star + 2(M_0 + M)\eta]r_\star \\
&= p(r_\star)r_\star = r_\star,
\end{aligned} \tag{2.20}$$

which shows $x_1 \in U(x^\star, r_\star)$, and estimate (2.10) holds for $n = 0$.

Let us assume $x_n \in U(x^\star, r_\star)$ for $n = 0, 1, \ldots, k$, and estimate (2.10) holds for $n = 0, 1, \ldots, k-1$, and $k \geq 1$ is an integer. We shall show: $x_{k+1} \in U(x^\star, r_\star)$, and estimate (2.10) holds for $n = k$.

By simply replacing above x_{-1}, x_0 by x_{k-1}, x_k, respectively, we get

$$
\begin{array}{rcl}
\|I - LA_k^T A_k\| & \leq & B(\|A_k\| + \|A_\star\|)\|A_k - A_\star\| \\
& \leq & B(2\|A_\star\| + \|A_k - A_\star\|)\|A_k - A_\star\| \\
& \leq & B\big(2\alpha + M_0(\|x_k - x^\star\| + \|x_{k-1} - x^\star\|) + M\|x_k - x_{k-1}\|\big) \\
& & \times\big(M_0(\|x_k - x^\star\| + \|x_{k-1} - x^\star\|) + M\|x_k - x_{k-1}\|\big) \\
& \leq & 4(M_0 + M)B(\alpha + (M_0 + M)r_\star)r_\star < 1.
\end{array}
\tag{2.21}
$$

Hence, again $(A_k^T A_k)^{-1}$ exists, and

$$
\|(A_k^T A_k)^{-1}\| \quad \leq \quad g_k \leq g(r_\star). \tag{2.22}
$$

That is, iterate x_{k+1} is well defined. Moreover, as in (2.20) we obtain in turn:

$$
\begin{array}{rcl}
\|x_{k+1} - x^\star\| & = & \|(A_k^T A_k)^{-1} A_k^T\big(F(x_k) - F(x^\star) + F(x^\star) - A_k(x_k - x^\star)\big)\| \\
& \leq & g_k\|(A_k^T - A_\star^T + A_\star^T)\big(([x_k, x^\star; F] - A_k)(x_k - x^\star) + F(x^\star)\big)\| \\
& \leq & g_k\big((\|A_k - A_\star\| + \|A_\star\|)\|[x_k, x^\star; F] - A_k\|\|x_k - x^\star\| \\
& & +\|A_k - A_\star\|\|F(x^\star)\|\big) \\
& \leq & g_k\big[\big(\alpha + M_0(\|x_k - x^\star\| + \|x_{k-1} - x^\star\|) + M\|x_k - x_{k-1}\|\big) \\
& & \times(M\|x_k - x^\star\| + N\|x_k - x_{k-2}\|\|x_k - x_{k-1}\|)\|x_k - x^\star\| \\
& & +\big(M_0(\|x_k - x^\star\| + \|x_{k-1} - x^\star\|) + M\|x_k - x_{k-1}\|\big)\eta\big] \\
& < & g(r_\star)[(\alpha + 2(M_0 + M)r_\star)(M + 4Nr_\star)r_\star + 2(M_0 + M)\eta]r_\star \\
& = & p(r_\star)r_\star = r_\star,
\end{array}
\tag{2.23}
$$

which shows $x_{k+1} \in U(x^\star, r_\star)$, and estimate (2.10) holds for $n = k$.

By induction, iterative process $\{x_n\}$ generated by (TPM) for A_n given by (1.3) is well defined, remains in $U(x^\star, r_\star)$ for all $n \geq 0$, and estimate (2.10) holds for all $n \geq 0$.

Next, we shall show $x_n \to x^\star$ as $n \to \infty$.

Define functions a, and b on $[0, r_\star]$ by:

$$
a(r) = [(\alpha + 2(M_0 + M)r)(M + 4Nr)r + (M_0 + M)\eta]g(r), \tag{2.24}
$$

and

$$
b(r) = (M_0 + M)\eta g(r). \tag{2.25}
$$

By the choice of $r_\star$, we get

$$
a(r_\star) \geq 0, \quad b(r_\star) \geq 0, \quad a(r_\star) + b(r_\star) = 1. \tag{2.26}
$$

Using estimate (2.10), the definition of the C' constants and functions a, b, we obtain for $n \geq 0$

$$
e_{n+1} \leq a(r_\star)e_n + b(r_\star)e_{n-1}. \tag{2.27}
$$

On the other hand, for the real number $r_\star > 0$, and initial points x_0, x_{-1} belong to $U(x^\star, r_\star)$, there must exist a real number r', such that $0 < r' < r_\star$,

$x_0, x_{-1} \in U(x^\star, r')$, and all estimates about $\{x_n\}$ above will be true if $r_\star$ is replaced by r'. Specially, we get from (2.27) for $n \geq 0$:

$$e_{n+1} \leq ae_n + be_{n-1}, \tag{2.28}$$

where,

$$a = a(r'), \quad b = b(r'). \tag{2.29}$$

Clearly we also have

$$a \geq 0, \quad b \geq 0, \quad a + b < a(r_\star) + b(r_\star) = 1. \tag{2.30}$$

Define sequences $\{\theta_n\}$ and $\{\rho_n\}$ as:

$$\theta_n = \frac{e_n}{r'}, \quad n = -1, 0, 1, \ldots, \tag{2.31}$$

$$\rho_{-1} = \theta_{-1}, \quad \rho_0 = \theta_0, \quad \rho_{n+1} = a\rho_n + b\rho_{n-1}, \quad n = 0, 1, 2, \ldots. \tag{2.32}$$

Dividing two sides of inequality (2.28) by r' gives

$$\theta_{n+1} \leq a\theta_n + b\theta_{n-1}, \quad n = 0, 1, 2, \ldots. \tag{2.33}$$

By the definition of sequence $\{\rho_n\}$, we get

$$0 \leq \theta_n \leq \rho_n, \quad n = -1, 0, 1, \ldots. \tag{2.34}$$

Moreover, it is known that an explicit formula of sequence $\{\rho_n\}$ can be given by:

$$\rho_n = \omega_1\lambda_1^n + \omega_2\lambda_2^n, \quad n = -1, 0, 1, \ldots, \tag{2.35}$$

where,

$$\lambda_1 = \frac{a - \sqrt{a^2 + 4b}}{2}, \quad \lambda_2 = \frac{a + \sqrt{a^2 + 4b}}{2}, \tag{2.36}$$

and

$$\omega_1 = \frac{\lambda_2^{-1}\rho_0 - \rho_{-1}}{\lambda_2^{-1} - \lambda_1^{-1}}, \quad \omega_2 = \frac{\rho_{-1} - \lambda_1^{-1}\rho_0}{\lambda_2^{-1} - \lambda_1^{-1}}. \tag{2.37}$$

Note that

$$0 \leq |\lambda_1| \leq |\lambda_2| < \frac{a + \sqrt{a^2 + 4(1-a)}}{2} = \frac{a + 2 - a}{2} = 1. \tag{2.38}$$

Using (2.34) and (2.38), we deduce that $\{\theta_n\} \to 0, (n \to \infty)$, that is, $x_n \to x^\star, (n \to \infty)$. That completes the proof of the Theorem 2.1. □

In the case of zero residual ($\eta = 0$), we have the special case of Theorem 2.1.

Theorem 2.2. *Let $F : \mathbb{R}^i \to \mathbb{R}^j$ be a continuously differentiable mapping defined on an open convex subset D of $\mathbb{R}^i$ with values in $\mathbb{R}^j$.*
Assume:
the problem (1.1) *has a solution $x^\star \in D$, and the inverse operator L exists, and*

$$\|L\| \leq B;$$

F has divided differences of order one and two, and for all $x, y, z, v \in D$:

$$\|[x, y; F] - [x, z; F]\| \leq M\|y - z\|,$$

$$\|[x, y; F] - [x^\star, x^\star; F]\| \leq M_0(\|x - x^\star\| + \|y - x^\star\|),$$

and

$$\|[v, x, y; F] - [z, x, y; F]\| \leq N\|v - z\|;$$

the following hold

$$\|F'(x^\star)\| \leq \alpha,$$

$$4BM_0\eta < 1,$$

and

$$U(x^\star, r_\star) \subseteq D,$$

where, $r_\star$ is the unique positive root of polynomial q given by

$$q(r) = B[(\alpha + 2(M_0 + M)r)(M + 4Nr)r] + 4(M_0 + M)B(\alpha + (M_0 + M)r)r - 1.$$

Then, for $x_{-2}, x_{-1}, x_0 \in U(x^\star, r_\star)$, iterative process $\{x_n\}$ generated by (TPM) for A_n given by (1.3) *is well defined, remains in $U(x^\star, r_\star)$ for all $n \geq 0$, and converges to $x^\star$.*

Moreover, the following error estimates hold for all $n \geq 0$:

$$e_{n+1} \leq \overline{g}_n \overline{\zeta}_n \leq g_n \zeta_n, \tag{2.39}$$

$$e_{n+1} \leq C e_{n-2} e_{n-1} e_n, \tag{2.40}$$

where, $\overline{\zeta}_n$, ζ_n are obtained from $\overline{\delta}_n$, δ_n, respectively by letting $\eta = 0$ in (2.13) *and* (2.14)*, and $C > 0$.*

Furthermore, the order of convergence of (TPM) is $1.839\ldots$*.*

Proof. All claims but (2.40) have been established in Theorem 2.1.

In view of (2.39), there exists a natural number N_0, and constant $C > 0$ such that (2.40) holds for all $n \geq N_0$. This shows that the $R-$order of convergence of (TPM) is equal to the unique positive root of equation

$$t^3 - t^2 - t - 1 = 0, \tag{2.41}$$

which is $1.839\ldots$. That completes the proof of the Theorem 2.2. $\square$

Remark 2.3. (a) Hypotheses (2.2) and (2.3) can be replaced by the stronger but popular condition (yet not needed in the proof of Theorem 2.1 and 2.2)

$$\|[x,y;F]-[u,v;F]\| \le K(\|x-u\|+\|y-v\|).$$

Note however that

$$M \le K,$$
$$M_0 \le K,$$

hold in general and $\frac{K}{M}$ and $\frac{K}{M_0}$ can be arbitrarily large [3], [4].

(b) A popular choice for the divided difference is given by

$$[x,y;F] = \int_0^1 F'(y+t(x-y))dt.$$

Remark 2.4. A similar result was given in [18, Theorem 2]. However, the proof is not complete. Indeed, they arrive at the estimate

$$\|x_{n+1}-x^\star\| < \|x_{n-1}-x^\star\| < R_\star,$$

which obviously shows $x_{n+1} \in U(x^\star, R_\star)$. However, it is not shown that $x_n \to x^\star$ as $n \to \infty$.

Remark 2.5. We arrived at estimate (2.21) using only (2.2), and (2.3). Instead, if we use (2.2), (2.3), and (2.5), we get

$$\begin{aligned}
\|A_k - A_\star\| &= \|[x^\star,x^\star;F]-[x_k,x^\star;F]+[x_{k-2},x^\star;F]-[x_{k-2},x_k;F] \\
&\quad +[x_k,x^\star;F]-[x_k,x_{k-1};F]-[x_{k-2},x^\star;F]+[x_{k-2},x_{k-1};F]\| \\
&= \|([x^\star,x^\star;F]-[x_k,x^\star;F])+([x_{k-2},x^\star;F]-[x_{k-2},x_k;F]) \\
&\quad +([x_k,x^\star,x_{k-1};F]-[x_{k-2},x^\star,x_{k-1};F])(x^\star-x_{k-1})\| \\
&\le (M_0+M)\|x_k-x^\star\|+N\|x_k-x_{k-2}\|\|x_{k-1}-x^\star\|.
\end{aligned}$$

Define

$$\begin{aligned}
&\overline{\xi}_k = (M_0+M)e_k + N\|x_k-x_{k-2}\|e_{k-1}, \\
&\xi_k = (M_0+M)e_k + N(e_k+e_{k-2})e_{k-1}, \\
&\xi(r) = (M_0+M+2Nr)r, \\
&\overline{G}_k = \tfrac{B}{1-B(2\alpha+\overline{\xi}_k)\overline{\xi}_k}, \\
&G_k = \tfrac{B}{1-B(2\alpha+\xi_k)\xi_k}, \\
&G(r) = \tfrac{B}{1-B(2\alpha+\xi(r))\xi(r)}.
\end{aligned}$$

Then, as in (2.21) we arrive at

$$\begin{aligned}
\|I-LA_k^TA_k\| &\le B(2\alpha+\overline{\xi}_k)\overline{\xi}_k \\
&\le B(2\alpha+\xi_k)\xi_k.
\end{aligned}$$

Moreover, define

$$q(r) = B[(\alpha+\xi(r))(M+4Nr)+(M_0+M+2Nr)\eta]+B(2\alpha+\xi(r))\xi(r)-1.$$

Then, with the above changes the conclusions of Theorem 2.1 hold in this setting. Let us call the new result Theorem 2.1′. We shall use in practice both results to obtain the largest radius of convergence, and the finer error estimates on the distances $\|x_n - x^\star\|$.

Not however that clearly, if

$$g(r) < G(r), \tag{2.42}$$

then Theorem 2.1 provides a larger convergence radius and finer error estimate than Theorem 2.1′.

If equality holds in (2.42), the results provide the same information. Otherwise Theorem 2.1′ is better than Theorem 2.1.

3. Numerical example

In this section, we give an example to show the application of our results.

Example 3.1. Consider function $F : D = \mathbb{R} \subseteq \mathbb{R} \to \mathbb{R}^2$ given by:

$$F(x) = \begin{pmatrix} x+1 \\ \lambda x^2 + x - 1 \end{pmatrix}, \tag{3.1}$$

where $\lambda \in \mathbb{R}$ is a parameter. It follows that

$$F'(x) = \begin{pmatrix} 1 \\ 2\lambda x + 1 \end{pmatrix}, \tag{3.2}$$

$$F''(x) = \begin{pmatrix} 0 \\ 2\lambda \end{pmatrix}, \tag{3.3}$$

and $x^\star = 0$ is a solution of function

$$\begin{aligned} F'(x)^T F(x) &= \begin{pmatrix} 1 & 2\lambda x + 1 \end{pmatrix} \begin{pmatrix} x+1 \\ \lambda x^2 + x - 1 \end{pmatrix} \\ &= x + 1 + (2\lambda x + 1)(\lambda x^2 + x - 1). \end{aligned} \tag{3.4}$$

That is, $x^\star = 0$ is a solution of problem (1.1). In view of

$$F(x^\star) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{3.5}$$

$$F'(x^\star) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3.6}$$

and

$$F'(x^\star)^T F'(x^\star) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2, \tag{3.7}$$

we can set constants B, η, and α in Theorem 2.1 as follows:

$$B = \frac{1}{2}, \quad \eta = \sqrt{2}, \quad \alpha = \sqrt{2}. \tag{3.8}$$

Moreover, using (3.2) and (3.3), we can get for all $x, y \in D \subseteq \mathbb{R}$:

$$\|F'(x) - F'(y)\| = \| \begin{pmatrix} 0 \\ 2\lambda(x-y) \end{pmatrix} \| = 2|\lambda||x-y|, \tag{3.9}$$

and

$$\|F''(x) - F''(y)\| = \| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \| = 0|x-y|. \tag{3.10}$$

Using divided difference of order $[x, y; F]$ as given in Remark 2.3 (b) (see also [14]), we can get for all $x, y, u, v \in D \subseteq \mathbb{R}$:

$$\begin{aligned} \|[x,y;F] - [u,v;F]\| &= \| \int_0^1 \big(F'(tx + (1-t)y)dt - F'(tu + (1-t)v)\big)dt\| \\ &\leq \int_0^1 2|\lambda||t(x-u) + (1-t)(y-v)|dt \\ &\leq |\lambda|(|x-u| + |y-v|). \end{aligned} \tag{3.11}$$

That is, we can set constant $M = M_0 = |\lambda|$ and $N = 0$ in Theorem 2.1. Hence, the function $q(r)$ can be given by:

$$\begin{aligned} q(r) &= B[(\alpha + 2(M_0 + M)r)(M + 4Nr)r + 2(M_0 + M)\eta] \\ &\quad +4(M_0 + M)B(\alpha + (M_0 + M)r)r - 1 \\ &= \tfrac{1}{2}(\sqrt{2} + 4|\lambda|r)(|\lambda|r) + 2\sqrt{2}|\lambda| + 4(\sqrt{2} + 2|\lambda|r)(|\lambda|r) - 1 \\ &= 10(|\lambda|r)^2 + \tfrac{9\sqrt{2}}{2}|\lambda|r + 2\sqrt{2}|\lambda| - 1, \end{aligned} \tag{3.12}$$

which has a unique positive root $r_\star$:

$$r_\star = \frac{\sqrt{322 - 320\sqrt{2}|\lambda|} - 9\sqrt{2}}{40|\lambda|}. \tag{3.13}$$

If we choose parameter λ such that

$$0 < |\lambda| < \frac{\sqrt{2}}{4}, \tag{3.14}$$

then $4BM\eta = 2\sqrt{2}|\lambda| < 1$, and $U(x^\star, r_\star) \subseteq D$. Therefore, all conditions in Theorem 2.1 are satisfied, and Theorem 2.1 applies.

References

[1] I. K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl. **298** (2004), 374–397.

[2] I .K. Argyros, *On a two-point Newton-like method of convergent order two*, Internat. J. Comput. Math. **82** (2005), 219–233.

[3] I. K. Argyros, *A Kantorovich-type analysis for a fast iterative method for solving nonlinear equations*, J. Math. Anal. and Applic. **332** (2007), 97–108.

[4] I. K. Argyros, *Computational Theory of Iterative Methods, Series: Studies in Computational Mathematics 15*, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co., New York, U.S.A., 2007.

[5] I. K. Argyros, *Convergence domains for some iterative processes in Banach spaces using outer or generalized inverses*, J. Comput. Anal. Appl. **1** (1999), 87–104.

[6] A. Ben-Israel and N. E. Greville, *Generalized Inverses: Theory and Applications, Pure and Applied Mathematics*, Wiley-interscience, John Wiley and Sons, London, New York, Sydney, 1974.

[7] K. M. Brown and J. M. Dennis Jr., *Derivative free analogues of the Levenberg-Marquardt and Gauss algorithms for nonlinear least squares approximation*, Numer. Math. **18** (1972), 289–297.

[8] X. Chen and M. Z. Nashed, *Convergence of Newton-like methods for singular operator equations using outer inverses*, Numer. Math. **66** (1993), 235–257.

[9] J. M. Dennis Jr. and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, New York, 1983.

[10] J. J. More, B. S. Garbow and K. E. Hillstrom, *Testing unconstrained optimization software*, ACM Trans. Math. Software **7** (1981), 17–41.

[11] H. Mühlig, *Lösung praktischer Approximationsaufgaben durch parameteridentifika- tion*, ZAMM **73** (1993), T837–839.

[12] J. M. Ortega and W. C. Rheinbolt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, U.S.A., 1970.

[13] F. A. Potra, *On an iterative algorithm of order 1.839... for solving nonlinear operator equations*, Numer. Funct. Anal. Optimiz. **7** (1) (1984-85), 75–106.

[14] H. M. Ren and Q. B. Wu, *The convergence ball of the Secant method under Hölder continuous divided differences*, J. Comput. Appl. Math. **194** (2006), 284–293.

[15] H. M. Ren and Q. B. Wu, *Mysovskii-type theorem for the Secant method under Hölder continuous Frchet derivative*, J. Math. Anal. Appl. **320** (2006), 415–424.

[16] H. Schwetlick, *Numerische Lösung nichtlinearer Gleichungen*, R. Oldenburg Verlag, Munchen Wien, 1979.

[17] S. M. Shakhno and O. P. Gnatyshyn, *Iterative-difference methods for solving nonlinear leastsquares problem, in: L. Arkeryd, J. Bergh, P. Brenner, R. Petersson (Eds.)*, Progress in Industrial Mathematics at ECMI'98, Teubner, Stuttgart, 1999, 287–294.

[18] S. M. Shakhno and O. P. Gnatyshyn, *On an iterative algorithm of order 1.839... for solving the nonlinear least squares problems*, Appl. Math. Comput. **161** (2005), 253–264.

[19] J. H. Wilkinson, *The Algebraic Eigenvalue Problem, Clarendon*, Oxford, 1965.

[20] T. Yamamoto, *Uniqueness of the solution in a Kantorovich-type theorem of HauBler for the Gauss-Newton method*, Japan J. Appl. Math. **6** (1989), 77–81.