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STABLE PERTURBED ITERATION PROCEDURES FOR SOLVING NEW STRONGLY NONLINEAR OPERATOR INCLUSIONS IN BANACH SPACES

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Abstract. The purpose of this paper is to introduce and study a new kind of strongly nonlinear operator inclusions involving generalized m-accretive mapping in Banach spaces. By using the resolvent operator technique associated with the generalized m-accretive mapping due to Huang and Fang, we prove the existence theorems of the solution for this kind of operator inclusions in Banach spaces, and discuss the convergence and stability of a new perturbed iterative algorithm with mixed errors for solving this class of nonlinear operator inclusions in Banach spaces. Our results improve and generalize the corresponding results of recent works.

1. Introduction

In this paper, we introduce and study the following new kind of strongly nonlinear operator inclusion problems involving generalized m-accretive mapping:

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Find $x \in X$ such that $p(x) \in Dom(M)$ and

$$f \in N(S(x), T(x), U(x)) + M(p(x)), \tag{1.1}$$

where f is an any given element on X, a real Banach space, $S, T, U, p: X \to X$ and $N: X \times X \times X \to X$ are single-valued mappings and $M: X \to 2^X$ is a generalized m-accretive mapping, 2^X denotes the family of all the nonempty subsets of X.

We remark that for a suitable choice of f, the mappings N, η, S, T, U, M, p and the space X, a number of known new classes variational inequalities, variational inclusions and corresponding optimization problems can be obtained as special cases of nonlinear quasi-variational inclusion problem (1.1). Moreover, these classes variational inclusions provide us a general and unified framework for studying a wide range of interesting and important problems arising in mathematics, physics, engineering sciences and economics finance, etc. See for more details [1, 3, 5, 7, 10, 11, 17, 18, 20] and the references therein.

The study of such types of problems is motivated by an increasing interest in the operator inclusion problems with applications in Banach spaces. It is well known that if $T: X \to 2^X$ is a multi-valued operator, where 2^X denotes the family of all the nonempty subsets of X, then the following operator inclusion problem of finding $x \in X$ such that

$$0 \in T(u) \tag{1.2}$$

has been studied extensively because of its role in the modelization of unilateral problems, nonlinear dissipative systems, convex optimizations, equilibrium problems, knowledge engineering, etc. For details, we can refer to [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 20] and the references therein. Concerning the development of iterative algorithms for the problem (1.2) in the literature, a very common assumption is that T is a maximal monotone operator or m-accretive operator. When T is maximal monotone or m-accretive, many iterative algorithms have been constructed to approximate the solutions of the problem (1.2).

Recently, Huang and Fang [8] first introduced the concept of a generalized m-accretive mapping, which is a generalization of an m-accretive mapping, and gave the definition and properties of the resolvent operator for the generalized m-accretive mapping in Banach space. Furthermore, Bi et al. [2] introduced and studied some new class of nonlinear variational inclusions involving generalized m-accretive mappings in Banach spaces, they also obtained some new corresponding existence and convergence results (see, [6, 9]). On the other hand, many authors discussed stability of the iterative sequence generated by the algorithm for solving the problems what they studied. See, for example, [4, 7, 10, 11, 12, 14, 15, 19, 20]) and the references therein.

Motivated and inspired by the above works, in this paper, we shall introduce and study the new kind of strongly nonlinear operator inclusion problems (1.1) involving generalized m-accretive mapping in Banach spaces. By using the resolvent operator technique for generalized m-accretive mapping due to Huang and Fang [8, 9], we prove the existence theorems of the solution for this kind of operator inclusion problems in Banach spaces, and discuss the convergence and stability of a new perturbed iterative algorithm for solving this class of nonlinear operator inclusion problems in Banach spaces. Our results improve and generalize the corresponding results of [3, 7, 10, 11, 13].

2. Preliminaries

Throughout this paper, let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex (see, for example, [16]). If $X = H$ is a Hilbert space, then J_2 becomes the identity mapping of H. In what follows we shall denote the single-valued generalized duality mapping by j_q.

The modules of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\{\frac{1}{2}\|x + y\| + \|x - y\| - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called uniformly smooth if $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$ and X is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_X \leq ct^q$, where $q > 1$ is a real number.

It is well known that Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q-uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [16] proved the following result:

Lemma 2.1. *Let $q > 1$ be a given real number and X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$, $j_q(x) \in J_q(x)$, there holds* *the following inequality*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

Definition 2.2. Let X be a q-uniformly smooth Banach space. Then the mapping $g : X \to X$ is said to be

(i) α-strongly accretive, if for any $x, y \in X$, there exist $j_q(x-y) \in J_q(x-y)$ and constant $\alpha > 0$ such that

$$\langle g(x) - g(y), j_q(x-y)\rangle \geq \alpha\|x-y\|^q;$$

(ii) β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|g(x) - g(y)\| \leq \beta\|x-y\|, \quad \forall x, y \in X.$$

Definition 2.3. Let X be a q-uniformly smooth Banach space and $h, g : X \to X$ be two single-valued mappings. Then the mapping $N : X \times X \times X \to X$ is said to be

(i) σ-strongly accretive with respect to h in the first argument, if for any $x, y \in X$, there exist $j_q(x-y) \in J_q(x-y)$ and constant $\sigma > 0$ such that

$$\langle N(h(x), \cdot, \cdot) - N(h(y), \cdot, \cdot), j_q(x-y)\rangle \geq \sigma\|x-y\|^q;$$

(ii) ς-relaxed accretive with respect to g in the second argument, if for any $x, y \in X$, there exist $j_q(x-y) \in J_q(x-y)$ and constant $\varsigma > 0$ such that

$$\langle N(\cdot, g(x), \cdot) - N(\cdot, g(y), \cdot), j_q(x-y)\rangle \geq -\varsigma\|x-y\|^q;$$

(iii) ϵ-Lipschitz continuous with respect to the first argument, if there exists constant $\epsilon > 0$ such that

$$\|N(x, \cdot, \cdot) - N(y, \cdot, \cdot)\| \leq \epsilon\|x-y\|, \quad \forall x, y \in X.$$

Similarly, we can define the ξ, γ-Lipschitz continuity in the second and third argument of $N(\cdot, \cdot, \cdot)$, respectively.

Definition 2.4. ([8]) Let $\eta : X \times X \to X^*$ be a single-valued mapping and $A : X \to 2^X$ be a multi-valued mapping. Then A is said to be

(i) η-accretive if

$$\langle u - v, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in X, u \in A(x), v \in A(y);$$

(ii) generalized m-accretive if M is η-accretive and $(I + \lambda M)(X) = X$ for all (equivalently, for some) $\lambda > 0$.

Remark 2.5. Huang and Fang gave one example of the generalized m-accretive mapping in [8]. If $X = X^* = H$ is a Hilbert space, then (i), (ii) of Definition reduce to the definition of η-monotonicity and maximal η-monotonicity respectively; if X is uniformly smooth and $\eta(x,y) = J_2(x-y)$, then (i) and (ii) of Definition reduce to the definitions of accretivity and m-accretivity in uniformly smooth Banach spaces, respectively (see [8, 9]).

Definition 2.6. The mapping $\eta : X \times X \to X^*$ is said to be

(i) δ-strongly monotone, if there exists a constant $\delta > 0$ such that

$$\langle x-y, \eta(x,y)\rangle \geq \delta\|x-y\|^2, \quad \forall x,y \in X;$$

(ii) τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x-y\|, \quad \forall x,y \in X.$$

In [8], Huang and Fang show that for any $\rho > 0$, inverse mapping $(I+\rho A)^{-1}$ is single-valued, if $\eta : X \times X \to X^*$ is strict monotone and $A : X \to 2^X$ is a generalized m-accretive mapping, where I is the identity mapping. Based on this fact, Huang and Fang [8] gave the following definition:

Definition 2.7. Let $A : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_A^ρ for A is defined as follows:

$$J_A^\rho(z) = (I+\rho A)^{-1}(z), \quad \forall z \in X,$$

where $\rho > 0$ is a constant and $\eta : X \times X \to X^*$ is a strictly monotone mapping.

Lemma 2.8. ([8, 9]) *Let X be a q-uniformly smooth Banach space and $\eta : X \times X \to X^*$ be τ-Lipschitz continuous and δ-strongly monotone, and $A : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_A^ρ for A is $\frac{\tau}{\delta}$-Lipschitz continuous, i.e.,*

$$\|J_A^\rho(x) - J_A^\rho(y)\| \leq \frac{\tau}{\delta}\|x-y\|, \quad \forall x,y \in X.$$

3. Existence Theorems

In this section, we shall give the existence theorems of problem (1.1). Firstly, from the definition of J_M^ρ, we have the following lemma:

Lemma 3.1. *x is the solution of problem* (1.1) *if and only if*

$$p(x) = J_M^\rho[p(x) - \rho(N(S(x),T(x),U(x)) - f)]. \tag{3.1}$$

Theorem 3.2. *Assume that X is a q-uniformly smooth Banach space, mappings $S, T, U : X \to X$ is κ, μ, ν-Lipschitz continuous, respectively. Let $p : X \to X$ be α-strongly accretive and β-Lipschitz continuous, $N : X \times X \times X \to X$ be σ-strongly accretive with respect to S in the first argument and ς-relaxed accretive with respect to T in the second argument, and ϵ, ξ, γ-Lipschitz continuous in the first, second and third argument, respectively. Let $\eta : X \times X \to X^*$ be τ-Lipschitz continuous and δ-strongly monotone, and $M : X \to 2^X$ be generalized m-accretive. If*

$$\begin{cases} h = (1 + \frac{\tau}{\delta})(1 - q\alpha + c_q \beta^q)^{\frac{1}{q}} < 1, \\ \tau[(1 - q\rho(\sigma - \varsigma) + c_q \rho^q (\epsilon\kappa + \xi\mu)^q)^{\frac{1}{q}} + \rho\gamma\nu] < \delta(1 - h), \end{cases} \tag{3.2}$$

where c_q is the same as in Lemma 2.1, then problem (1.1) has a unique solution x^*.

Proof. From Lemma 3.1, problem (1.1) is equivalent to the fixed problem (3.1), equation (3.1) can be rewritten as follows:

$$x = x - p(x) - J_M^\rho[p(x) - \rho(N(S(x), T(x), U(x)) - f)].$$

For every $x \in X$, take

$$Q(x) = x - p(x) - J_M^\rho[p(x) - \rho(N(S(x), T(x), U(x)) - f)]. \tag{3.3}$$

Then x^* is the unique solution of problem (1.1) if and only if x^* is the unique fixed point of Q. In fact, it follows from (3.3) and Lemma 2.8 that

$$\begin{aligned} &\|Q(x) - Q(y)\| \\ &\le \|x - y - (p(x) - p(y))\| \\ &\quad + \|J_M^\rho[p(x) - \rho(N(S(x), T(x), U(x)) - f)] \\ &\quad - J_M^\rho[p(y) - \rho(N(S(y), T(y), U(y)) - f)]\| \\ &\le (1 + \frac{\tau}{\delta})\|x - y - (p(x) - p(y))\| \\ &\quad + \frac{\tau}{\delta}\{\|x - y - \rho[(N(S(x), T(x), U(x)) - N(S(y), T(x), U(x))) \\ &\quad + (N(S(y), T(x), U(x)) - N(S(y), T(y), U(x)))]\| \\ &\quad + \rho\|N(S(y), T(y), U(x)) - N(S(y), T(y), U(y))\|\}. \end{aligned} \tag{3.4}$$

By the hypothesis of S, T, U, p, N and Lemma 2.1, now we know there exists $c_q > 0$ such that

$$\|x - y - (p(x) - p(y))\|^q \le (1 - q\alpha + c_q\beta^q)\|x - y\|^q, \tag{3.5}$$

$$\|N(S(y), T(y), U(x)) - N(S(y), T(y), U(y))\| \le \gamma\nu\|x - y\|, \tag{3.6}$$

$$
\begin{aligned}
&\|x-y-\rho[(N(S(x),T(x),U(x))-N(S(y),T(x),U(x)))\\
&\quad+(N(S(y),T(x),U(x))-N(S(y),T(y),U(x)))]\|^q\\
&\le \|x-y\|^q\\
&\quad-q\rho\langle(N(S(x),T(x),U(x))-N(S(y),T(x),U(x)))\\
&\quad+(N(S(y),T(x),U(x))-N(S(y),T(y),U(x))),j_q(x-y)\rangle\\
&\quad+c_q\rho^q\|(N(S(x),T(x),U(x))-N(S(y),T(x),U(x)))\\
&\quad+(N(S(y),T(x),U(x))-N(S(y),T(y),U(x)))\|^q\\
&\le \|x-y\|^q\\
&\quad-q\rho[\langle N(S(x),T(x),U(x))-N(S(y),T(x),U(x)),j_q(x-y)\rangle\\
&\quad+\langle N(S(y),T(x),U(x))-N(S(y),T(y),U(x)),j_q(x-y)\rangle]\\
&\quad+c_q\rho^q[\|N(S(x),T(x),U(x))-N(S(y),T(x),U(x))\|\\
&\quad+\|N(S(y),T(x),U(x))-N(S(y),T(y),U(x))\|]^q\\
&\le [1-q\rho(\sigma-\varsigma)+c_q\rho^q(\epsilon\kappa+\xi\mu)^q]\|x-y\|^q. \qquad (3.7)
\end{aligned}
$$

Combining (3.4)-(3.7), we get

$$\|Q(x)-Q(y)\|\le\theta\|x-y\|, \tag{3.8}$$

where

$$
\begin{aligned}
\theta&=h+\frac{\tau}{\delta}[(1-q\rho(\sigma-\varsigma)+c_q\rho^q(\epsilon\kappa+\xi\mu)^q)^{\frac{1}{q}}+\rho\gamma\nu],\\
h&=(1+\frac{\tau}{\delta})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}}.
\end{aligned} \tag{3.9}
$$

It follows from (3.2) that $0<\theta<1$ and so $Q:X\to X$ is a contractive mapping, i.e., Q has a unique fixed point in X. This completes the proof. $\square$

Remark 3.3. If X is 2-uniformly smooth Banach space and there exists $\rho>0$ such that

$$
\begin{cases}
h=(1+\frac{\tau}{\delta})\sqrt{1-2\alpha+c_2\beta^2}<1,\\
0<\rho<\frac{\delta(1-h)}{\tau\gamma\nu}, \quad \gamma\nu<\sqrt{c_2}(\epsilon\kappa+\xi\mu),\\
\tau(\sigma-\varsigma)>\delta\gamma\nu(1-h)+\sqrt{[c_2(\epsilon\kappa+\xi\mu)^2-\gamma^2\nu^2][\tau^2-\delta^2(1-h)^2]},\\
|\rho-\frac{\tau(\sigma-\varsigma)+\delta\gamma\nu(h-1)}{\tau[c_2(\epsilon\kappa+\xi\mu)^2-\gamma^2\nu^2]}|<\frac{[\tau(\sigma-\varsigma)-\delta\gamma\nu(1-h)]^2-[c_2(\epsilon\kappa+\xi\mu)^2-\gamma^2\nu^2][\tau^2-\delta^2(1-h)^2]}{\tau[c_2(\epsilon\kappa+\xi\mu)^2-\gamma^2\nu^2]},
\end{cases}
$$

then (3.2) holds. We note that Hilbert space and L_p rem (or l_p) $(2\le p<\infty)$ spaces are 2-uniformly Banach spaces.

Remark 3.4. Conditions (3.2) in Theorem 3.2 hold for some suitable value of constants, for example, $q=2$, $\tau=0.25,\delta=0.38,\beta=0.75,\alpha=0.65,c_2=0.65,\gamma=0.15,\nu=0.45,\kappa=0.45,\epsilon=0.85,\mu=0.85,\xi=0.72,\sigma=0.75,\varsigma=0.150$, $\rho=0.8648$ and the contractive coefficient $\theta=0.9010<1$.

4. Perturbed Algorithms and Stability

In this section, by using the following definition and lemma, we construct a new perturbed iterative algorithm with mixed errors for solving problem (1.1) and prove the convergence and stability of the iterative sequence generated by the algorithm.

Definition 4.1. Let S be a selfmap of X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\} \subset X$ and let $\epsilon_n = \|u_{n+1} - h(S, u_n)\|$. If $\lim \epsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure defined by

$$x_{n+1} = h(S, x_n)$$

is said to be S-stable or stable with respect to S.

Lemma 4.2. ([13]) *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$ $(n \to \infty)$.

The relation (3.1) allows us to suggest the following perturbed iterative algorithm with mixed errors.

Algorithm 4.3. *Step* 1. *Choose $x_0 \in X$.*
Step 2. *Let*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[y_n - p(y_n) \\ \qquad\quad + J_M^{\rho}(p(y_n) - \rho(N(S(y_n), T(y_n), U(y_n)) - f))] + \alpha_n u_n + \omega_n, \\ y_n = (1 - \beta_n)x_n + \beta_n[x_n - p(x_n) \\ \qquad\quad + J_M^{\rho}(p(x_n) - \rho(N(S(x_n), T(x_n), U(x_n)) - f))] + v_n. \end{cases} \tag{4.1}$$

Step 3. *Choose sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ such that for $n \geq 0$, $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0, 1]$, $\{u_n\}, \{v_n\}, \{\omega_n\}$ are sequences in X satisfying the following conditions:*

(i) $u_n = u'_n + u''_n$;

(ii) $\lim_{n\to\infty} \|u'_n\| = \lim_{n\to\infty} \|v_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|u''_n\| < \infty, \quad \sum_{n=0}^{\infty} \|\omega_n\| < \infty$.

Step 4. *If* x_{n+1}, y_n, α_n, β_n, u_n, v_n *and* ω_n *satisfy* (4.1) *to sufficient accuracy, go to Step* 5*; otherwise, set* $n := n+1$ *and return to Step* 2.
Step 5. *Let* $\{z_n\}$ *be any sequence in* X *and define* $\{\varepsilon_n\}$ *by*

$$\begin{cases} \varepsilon_n = \|z_{n+1} - \{(1-\alpha_n)z_n + \alpha_n[t_n - p(t_n) \\ \qquad + J_M^\rho(p(t_n) - \rho(N(S(t_n), T(t_n), U(t_n)) - f))] + \alpha_n u_n + \omega_n\}\|, \\ t_n = (1-\beta_n)z_n + \beta_n[z_n - p(z_n) \\ \qquad + J_M^\rho(p(z_n) - \rho(N(S(z_n), T(z_n), U(z_n)) - f))] + v_n. \end{cases} \tag{4.2}$$

Step 6. *If* ε_n, z_{n+1}, t_n, α_n, β_n, u_n, v_n *and* ω_n *satisfy* (4.2) *to sufficient accuracy, stop; otherwise, set* $n := n+1$ *and return to Step* 3.

Theorem 4.4. *Suppose that* X, S, T, U, p, N, η *and* M *are the same as in Theorem* 3.2, θ *is defined by* (3.9). *If* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and condition* (3.2) *holds, then the perturbed iterative sequence* $\{x_n\}$ *defined by* (4.1) *converges strongly to the unique solution of problem* (1.1). *Moreover, if there exists* $a \in (0, \alpha_n]$ *for all* $n \geq 0$, *then* $\lim_{n\to\infty} z_n = x^*$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$, *where* ε_n *is defined by* (4.2).

Proof. From Theorem 3.2, we know that problem (1.1) has a unique solution $x^* \in X$. It follows from (4.1) and the proof of (3.8) in Theorem 3.2 that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\| + \alpha_n(\|u_n'\| + \|u_n''\|) + \|\omega_n\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\| + \alpha_n\|u_n'\| + (\|u_n''\| + \|\omega_n\|), \end{aligned} \tag{4.3}$$

where θ is the same as in (3.9). Similarly, we have

$$\|y_n - x^*\| \leq (1 - \beta_n + \beta_n\theta)\|x_n - x^*\| + \|v_n\|. \tag{4.4}$$

Combining (4.3)-(4.4), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| \leq\ & [1 - \alpha_n(1 - \theta(1 - \beta_n + \beta_n\theta))]\|x_n - x^*\| \\ & + \alpha_n(\|u_n'\| + \theta\|v_n\|) + (\|u_n''\| + \|\omega_n\|). \end{aligned} \tag{4.5}$$

Since $\theta < 1$, $0 < \beta_n \leq 1$ $(n \geq 0)$, we have $1 - \beta_n + \beta_n\theta < 1$ and $1 - \theta(1 - \beta_n + \beta_n\theta) > 1 - \theta > 0$. Therefore, (4.5) implies

$$\begin{aligned} \|x_{n+1} - x^*\| \leq\ & [1 - \alpha_n(1-\theta)]\|x_n - x^*\| \\ & + \alpha_n(1-\theta) \cdot \frac{1}{1-\theta}(\|u_n'\| + \theta\|v_n\|) + (\|u_n''\| + \|\omega_n\|). \end{aligned} \tag{4.6}$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, it follows from Lemma 4.2 and (4.6) that $\|x_n - x^*\| \to 0 (n \to \infty)$, i.e., $\{x_n\}$ converges strongly to the unique solution x^* of the problem (1.1).

Now we prove the second conclusion. By (4.2), we know

$$\begin{aligned}\|z_{n+1}-x^*\| \;\le\; & \|(1-\alpha_n)z_n+\alpha_n[t_n-p(t_n)\\ & +J_M^{\rho}(p(t_n)-\rho(N(S(t_n),T(t_n),U(t_n))-f))]\\ & +\alpha_n u_n+\omega_n-x^*\|+\varepsilon_n. \end{aligned}\tag{4.7}$$

As the proof of inequality (4.6), we have

$$\begin{aligned}&\|(1-\alpha_n)z_n\\ &\quad+\alpha_n[t_n-p(t_n)+J_M^{\rho}(p(t_n)-\rho(N(S(t_n),T(t_n),U(t_n))-f))]\\ &\quad+\alpha_n u_n+\omega_n-x^*\|\\ &\le[1-\alpha_n(1-\theta)]\|z_n-x^*\|\\ &\quad+\alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|u_n'\|+\theta\|v_n\|)+(\|u_n''\|+\|\omega_n\|).\end{aligned}\tag{4.8}$$

Since $0<a\le\alpha_n$, it follows from (4.7) and (4.8) that

$$\begin{aligned}&\|z_{n+1}-x^*\|\\ &\le[1-\alpha_n(1-\theta)]\|z_n-x^*\|\\ &\quad+\alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|u_n'\|+\theta\|v_n\|)+(\|u_n''\|+\|\omega_n\|)+\varepsilon_n\\ &\le[1-\alpha_n(1-\theta)]\|z_n-x^*\|\\ &\quad+\alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|u_n'\|+\theta\|v_n\|+\frac{\varepsilon_n}{a})+(\|u_n''\|+\|\omega_n\|).\end{aligned}$$

Suppose that $\lim\epsilon_n=0$. Then from $\sum_{n=0}^{\infty}\alpha_n=\infty$ and Lemma 4.2, we have $\lim z_n=x^*$.

Conversely, if $\lim z_n=x^*$, then we get

$$\begin{aligned}\varepsilon_n \;=\;& \|z_{n+1}-\{(1-\alpha_n)z_n+\alpha_n[t_n-p(t_n)\\ &+J_M^{\rho}(p(t_n)-\rho(N(S(t_n),T(t_n),U(t_n))-f))]+\alpha_n u_n+\omega_n\}\|\\ \le\;& \|z_{n+1}-x^*\|+\|(1-\alpha_n)z_n+\alpha_n[t_n-p(t_n)\\ &+J_M^{\rho}(p(t_n)-\rho(N(S(t_n),T(t_n),U(t_n))-f)))+\alpha_n u_n+\omega_n-x^*\|\\ \le\;& \|z_{n+1}-x^*\|+[1-\alpha_n(1-\theta)]\|z_n-x^*\|\\ &+\alpha_n(\|u_n'\|+\theta\|v_n\|)+(\|u_n''\|+\|\omega_n\|)\to 0\ (n\to\infty).\end{aligned}$$

This completes the proof. □

Remark 4.5. If $u_n=v_n=\omega_n=0$ $(n\ge 0)$ in Algorithm 4.3, then the conclusions of Theorem 4.4 also hold. The results of Theorems 3.1 and 4.1 improve and generalize the corresponding results of [3, 7, 10, 11, 13].

5. Conclusions

In this paper, we first introduce a new kind of strongly nonlinear operator inclusion problems (i.e., quasi-variational inclusions) involving generalized m-accretive mapping in Banach spaces and analyze the generalized m-accretive mapping introduced by Huang and Fang (see [8, 9]), the definition of the resolvent operator for the generalized m-accretive mapping in Banach space and some correlative results. Secondly, by using the resolvent operator technique for generalized m-accretive mapping, we translate the kind of strongly nonlinear operator inclusion problems involving generalized m-accretive mapping into a fixed point problem, and prove the existence and uniqueness of the solution for this kind operator inclusion problems. Finally, we construct a new class of perturbed iterative algorithm with mixed errors for solving this kind of strongly nonlinear operator inclusion problems in Banach spaces and discuss the convergence and stability of the iterative sequence generated by the algorithm. Our results improve and generalize the corresponding results of [3, 7, 10, 11, 13].

On above studied a new kind of strongly nonlinear operator inclusion problems involving generalized m-accretive mapping in Banach spaces, we can completely consider three-step perturbed iterative algorithm constructed by Wang et al. in [20] and easily obtain the same results as Theorem 4.4.

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