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SOME GENERALIZATIONS OF ENESTRÖM-KAKEYA THEOREM

N. A. Rather[1] **and M. A. Shah**[2]

[1]Department of Mathematics
University of Kashmir, Hazratbal Srinagar, India
e-mail: `dr.narather@gmail.com`

[2]Department of Mathematics
University of Kashmir, Hazratbal Srinagar, India
e-mail: `mushtaqa022@gmail.com`

Abstract. Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_j \geq a_{j-1}$, $a_0 > 0$, $j = 1, 2, \cdots, n$ is a polynomial of degree n. Then by a classical result of Eneström-Kakeya, all the zeros of $P(z)$ lie in $|z| \leq 1$. In this paper, we prove some generalizations of this result.

1. Introduction

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Then concerning the distribution of zeros of $P(z)$, Eneström and Kakeya [10, 11] proved the following interesting result.

Theorem 1.1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0. \tag{1.1}$$

Then $P(z)$ has all its zeros in $|z| \leq 1$.

In the literature [1-11], there exist several extensions and generalizations of this theorem. Joyal *et al.* [9] extended Theorem 1.1 to the polynomials whose coefficients are monotonic but not necessarily non-negative. In fact, they proved the following result.

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Theorem 1.2. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that*

$$a_n\geq a_{n-1}\geq\cdots\geq a_1\geq a_0.$$

Then $P(z)$ has all its zeros in the disk

$$|z|\leq\frac{1}{|a_n|}\left(|a_n|-a_0+|a_0|\right).$$

Govil and Rahman [8] extended the result to the class of polynomial with complex coefficients by proving the following interesting result.

Theorem 1.3. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|arg\ \ a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\quad 0\leq j\leq n$$

and

$$|a_n|\geq|a_{n-1}|\geq\cdots\geq|a_1|\geq|a_0|.$$

Then $P(z)$ has all its zeros in the disk

$$|z|\leq(Sin\alpha+Cos\alpha)+\frac{2Sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|.$$

Aziz and Zargar [2] relaxed the hypothesis of Theorem 1.1 and proved the following:

Theorem 1.4. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $k\geq1$,*

$$ka_n\geq a_{n-1}\geq\cdots\geq a_1\geq a_0>0. \tag{1.2}$$

Then $P(z)$ has all its zeros in $|z+k-1|\leq k$.

2. Main results

In this paper, we prove some generalizations of the Eneström-Kakeya theorem. In this direction we first present the following result which is a generalization of Theorem 1.2.

Theorem 2.1. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. If $Re\ a_j=\alpha_j$ and $Im\ a_j=\beta_j$ for $j=0,1,2,\cdots,n$, such that for some real $t\geq0$, and $0\leq\lambda\leq n-1$,*

$$\alpha_n-t\leq\alpha_{n-1}\leq\cdots\leq\alpha_\lambda,\quad\alpha_\lambda\geq\alpha_{\lambda-1}\geq\cdots\geq\alpha_1\geq\alpha_0,$$

then all the zeros of $P(z)$ *lie in*

$$\left|z-\frac{t}{a_n}\right| \le \frac{1}{|a_n|}\left\{2\alpha_\lambda-(\alpha_n-t)-\alpha_0+|\alpha_0|+\sum_{i=1}^{n}|\beta_i-\beta_{i-1}|+|\beta_0|\right\}. \quad (2.1)$$

Proof. Proof follows from next Theorem 2.2 as a special case. □

If the imaginary parts of the coefficients are also monotonic and non-negative, then we obtain the following result.

Corollary 2.1. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. If* $Re\ a_j=\alpha_j$ *and* $Im\ a_j=\beta_j$ *for* $j=0,1,2,\cdots,n$*, such that for some real* $t\ge 0$*, and* $0\le\lambda\le n-1$*,*

$$\alpha_n-t\le\alpha_{n-1}\le\cdots\le\alpha_\lambda, \quad \alpha_\lambda\ge\alpha_{\lambda-1}\ge\cdots\ge\alpha_1\ge\alpha_0,$$

and

$$\beta_n\ge\beta_{n-1}\ge\cdots\ge\beta_1\ge\beta_0>0,$$

then all the zeros of $P(z)$ *lie in*

$$\left|z-\frac{t}{a_n}\right| \le \frac{1}{|a_n|}\left\{2\alpha_\lambda-(\alpha_n-t)-\alpha_0+|\alpha_0|+\beta_n\right\}. \quad (2.2)$$

Remark 2.1. Taking $t=(1-k)\alpha_n$, $0<k\le 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. If* $Re\ a_j=\alpha_j$ *and* $Im\ a_j=\beta_j$ *for* $j=0,1,2,\cdots,n$*, such that for some* $0<k\le 1$, $0\le\lambda\le n-1$*,*

$$k\alpha_n\le\alpha_{n-1}\le\cdots\le\alpha_\lambda, \quad \alpha_\lambda\ge\alpha_{\lambda-1}\ge\cdots\ge\alpha_1\ge\alpha_0,$$

then all the zeros of $P(z)$ *lie in*

$$\left|z-\frac{\alpha_n}{a_n}(1-k)\right| \le \frac{1}{|a_n|}\left\{2\alpha_\lambda-k\alpha_n-\alpha_0+|\alpha_0|+\sum_{i=1}^{n}|\beta_i-\beta_{i-1}|+|\beta_0|\right\}. \quad (2.3)$$

If $\alpha_0>0$, then we get the following result.

Corollary 2.3. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. If* $Re\ a_j=\alpha_j$ *and* $Im\ a_j=\beta_j$ *for* $j=0,1,2,\cdots,n$*, such that for some real* $t\ge 0$*, and* $0\le\lambda\le n-1$*,*

$$\alpha_n-t\le\alpha_{n-1}\le\cdots\le\alpha_\lambda, \quad \alpha_\lambda\ge\alpha_{\lambda-1}\ge\cdots\ge\alpha_1\ge\alpha_0>0,$$

then all the zeros of $P(z)$ lie in

$$\left|z-\frac{t}{a_n}\right| \leq \frac{1}{|a_n|}\left\{2\alpha_\lambda-(\alpha_n-t)+\sum_{i=1}^{n}|\beta_i-\beta_{i-1}|+|\beta_0|\right\}. \tag{2.4}$$

Instead of proving Theorem 2.1, we prove the following more generalized result.

Theorem 2.2. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re\ a_j=\alpha_j$ and $Im\ a_j=\beta_j$ for $j=0,1,2,\cdots,n$, such that for some real $t,s\geq 0$, and $0\leq\lambda\leq n-1$,*

$$\alpha_n-t\leq\alpha_{n-1}\leq\cdots\leq\alpha_\lambda,\quad \alpha_\lambda\geq\alpha_{\lambda-1}\geq\cdots\geq\alpha_1\geq\alpha_0-s,$$

then all the zeros of $P(z)$ lie in

$$\left|z-\frac{t}{a_n}\right|\leq\frac{1}{|a_n|}\left\{2\alpha_\lambda-(\alpha_n-t)-(\alpha_0-s)+(1+s)|\alpha_0|+\sum_{i=1}^{n}|\beta_i-\beta_{i-1}|+|\beta_0|\right\}. \tag{2.5}$$

Proof. Consider the polynomial

$$\begin{aligned}
F(z)&=(1-z)P(z)\\
&=(1-z)\left(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0\right)\\
&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0)z+a_0\\
&=-z^n(a_nz-t)+\{(\alpha_n-t-\alpha_{n-1})z^n+\cdots+(\alpha_1-\alpha_0+s)z-s\alpha_0z+\alpha_0\}\\
&\quad+i\{(\beta_n-\beta_{n-1})z^n+\cdots+(\beta_1-\beta_0)z+\beta_0\}.
\end{aligned}$$

This gives

$$\begin{aligned}
|F(z)|\geq{}&|z|^n|a_nz-t|-\Big\{|\alpha_n-t-\alpha_{n-1}||z|^n+|\alpha_{n-1}-\alpha_{n-2}||z|^{n-1}+\cdots\\
&+|\alpha_{\lambda+1}-\alpha_\lambda||z|^{\lambda+1}+|\alpha_\lambda-\alpha_{\lambda-1}||z|^\lambda+\cdots+|\alpha_1-(\alpha_0-s)||z|\\
&+s|\alpha_0||z|+|\alpha_0|+|\beta_n-\beta_{n-1}||z|^n+|\beta_{n-1}-\beta_{n-2}||z|^{n-1}+\cdots\\
&+|\beta_1-\beta_0||z|+|\beta_0|\Big\}\\
={}&|z|^n\Bigg[|a_nz-t|-\Bigg\{|\alpha_n-t-\alpha_{n-1}|+\frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|}+\cdots\\
&+\frac{|\alpha_{\lambda+1}-\alpha_\lambda|}{|z|^{n-\lambda-1}}+\frac{|\alpha_\lambda-\alpha_{\lambda-1}|}{|z|^{n-\lambda}}+\cdots+\frac{|\alpha_1-(\alpha_0-s)|}{|z|^{n-1}}+\frac{s|\alpha_0|}{|z|^{n-1}}\\
&+\frac{|\alpha_0|}{|z|^n}+|\beta_n-\beta_{n-1}|+\frac{|\beta_{n-1}-\beta_{n-2}|}{|z|}+\cdots+\frac{|\beta_1-\beta_0|}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\Bigg\}\Bigg].
\end{aligned}$$

Now, let $|z| \geq 1$, so that $\frac{1}{|z|^{n-j}} \leq 1$, $0 \leq j \leq n$. Then we have

$$\begin{aligned}
|F(z)| &\geq |z|^n \Bigg[|a_n z - t| - \Bigg\{ |\alpha_n - t - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \cdots \\
&\quad + |\alpha_{\lambda+1} - \alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + \cdots + |\alpha_1 - (\alpha_0 - s)| + (1+s)|\alpha_0| \\
&\quad + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \cdots + |\beta_1 - \beta_0| + |\beta_0| \Bigg\} \Bigg] \\
&= |z|^n \Bigg[|a_n z + t| - \Bigg\{ - \alpha_n + t + \alpha_{n-1} - \alpha_{n-1} + \alpha_{n-2} - \cdots \\
&\quad - \alpha_{\lambda+1} + \alpha_\lambda + \alpha_\lambda - \alpha_{\lambda-1} + \cdots + \alpha_1 - (\alpha_0 - s) + (1+s)|\alpha_0| \\
&\quad + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\beta_0| \Bigg\} \Bigg] \\
&= |z|^n \Bigg[|a_n z - t| - \Bigg\{ - \alpha_n + t + 2\alpha_\lambda - (\alpha_0 - s) + (1+s)|\alpha_0| \\
&\quad + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\beta_0| \Bigg\} \Bigg] \\
&> 0.
\end{aligned}$$

If

$$|a_n z - t| > \left\{ -\alpha_n + t + 2\alpha_\lambda - (\alpha_0 - s) + (1+s)|\alpha_0| + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\beta_0| \right\},$$

i.e,

$$\left| z - \frac{t}{a_n} \right| > \frac{1}{|a_n|} \left\{ 2\alpha_\lambda + t - \alpha_n - \alpha_0 + s + (1+s)|\alpha_0| + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\beta_0| \right\},$$

then all the zeros of $F(z)$ whose modulus is greater than or equal to 1 lie in

$$\left| z - \frac{t}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda + t - \alpha_n - \alpha_0 + s + (1+s)|\alpha_0| + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\beta_0| \right\}.$$

But those zeros of $F(z)$ whose modulus is less than 1 already satisfy the above inequality and all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $F(z)$ and hence of $P(z)$ lie in

$$\left| z - \frac{t}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda + t - \alpha_n - \alpha_0 + s + (1+s)|\alpha_0| + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\beta_0| \right\}.$$

This completes the proof. □

As in Theorem 2.1, if the imaginary parts of the coefficients are also monotonic and non-negative, then we obtain the following corresponding result.

Corollary 2.4. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re\ a_j = \alpha_j$ and $Im\ a_j = \beta_j$ for $j = 0, 1, 2, \cdots, n$, such that for some real $t, s \geq 0$, and $0 \leq \lambda \leq n-1$,*

$$\alpha_n - t \leq \alpha_{n-1} \leq \cdots \leq \alpha_\lambda, \quad \alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 - s,$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z - \frac{t}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda - (\alpha_n - t) - (\alpha_0 - s) + (1+s)|\alpha_0| + \beta_n \right\}. \tag{2.6}$$

Remark 2.2. For $s = 0$, Theorem 2.2 reduces to Theorem 2.1. For $t = 0$, Theorem 2.2 reduces to the following result.

Corollary 2.5. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $Re\ a_j = \alpha_j$ and $Im\ a_j = \beta_j$ for $j = 0, 1, 2, \cdots, n$, such that for some real s, and $0 \leq \lambda \leq n-1$,*

$$\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_\lambda, \quad \alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 - s,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda - \alpha_n - (\alpha_0 - s) + (1+s)|\alpha_0| + \sum_{i=1}^{n} |\beta_i - \beta_{i-1}| + |\beta_0| \right\}. \tag{2.7}$$

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