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RESULTS FOR COUPLED FIXED POINTS IN PARTIAL METRIC SPACES UNDER A CONTRACTIVE TYPE CONDITION

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Abstract. In this paper, we prove some coupled fixed point results within the framework of partial metric spaces and give some consequences of the main result. We do this by focusing on partial metric spaces. In addition, we provide some illustrations to back up the conclusion we reached. The results that were obtained in this paper extend and generalize a number of results that were previously published in the relevant literature.

1. Introduction

For any partially ordered set X, Bhashkar and Lakshmikantham introduced in 2006 the concept of a "coupled fixed point" of the mapping $F\colon X \times X \to X$. Coupled fixed point theorems in metric spaces have been proven by a number of authors, including Ciric and Lakshmikantham [6], Sabetghadam *et al.* [15] and Olaleru *et al.* [14] (see also: [10], [13]).

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A wide variety of generalizations of metric spaces have been proposed and implemented in the literature. These overarching rules are established by utilizing alternative contractive conditions or by imposing an additional condition on the surrounding spaces. A partial metric space, which was first introduced by Matthews in 1992 [11, 12], is one example of such a generalization. Model building in the theory of computation relies heavily on partial metric spaces (for examples, see [7], [14], and others). To put it another way, the distance between two points in the self might not be zero in partial metric spaces. By introducing the concept of partial metric space, Matthews was able to prove the Banach fixed point theorem ([3]) in the partial metric setting.

As an extension of previous work by Sabetghadam *et al.* [15] and Aydi [1] in the context of partial metric spaces, the goal of this paper is to prove some coupled fixed point results under contractive type condition.

2. Preliminaries

In the following section, we define some terms and introduce some fundamental ideas regarding partial metric spaces.

Definition 2.1. ([1]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F\colon X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Example 2.2. Let $X = [0, +\infty)$ and $F\colon X \times X \to X$ defined by $F(x, y) = \frac{x+y}{3}$ for all $x, y \in X$. One can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.3. Let $X = [0, +\infty)$ and $F\colon X \times X \to X$ be defined by $F(x, y) = \frac{x+y}{2}$ for all $x, y \in X$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Definition 2.4. ([12]) Let X be a nonempty set and $p\colon X \times X \to \mathbb{R}^+$ be a self-mapping of X such that for all $x, y, z \in X$ the followings are satisfied:

$(P1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
$(P2)$ $p(x, x) \leq p(x, y)$,
$(P3)$ $p(x, y) = p(y, x)$,
$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is called a partial metric on X and the pair (X, p) is called a partial metric space (in short PMS).

Remark 2.5. It is clear that if $p(x, y) = 0$, then from $(P1)$, $(P2)$ and $(P3)$, $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

If p is a partial metric on X, then the function $p^s\colon X\times X\to\mathbb{R}^+$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y), \tag{2.1}$$

is a metric on X.

Example 2.6. ([2]) Let $X=\mathbb{R}^+$, where $\mathbb{R}^+=[0,+\infty)$ and $p\colon X\times X\to\mathbb{R}^+$ be given by $p(x,y)=\max\{x,y\}$ for all $x,y\in\mathbb{R}^+$. Then $(\mathbb{R}^+,p)$ is a partial metric space.

Example 2.7. ([2]) Let I denote the set of all intervals $[a,b]$ for any real numbers $a\leq b$. Let $p\colon I\times I\to[0,\infty)$ be a function such that

$$p\Big([a,b],[c,d]\Big)=\max\{b,d\}-\min\{a,c\}.$$

Then (I,p) is a partial metric space.

Example 2.8. ([5]) Let $X=\mathbb{R}$ and $p\colon X\times X\to\mathbb{R}^+$ be given by $p(x,y)=e^{\max\{x,y\}}$ for all $x,y\in\mathbb{R}$. Then (X,p) is a partial metric space.

Various applications of this space has been extensively investigated by many authors (see, Künzi [9] and Valero [16] for details).

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x,\varepsilon):x\in X,\varepsilon>0\}$ where

$$B_p(x,\varepsilon)=\{y\in X:p(x,y)<p(x,x)+\varepsilon\}$$

for all $x\in X$ and $\varepsilon>0$.

Similarly, closed p-ball is defined as

$$B_p[x,\varepsilon]=\{y\in X:p(x,y)\leq p(x,x)+\varepsilon\}$$

for all $x\in X$ and $\varepsilon>0$.

Definition 2.9. ([11]) Let (X,p) be a partial metric space. Then

(1) a sequence $\{a_n\}$ in (X,p) is said to be convergent to a point $a\in X$ if and only if $p(a,a)=\lim_{n\to\infty}p(a_n,a)$;

(2) a sequence $\{a_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty}p(a_m,a_n)$ exists and is finite;

(3) (X,p) is said to be complete if every Cauchy sequence $\{a_n\}$ in X converges to a point $a\in X$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty}p(a_m,a_n)=\lim_{n\to\infty}p(a_n,a)=p(a,a);$$

(4) a mapping $g\colon X\to X$ is said to be continuous at $a_0\in X$ if for every $\varepsilon>0$, there exists $\alpha>0$ such that $g\Big(B_p(a_0,\alpha)\Big)\subset B_p\Big(g(a_0),\varepsilon\Big)$.

Lemma 2.10. ([1, 11, 12]) *Let (X,p) be a partial metric space. Then*

(a) *a sequence* $\{a_n\}$ *in* (X,p) *is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space* (X,p^s),

(b) *a partial metric space* (X,p) *is complete if and only if the metric space* (X,p^s) *is complete, furthermore,* $\lim_{n\to\infty} p^s(a_n,a)=0$ *if and only if*

$$p(a,a)=\lim_{n\to\infty} p(a_n,a)=\lim_{n,m\to\infty} p(a_n,a_m). \tag{2.2}$$

Lemma 2.11. ([8]) *Let* (X,p) *be a partial metric space.*

(i) *If for all* $a,b\in X$, $p(a,b)=0$, *then* $a=b$;

(ii) *If* $a\neq b$, *then* $p(a,b)>0$.

Sabetghadam *et al.* [15] obtained the following.

Theorem 2.12. *Let* (X,d) *be a complete cone metric space. Suppose that the mapping* $F\colon X\times X\to X$ *satisfies the following contractive condition for all* $x,y,u,v\in X$

$$d(F(x,y),F(u,v))\leq kd(x,u)+ld(y,v), \tag{2.3}$$

where k,l *are nonnegative constants with* $k+l<1$. *Then* F *has a unique coupled fixed point.*

Recently, Aydi [1] obtained the following results in a partial metric space.

Theorem 2.13. ([1]) *Let* (X,p) *be a complete partial metric space. Suppose that the mapping* $F\colon X\times X\to X$ *satisfies one of the following contractive conditions* (Υ_1), (Υ_2), (Υ_3):

(Υ_1) *for all* $x,y,u,v\in X$ *and nonnegative constants* k,l *with* $k+l<1$,

$$p(F(x,y),F(u,v))\leq kp(x,u)+lp(y,v), \tag{2.4}$$

(Υ_2) *for all* $x,y,u,v\in X$ *and nonnegative constants* k,l *with* $k+l<1$,

$$p(F(x,y),F(u,v))\leq kp(F(x,y),x)+lp(F(u,v),u), \tag{2.5}$$

(Υ_3) *for all* $x,y,u,v\in X$ *and nonnegative constants* k,l *with* $k+2l<1$,

$$p(F(x,y),F(u,v))\leq kp(F(x,y),u)+lp(F(u,v),x). \tag{2.6}$$

Then F *has a unique coupled fixed point.*

3. Main results

Within the context of partial metric spaces, we will demonstrate some specific coupled fixed point theorems that are unique to this section.

Theorem 3.1. *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$\begin{aligned} p(F(x,y),F(u,v)) \leq\ & a_1 p(x,u) + a_2 p(y,v) + a_3 p(F(x,y),x) \\ & + a_4 p(F(u,v),u) + a_5 p(F(x,y),u) \\ & + a_6 p(F(u,v),x), \end{aligned} \tag{3.1}$$

where $a_1, a_2, \dots, a_6$ are nonnegative constants with $a_1+a_2+a_3+a_4+a_5+2a_6 < 1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then, from equation (3.1) and using $(P3)$, $(P4)$, we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq a_1 p(x_{n-1}, x_n) + a_2 p(y_{n-1}, y_n) + a_3 p(F(x_{n-1}, y_{n-1}), x_{n-1}) \\ &\quad + a_4 p(F(x_n, y_n), x_n) + a_5 p(F(x_{n-1}, y_{n-1}), x_n) \\ &\quad + a_6 p(F(x_n, y_n), x_{n-1}) \\ &= a_1 p(x_{n-1}, x_n) + a_2 p(y_{n-1}, y_n) + a_3 p(x_n, x_{n-1}) \\ &\quad + a_4 p(x_{n+1}, x_n) + a_5 p(x_n, x_n) + a_6 p(x_{n+1}, x_{n-1}) \\ &\leq a_1 p(x_{n-1}, x_n) + a_2 p(y_{n-1}, y_n) + a_3 p(x_n, x_{n-1}) \\ &\quad + a_4 p(x_{n+1}, x_n) + a_5 p(x_n, x_{n+1}) \\ &\quad + a_6 p(x_n, x_{n+1}) + a_6 p(x_n, x_{n-1}). \end{aligned} \tag{3.2}$$

Similarly, we have

$$\begin{aligned} p(y_n, y_{n+1}) &= p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq a_1 p(y_{n-1}, y_n) + a_2 p(x_{n-1}, x_n) + a_3 p(F(y_{n-1}, x_{n-1}), y_{n-1}) \\ &\quad + a_4 p(F(y_n, x_n), y_n) + a_5 p(F(y_{n-1}, x_{n-1}), y_n) \\ &\quad + a_6 p(F(y_n, x_n), y_{n-1}) \\ &= a_1 p(y_{n-1}, y_n) + a_2 p(x_{n-1}, x_n) + a_3 p(y_n, y_{n-1}) \\ &\quad + a_4 p(y_{n+1}, y_n) + a_5 p(y_n, y_n) + a_6 p(y_{n+1}, y_{n-1}) \\ &\leq a_1 p(y_{n-1}, y_n) + a_2 p(x_{n-1}, x_n) + a_3 p(y_n, y_{n-1}) \\ &\quad + a_4 p(y_{n+1}, y_n) + a_5 p(y_n, y_{n+1}) \\ &\quad + a_6 p(y_n, y_{n+1}) + a_6 p(y_n, y_{n-1}). \end{aligned} \tag{3.3}$$

Therefore, by setting

$$D_n = p(x_n, x_{n+1}) + p(y_n, y_{n+1}), \tag{3.4}$$

we have

$$\begin{aligned}
D_n &= p(x_n, x_{n+1}) + p(y_n, y_{n+1}) \\
&\le a_1 p(x_{n-1}, x_n) + a_2 p(y_{n-1}, y_n) + a_3 p(x_n, x_{n-1}) \\
&\quad + a_4 p(x_{n+1}, x_n) + a_5 p(x_n, x_{n+1}) \\
&\quad + a_6 p(x_n, x_{n+1}) + a_6 p(x_n, x_{n-1}) \\
&\quad + a_1 p(y_{n-1}, y_n) + a_2 p(x_{n-1}, x_n) + a_3 p(y_n, y_{n-1}) \\
&\quad + a_4 p(y_{n+1}, y_n) + a_5 p(y_n, y_{n+1}) \\
&\quad + a_6 p(y_n, y_{n+1}) + a_6 p(y_n, y_{n-1}) \\
&= (a_1 + a_2 + a_3 + a_6) p(x_{n-1}, x_n) + (a_1 + a_2 + a_3 + a_6) p(y_{n-1}, y_n) \\
&\quad + (a_4 + a_5 + a_6) p(x_n, x_{n+1}) + (a_4 + a_5 + a_6) p(y_n, y_{n+1}) \\
&= (a_1 + a_2 + a_3 + a_6) D_{n-1} + (a_4 + a_5 + a_6) D_n.
\end{aligned}$$

Hence, we have

$$(1 - a_4 - a_5 - a_6) D_n \le (a_1 + a_2 + a_3 + a_6) D_{n-1}$$

or

$$\begin{aligned}
D_n &\le \Big(\frac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_5 - a_6}\Big) D_{n-1} \\
&= \mu D_{n-1}. \qquad (3.5)
\end{aligned}$$

Since by hypothesis $a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 < 1$, $\mu = \Big(\frac{a_1+a_2+a_3+a_6}{1-a_4-a_5-a_6}\Big) < 1$. Consequently, for each $n \in \mathbb{N}$, we obtain

$$D_n \le \mu D_{n-1} \le \mu^2 D_{n-2} \le \cdots \le \mu^n D_0. \qquad (3.6)$$

If $D_0 = 0$, then $p(x_0, x_1) + p(y_0, y_1) = 0$. Hence, from Remark 2.5, we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$, means that (x_0, y_0) is a coupled fixed point of F. Now, we assume that $D_0 > 0$. For each $n \ge m$, where $n, m \in \mathbb{N}$, we have, by using condition $(P4)$

$$\begin{aligned}
p(x_n, x_m) &\le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots + p(x_{m+1}, x_m) \\
&\quad - p(x_{n-1}, x_{n-1}) - p(x_{n-2}, x_{n-2}) - \cdots - p(x_{m+1}, x_{m+1}) \\
&\le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots + p(x_{m+1}, x_m). \qquad (3.7)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
p(y_n, y_m) &\le p(y_n, y_{n-1}) + p(y_{n-1}, y_{n-2}) + \cdots + p(y_{m+1}, y_m) \\
&\quad - p(y_{n-1}, y_{n-1}) - p(y_{n-2}, y_{n-2}) - \cdots - p(y_{m+1}, y_{m+1}) \\
&\le p(y_n, y_{n-1}) + p(y_{n-1}, y_{n-2}) + \cdots + p(y_{m+1}, y_m). \qquad (3.8)
\end{aligned}$$

Thus,

$$\begin{aligned} p(x_n,x_m)+p(y_n,y_m) &\leq D_{n-1}+D_{n-2}+\cdots+D_m \\ &\leq (\mu^{n-1}+\mu^{n-2}+\cdots+\mu^m)D_0 \\ &\leq \frac{\mu^m}{1-\mu}D_0. \end{aligned} \tag{3.9}$$

By definition of metric p^s, we have $p^s(x,y) \leq 2p(x,y)$, therefore for any $n \geq m$

$$\begin{aligned} p^s(x_n,x_m)+p^s(y_n,y_m) &\leq 2p(x_n,x_m)+2p(y_n,y_m) \\ &\leq \frac{2\mu^m}{1-\mu}D_0, \end{aligned} \tag{3.10}$$

which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,p^s) because $0 \leq \mu < 1$, where $\mu = a_1+a_2+a_3+a_4+a_5+2a_6$. Since the partial metric space (X,p) is complete, by Lemma 2.10, the metric space (X,p^s) is complete, so there exist $u',v' \in X$ such that

$$\lim_{n\to\infty} p^s(x_n,u') = \lim_{n\to\infty} p^s(y_n,v') = 0. \tag{3.11}$$

Again, from Lemma 2.10, we obtain

$$p(u',u') = \lim_{n\to\infty} p(x_n,u') = \lim_{n\to\infty} p(x_n,x_n) \tag{3.12}$$

and

$$p(v',v') = \lim_{n\to\infty} p(y_n,v') = \lim_{n\to\infty} p(y_n,y_n). \tag{3.13}$$

But, from condition $(P2)$ and equation (3.6), we have

$$p(x_n,x_n) \leq p(x_n,x_{n+1}) \leq D_n \leq \mu^n D_0 \tag{3.14}$$

and since $0 \leq \mu < 1$, hence letting $n \to \infty$, we get $\lim_{n\to\infty} p(x_n,x_n) = 0$. It follows that

$$p(u',u') = \lim_{n\to\infty} p(x_n,u') = \lim_{n\to\infty} p(x_n,x_n) = 0. \tag{3.15}$$

Similarly, we obtain

$$p(v',v') = \lim_{n\to\infty} p(y_n,v') = \lim_{n\to\infty} p(y_n,y_n) = 0. \tag{3.16}$$

Therefore, using equation (3.1) and by $(P3), (P4)$, we have

$$\begin{aligned}
p(F(u',v'),u') &\le p(F(u',v'),x_{n+1}) + p(x_{n+1},u') - p(x_{n+1},x_{n+1}),\\
&\le p(F(u',v'),x_{n+1}) + p(x_{n+1},u')\\
&= p(F(u',v'),F(x_n,y_n)) + p(x_{n+1},u')\\
&= p(F(x_n,y_n),F(u',v')) + p(x_{n+1},u')\\
&\le a_1p(x_n,u') + a_2p(y_n,v') + a_3p(F(x_n,y_n),x_n)\\
&\quad + a_4p(F(u',v'),u') + a_5p(F(x_n,y_n),u')\\
&\quad + a_6p(F(u',v'),x_n) + p(x_{n+1},u')\\
&= a_1p(x_n,u') + a_2p(y_n,v') + a_3p(x_{n+1},x_n)\\
&\quad + a_4p(F(u',v'),u') + a_5p(x_{n+1},u')\\
&\quad + a_6p(F(u',v'),x_n) + p(x_{n+1},u'). \qquad (3.17)
\end{aligned}$$

Letting $n \to \infty$ in equation (3.17) and using equations (3.15), (3.16), we obtain

$$\begin{aligned}
p(F(u',v'),u') &\le (a_4+a_6)p(F(u',v'),u')\\
&\le (a_1+a_2+a_3+a_4+a_5+2a_6)p(F(u',v'),u')\\
&< p(F(u',v'),u'),
\end{aligned}$$

which is a contradiction. Hence, we have

$$p(F(u',v'),u') = 0,$$

so $F(u',v') = u'$.

Similarly, we can prove that $F(v',u') = v'$. This shows that (u',v') is a coupled fixed point of F.

Now, we will prove the uniqueness. If (u_1,v_1) is another coupled fixed point of F, that is, $F(u_1,v_1) = u_1$ and $F(v_1,u_1) = v_1$ such that $(u',v') \neq (u_1,v_1)$, then

$$\begin{aligned}
p(u',u_1) &= p(F(u',v'),F(u_1,v_1))\\
&\le a_1p(u',u_1) + a_2p(v',v_1) + a_3p(F(u',v'),u')\\
&\quad + a_4p(F(u_1,v_1),u_1) + a_5p(F(u',v'),u_1)\\
&\quad + a_6p(F(u_1,v_1),u')\\
&= a_1p(u',u_1) + a_2p(v',v_1) + a_3p(u',u')\\
&\quad + a_4p(u_1,u_1) + a_5p(u',u_1) + a_6p(u_1,u'). \qquad (3.18)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}p(v', v_1) &= p(F(v', u'), F(v_1, u_1))\\ &\le a_1 p(v', v_1) + a_2 p(u', u_1) + a_3 p(F(v', u'), v')\\ &\quad + a_4 p(F(v_1, u_1), v_1) + a_5 p(F(v', u'), v_1)\\ &\quad + a_6 p(F(v_1, u_1), v')\\ &= a_1 p(v', v_1) + a_2 p(u', u_1) + a_3 p(v', v')\\ &\quad + a_4 p(v_1, v_1) + a_5 p(v', v_1) + a_6 p(v_1, v'). \end{aligned} \tag{3.19}$$

Now from (3.18) and (3.19) and using equations (3.15), (3.16) and condition $(P3)$ in equation (3.17), we get

$$\begin{aligned}p(u', u_1) + p(v', v_1) &\le a_1 p(u', u_1) + a_2 p(v', v_1) + a_3 p(u', u')\\ &\quad + a_4 p(u_1, u_1) + a_5 p(u', u_1) + a_6 p(u_1, u')\\ &\quad + a_1 p(v', v_1) + a_2 p(u', u_1) + a_3 p(v', v')\\ &\quad + a_4 p(v_1, v_1) + a_5 p(v', v_1) + a_6 p(v_1, v')\\ &= (a_1 + a_2 + a_5 + a_6) p(u', u_1)\\ &\quad + (a_1 + a_2 + a_5 + a_6) p(v', v_1)\\ &= (a_1 + a_2 + a_5 + a_6)(p(u', u_1) + p(v', v_1))\\ &\le (a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6)(p(u', u_1) + p(v', v_1))\\ &< p(u', u_1) + p(v', v_1),\end{aligned}$$

which is a contradiction. Hence, we obtain $p(u', u_1) + p(v', v_1) = 0$, and so $u' = u_1$ and $v' = v_1$. Thus the coupled fixed point of the mapping F is unique. This completes the proof. □

Theorem 3.2. *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$\begin{aligned}p(F(x, y), F(u, v)) &\le \frac{\alpha}{2}[p(x, u) + p(y, v)] + \frac{\beta}{2}[p(x, F(x, y))\\ &\quad + p(u, F(u, v)) + p(y, F(y, x)) + p(v, F(v, u))]\\ &\quad + \gamma \Delta((x, y), (u, v)),\end{aligned} \tag{3.20}$$

where α, β, γ are nonnegative constants with $\alpha + 2\beta + \gamma < 1$ and

$$\Delta((x, y), (u, v))) = \min\left\{\frac{p(u, F(u, v))}{1 + p(x, F(x, y))}, \frac{p(v, F(v, u))}{1 + p(y, F(y, x))}\right\}. \tag{3.21}$$

Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then, from equation (3.20), (3.21) and using $(P3)$, $(P4)$, we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \frac{\alpha}{2}[p(x_{n-1}, x_n) + p(y_{n-1}, y_n)] \\ &\quad + \frac{\beta}{2}[p(x_{n-1}, F(x_{n-1}, y_{n-1})) \\ &\quad + p(x_n, F(x_n, y_n)) + p(y_{n-1}, F(y_{n-1}, x_{n-1})) \\ &\quad + p(y_n, F(y_n, x_n))] + \gamma\Delta((x_{n-1}, y_{n-1}), (x_n, y_n)) \\ &= \frac{\alpha}{2}[p(x_{n-1}, x_n) + p(y_{n-1}, y_n)] + \frac{\beta}{2}[p(x_{n-1}, x_n) \\ &\quad + p(x_n, x_{n+1}) + p(y_{n-1}, y_n) + p(y_n, y_{n+1})] \\ &\quad + \gamma\Delta((x_{n-1}, y_{n-1}), (x_n, y_n)), \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \Delta((x_{n-1}, y_{n-1}), (x_n, y_n)) &= \min\Big\{\frac{p(x_n, F(x_n, y_n))}{1 + p(x_{n-1}, F(x_{n-1}, y_{n-1}))}, \\ &\qquad \frac{p(y_n, F(y_n, x_n))}{1 + p(y_{n-1}, F(y_{n-1}, x_{n-1}))}\Big\} \\ &= \min\Big\{\frac{p(x_n, x_{n+1})}{1 + p(x_{n-1}, x_n)}, \frac{p(y_n, y_{n+1})}{1 + p(y_{n-1}, y_n)}\Big\}. \end{aligned} \tag{3.23}$$

• If $\Delta((x_{n-1}, y_{n-1}), (x_n, y_n)) = \frac{p(x_n, x_{n+1})}{1+p(x_{n-1}, x_n)}$, then from equation (3.22), we obtain

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \frac{\alpha}{2}[p(x_{n-1}, x_n) + p(y_{n-1}, y_n)] + \frac{\beta}{2}[p(x_{n-1}, x_n) \\ &\quad + p(x_n, x_{n+1}) + p(y_{n-1}, y_n) + p(y_n, y_{n+1})] \\ &\quad + \gamma\Big(\frac{p(x_n, x_{n+1})}{1 + p(x_{n-1}, x_n)}\Big) \\ &\leq \frac{\alpha}{2}[p(x_{n-1}, x_n) + p(y_{n-1}, y_n)] + \frac{\beta}{2}[p(x_{n-1}, x_n) \\ &\quad + p(x_n, x_{n+1}) + p(y_{n-1}, y_n) + p(y_n, y_{n+1})] \\ &\quad + \gamma p(x_n, x_{n+1}). \end{aligned} \tag{3.24}$$

Similarly, we have

$$\begin{aligned}
p(y_n, y_{n+1}) &= p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
&\leq \frac{\alpha}{2}[p(y_{n-1}, y_n) + p(x_{n-1}, x_n)] \\
&\quad + \frac{\beta}{2}[p(y_{n-1}, F(y_{n-1}, x_{n-1})) \\
&\quad + p(y_n, F(y_n, x_n)) + p(x_{n-1}, F(x_{n-1}, y_{n-1})) \\
&\quad + p(x_n, F(x_n, y_n))] + \gamma\Delta((y_{n-1}, x_{n-1}), (y_n, x_n)) \\
&= \frac{\alpha}{2}[p(y_{n-1}, y_n) + p(x_{n-1}, x_n)] + \frac{\beta}{2}[p(y_{n-1}, y_n) \\
&\quad + p(y_n, y_{n+1}) + p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \\
&\quad + \gamma\Delta((y_{n-1}, x_{n-1}), (y_n, x_n)),
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
\Delta((y_{n-1}, x_{n-1}), (y_n, x_n)) &= \min\Big\{\frac{p(y_n, F(y_n, x_n))}{1 + p(y_{n-1}, F(y_{n-1}, x_{n-1}))}, \\
&\qquad \frac{p(x_n, F(x_n, y_n))}{1 + p(x_{n-1}, F(x_{n-1}, y_{n-1}))}\Big\} \\
&= \min\Big\{\frac{p(y_n, y_{n+1})}{1 + p(y_{n-1}, y_n)}, \frac{p(x_n, x_{n+1})}{1 + p(x_{n-1}, x_n)}\Big\}.
\end{aligned} \tag{3.26}$$

• If $\Delta((y_{n-1}, x_{n-1}), (y_n, x_n)) = \frac{p(y_n, y_{n+1})}{1+p(y_{n-1}, y_n)}$, then from equation (3.25), we obtain

$$\begin{aligned}
p(y_n, y_{n+1}) &\leq \frac{\alpha}{2}[p(y_{n-1}, y_n) + p(x_{n-1}, x_n)] + \frac{\beta}{2}[p(y_{n-1}, y_n) \\
&\quad + p(y_n, y_{n+1}) + p(x_{n-1}, x_n) \\
&\quad + p(x_n, x_{n+1})] + \gamma\Big(\frac{p(y_n, y_{n+1})}{1 + p(y_{n-1}, y_n)}\Big) \\
&\leq \frac{\alpha}{2}[p(y_{n-1}, y_n) + p(x_{n-1}, x_n)] + \frac{\beta}{2}[p(y_{n-1}, y_n) \\
&\quad + p(y_n, y_{n+1}) + p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \\
&\quad + \gamma p(y_n, y_{n+1}).
\end{aligned} \tag{3.27}$$

Hence, by setting

$$H_n = p(x_n, x_{n+1}) + p(y_n, y_{n+1}), \tag{3.28}$$

we have

$$\begin{aligned}H_n &= p(x_n, x_{n+1}) + p(y_n, y_{n+1}) \\ &\le \frac{\alpha}{2}[p(x_{n-1}, x_n) + p(y_{n-1}, y_n)] + \frac{\beta}{2}[p(x_{n-1}, x_n) \\ &\quad + p(x_n, x_{n+1}) + p(y_{n-1}, y_n) + p(y_n, y_{n+1})] \\ &\quad + \gamma p(x_n, x_{n+1}) + \frac{\alpha}{2}[p(y_{n-1}, y_n) + p(x_{n-1}, x_n)] \\ &\quad + \frac{\beta}{2}[p(y_{n-1}, y_n) + p(y_n, y_{n+1}) + p(x_{n-1}, x_n) \\ &\quad + p(x_n, x_{n+1})] + \gamma p(y_n, y_{n+1}) \\ &= (\alpha+\beta)p(x_{n-1}, x_n) + (\alpha+\beta)p(y_{n-1}, y_n) \\ &\quad + (\beta+\gamma)p(x_n, x_{n+1}) + (\beta+\gamma)p(y_n, y_{n+1}) \\ &= (\alpha+\beta)[p(x_{n-1}, x_n) + p(y_{n-1}, y_n)] \\ &\quad + (\beta+\gamma)[p(x_n, x_{n+1}) + p(y_n, y_{n+1})] \\ &= (\alpha+\beta)H_{n-1} + (\beta+\gamma)H_n.\end{aligned}$$

Hence, we have

$$(1-\beta-\gamma)H_n \le (\alpha+\beta)H_{n-1}$$

or

$$\begin{aligned}H_n &\le \Big(\frac{\alpha+\beta}{1-\beta-\gamma}\Big)H_{n-1} \\ &= \delta H_{n-1},\end{aligned} \tag{3.29}$$

where $\delta = \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right) < 1$, since by hypothesis $\alpha + 2\beta + \gamma < 1$.

• Similarly, if we consider $\Delta((x_{n-1}, y_{n-1}), (x_n, y_n)) = \frac{p(y_n, y_{n+1})}{1+p(y_{n-1}, y_n)}$ and $\Delta((y_{n-1}, x_{n-1}), (y_n, x_n)) = \frac{p(x_n, x_{n+1})}{1+p(x_{n-1}, x_n)}$, then we get the same result as in equation (3.29). Thus, we have

$$H_n \le \delta H_{n-1}, \tag{3.30}$$

where $0 \le \delta = \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right) < 1$.

Consequently, for each $n \in \mathbb{N}$, we obtain

$$H_n \le \delta H_{n-1} \le \delta^2 H_{n-2} \le \cdots \le \delta^n H_0. \tag{3.31}$$

If $H_0 = 0$, then $p(x_0, x_1) + p(y_0, y_1) = 0$. Hence, from Remark 2.5, we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$, means that (x_0, y_0) is a coupled fixed point of F. Now, we assume that $H_0 > 0$. Rest of the proof follows from Theorem 3.1. This completes the proof. □

From Theorem 3.1, we obtain the following results.

Corollary 3.3. ([1, Theorem 2.1]) *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$p(F(x, y), F(u, v)) \leq kp(x, u) + lp(y, v),$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Proof. Follows from Theorem 3.1, by taking $a_1 = k$, $a_2 = l$ and $a_3 = a_4 = a_5 = a_6 = 0$. □

Corollary 3.4. ([1, Theorem 2.4]) *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$p(F(x, y), F(u, v)) \leq kp(F(x, y), x) + lp(F(u, v), u),$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Proof. Follows from Theorem 3.1, by taking $a_3 = k$, $a_4 = l$ and $a_1 = a_2 = a_5 = a_6 = 0$. □

Corollary 3.5. ([1, Theorem 2.5]) *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$p(F(x, y), F(u, v)) \leq kp(F(x, y), u) + lp(F(u, v), x),$$

where k, l are nonnegative constants with $k + 2l < 1$. Then F has a unique coupled fixed point.

Proof. Follows from Theorem 3.1, by taking $a_5 = k$, $a_6 = l$ and $a_1 = a_2 = a_3 = a_4 = 0$. □

It is worth noting that when the constants in Corollary 3.3 are equal, we have the following result.

Corollary 3.6. ([1, Corollary 2.2]) *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$p(F(x, y), F(u, v)) \leq \frac{k}{2}[p(x, u) + p(y, v)],$$

where $0 \leq k < 1$ is a constant. Then F has a unique coupled fixed point.

When the constants in Corollary 3.4 and Corollary 3.5 are equal, we get the following results.

Corollary 3.7. ([1, Corollary 2.6]) *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$p(F(x,y), F(u,v)) \leq \frac{k}{2}[p(F(x,y), x) + p(F(u,v), u)],$$

where $0 \leq k < 1$ is a constant. Then F has a unique coupled fixed point.

Corollary 3.8. ([1, Corollary 2.7]) *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$p(F(x,y), F(u,v)) \leq \frac{k}{2}[p(F(x,y), u) + p(F(u,v), x)],$$

where $0 \leq k < 2/3$ is a constant. Then F has a unique coupled fixed point.

Proof. The condition $0 \leq k < 2/3$ follows from the hypothesis on k and l given in Corollary 3.5. □

Remark 3.9. Theorem 3.1 and Theorem 3.2 extend the results of Sabetghadam *et al.* [15] from cone metric space to the setting of partial metric space.

Remark 3.10. Theorem 3.1 and Theorem 3.2 also generalize the results of Aydi [1].

Now, we give some examples in support of the results.

Example 3.11. Let $X = [0, +\infty)$ endowed with the usual partial metric p defined by $p\colon X \times X \to [0, +\infty)$ with $p(x, y) = \max\{x, y\}$. The partial metric space (X, p) is complete because (X, p^s) is complete. Indeed, for any $x, y \in X$,

$$\begin{aligned} p^s(x,y) &= 2p(x,y) - p(x,x) - p(y,y) \\ &= 2\max\{x,y\} - (x+y) \\ &= |x-y|. \end{aligned} \tag{3.32}$$

Thus, (X, p^s) is the Euclidean metric space which is complete. Consider the mapping $F\colon X \times X \to X$ defined by $F(x, y) = \frac{x+y}{6}$. Now, for any $x, y, u, v \in X$,

we have

$$p(F(x,y),F(u,v)) = \frac{1}{6}\max\{x+y,u+v\} \leq \frac{1}{6}[\max\{x,u\}+\max\{y,v\}] = \frac{1}{6}[p(x,u)+p(y,v)], \tag{3.33}$$

which is the contractive condition of Corollary 3.6 for $k = 1/3$. Therefore, by Corollary 3.6, F has a unique coupled fixed point, which is $(0,0)$. Note that if the mapping $F\colon X \times X \to X$ is given by $F(x,y) = \frac{x+y}{2}$, then F satisfies contractive condition of Corollary 3.6 for $k = 1$, that is,

$$p(F(x,y),F(u,v)) = \frac{1}{2}\max\{x+y,u+v\} \leq \frac{1}{2}[\max\{x,u\}+\max\{y,v\}] = \frac{1}{2}[p(x,u)+p(y,v)]. \tag{3.34}$$

In this case $(0,0)$ and $(1,1)$ are both coupled fixed points of F, and hence, the coupled fixed point of F is not unique. This shows that the condition $k < 1$ in Corollary 3.6, and hence $k + l < 1$ in Corollary 3.3 cannot be omitted in the statement of the aforesaid results.

Example 3.12. Let $X = [0,+\infty)$ endowed with the usual partial metric p defined by $p\colon X \times X \to [0,+\infty)$ with $p(x,y) = \max\{x,y\}$. The partial metric space (X,p) is complete because (X,p^s) is complete. Indeed, for any $x,y \in X$,

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2\max\{x,y\} - (x+y) = |x-y|.$$

Thus, (X,p^s) is the Euclidean metric space which is complete. Consider the mapping $F\colon X \times X \to X$ defined by $F(x,y) = \frac{2x+3y}{15}$.

(1) Let α,β,γ be nonnegative real numbers satisfying $\alpha,\beta,\gamma \geq 0$ with $\alpha + 2\beta + \gamma < 1$, and denote by $\mathbf{L}$ and $\mathbf{R}$, respectively, the left-hand and right-hand side of contractive condition (3.20). It is easy to check that all the conditions of Theorem 3.2 are satisfied for $\alpha,\beta,\gamma \geq 0$ with $\alpha+2\beta+\gamma < 1$ and that $(0,0)$ is a unique coupled fixed point of F.

Consider the example

$$\mathbf{L} \leq \frac{2x+3y}{15} \leq \frac{\alpha}{2}[p(x,u)+p(y,v)] \leq \mathbf{R}.$$

For example, if $(x,y)=(1,2)$, $(u,v)=(2,3)$ for all $(x,y),(u,v)\in X\times X$ and assume that $x\le y$, $u\le v$, $x\le u$ and $y\le v$, then

$$\mathbf{L}=p(F(x,y),F(u,v))=p\Big(\frac{2x+3y}{15},\frac{2u+3v}{15}\Big)=p\Big(\frac{8}{15},\frac{13}{15}\Big)$$
$$=\max\Big\{\frac{8}{15},\frac{13}{15}\Big\}=\frac{13}{15};$$

$$\mathbf{R}=\frac{\alpha}{2}[p(x,u)+p(y,v)]+\frac{\beta}{2}[p(x,F(x,y))+p(u,F(u,v))$$
$$+p(y,F(y,x))+p(v,F(v,u))]+\gamma\Delta((x,y),(u,v)).$$

Suppose $\alpha=\frac{1}{15}$, $\beta=\frac{15}{36}$, $\gamma=0$, then $\alpha,\beta,\gamma\ge 0$ with $\alpha+2\beta+\gamma<1$, we get $\mathbf{R}=\frac{9}{10}$. This implies that $\mathbf{L}\le\mathbf{R}$ and the given contractive condition is satisfied.

(2) Now, we discuss the inequality (3.1) of Theorem 3.1. Let $a_1,\dots,a_6$ be nonnegative real numbers satisfying $a_1,a_2,a_3,a_4,a_5,a_6\ge 0$ with $a_1+a_2+a_3+a_4+a_5+2a_6<1$. It is easy to check that all the conditions of Theorem 3.1 are satisfied for $a_1,a_2,a_3,a_4,a_5,a_6\ge 0$ with $a_1+a_2+a_3+a_4+a_5+2a_6<1$ and that $(0,0)$ is a unique coupled fixed point of F.

If we take $(x,y)=(1,2)$, $(u,v)=(2,3)$ for all $(x,y),(u,v)\in X\times X$ and assume that $x\le y$, $u\le v$, $x\le u$ and $y\le v$, then from L.H.S. of inequality (3.1), we have

$$\mathbf{L}.\mathbf{H}.\mathbf{S}.=p(F(x,y),F(u,v))=p\Big(\frac{2x+3y}{15},\frac{2u+3v}{15}\Big)=p\Big(\frac{8}{15},\frac{13}{15}\Big)$$
$$=\max\Big\{\frac{8}{15},\frac{13}{15}\Big\}=\frac{13}{15}$$

and

$$\mathbf{R}.\mathbf{H}.\mathbf{S}.=2a_1+3a_2+a_3+2a_4+2a_5+3a_6.$$

Thus, we have

$$\frac{13}{15}\le 2a_1+3a_2+a_3+2a_4+2a_5+3a_6.$$

The above inequality is satisfied for (i) $a_1=a_2=a_4=\frac{1}{15}$, $a_3=a_5=\frac{2}{15}$ and $a_6=0$, (ii) $a_1=a_2=a_3=a_6=\frac{1}{15}$, $a_4=\frac{2}{15}$ and $a_5=0$ and (iii) $a_1=a_2=a_6=\frac{1}{15}$, $a_3=a_5=\frac{2}{15}$ and $a_4=0$ etc. with $a_1+a_2+a_3+a_4+a_5+2a_6<1$. Thus all the conditions of Theorem 3.1 are satisfied for $a_1,a_2,a_3,a_4,a_5,a_6\ge 0$ with $a_1+a_2+a_3+a_4+a_5+2a_6<1$. This implies that $\mathbf{L}.\mathbf{H}.\mathbf{S}.\le\mathbf{R}.\mathbf{H}.\mathbf{S}.$ and that $(0,0)$ is a unique coupled fixed point of F.

4. Conclusion

In this paper, we establish a number of unique coupled fixed point theorems within the context of complete partial metric spaces, and we provide a number of corollaries to these main results. In addition, we provide some illustrations to illustrate the findings. In this paper, we obtain results that generalize and extend several results from the previous literature. In particular, our findings generalize and expand upon the related findings of Sabetghadam *et al.* [15] and Aydi [1].

References

[1] H. Aydi, *Some coupled fixed point results on partial metric spaces*, Int. J. Math. Math. Sci., (2011), Article ID 647091, 11 pages.

[2] H. Aydi, M. Abbas and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology and Its Appl., **159**(14) (2012), 3234–3242.

[3] S. Banach, *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math., **3** (1922), 133–181.

[4] T. Gnana Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. TMA, **65**(7) (2006), 1379–1393.

[5] S. Chandok, D. Kumar and M.S. Khan, *Some results in partial metric space using auxiliary functions*, Appl. Math. E-Notes, **15** (2015), 233–242.

[6] L. Ciric and V. Lakshmikantham, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. TMA,, **70**(12) (2009), 4341–4349.

[7] R. Heckmann, *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures, **7**(1-2) (1999), 71–83.

[8] E. Karapinar, W. Shatanawi and K. Tas, *Fixed point theorems on partial metric spaces involving rational expressions*, Miskolc Math. Notes, **14** (2013), 135–142.

[9] H.P.A. Künzi, *Nonsymmetric distances and their associated topologies about the origins of basic ideas in the area of asymptotic topology*, Handbook of the History Gen. Topology (eds. C.E. Aull and R. Lowen), Kluwer Acad. Publ., **3** (2001), 853–868.

[10] N.V. Luong and N.X. Thuan, *Coupled fixed points theorems for mixed monotone mappings and an application to integral equations*, Comput. Math. Appl., **62** (2011), 4238–4248.

[11] S.G. Matthews, *Partial metric topology*, Research Report 2012, Dept. Computer Science, University of Warwick, 1992.

[12] S.G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on Topology and its Applications, Ann. of the New York Acad. Sci., **728** (1994), 183–197.

[13] H.K. Nashine, J.K. Kim, A.K. Sharma and G.S. Saluja, *Some coupled fixed point without mixed monotone mappings*, Nonlinear Funct. Anal. Appl., **21**(2) (2016), 235–247.

[14] J.O. Olaleru, G.A. Okeke and H. Akewe, *Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces*, Int. J. Math. Model. Comput., **2**(2) (2012), 87–98.

[15] F. Sabetghadam, H.P. Mashiha and A.H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl., (2009), Article ID 125426, 8 pages.

[16] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topl., **6**(2) (2005), 229–240.