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HALPERN REGULARIZATION FOR SOLVING QUASI-EQUILIBRIUM PROBLEMS AND ZERO POINT PROBLEM IN HADAMARD SPACES

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Abstract. In this paper, we propose a Halpern regularization method and prove its strong convergence to a common element of the set of solutions of a quasi-equilibrium problem and a common zero of a finite family of monotone operators in Hadamard spaces. Finally, numerical experiments of our result are presented to further show its applicability.

1. Introduction

Let X be a metric space and $C \subseteq X$ be a nonempty, closed and convex set and $K(\cdot)$ be a multivalued mapping from C into itself such that for all $x \in C$, $K(x)$ is a nonempty, closed and convex subset of C, and let $f{:}X \times X \to \mathbb{R}$ be a bifunction. The quasi-equilibrium problem $QEP(f, K)$ consists of finding $x^* \in K(x^*)$, that is, a fixed point x^* of $K(\cdot)$, such that

$$f(x^*, y) \geq 0, \quad \forall\, y \in K(x^*). \tag{1.1}$$

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The set of all solutions of $QEP(f,K)$ is denoted by $S(f,K)$. Also, the set of all fixed points of K is denoted by $F(K)$. The associated Minty quasi-equilibrium problem is to find $x^* \in K(x^*)$ such that

$$f(y,x^*)\leq 0, \quad \forall\, y\in K(x^*).$$

If $K(x)=C$ for all $x\in C$, the quasi-equilibrium problem $QEP(f,K)$ becomes a classical equilibrium problem $EP(f,C)$, also the associated Minty quasi-equilibrium problem becomes a classical Minty equilibrium problem (see [12]).

An example of a quasi-equilibrium problem is a quasi-variational inequality problem. Let $K(\cdot)$ be a multivalued mapping from C into itself such that for all $x\in C$, $K(x)$ is a nonempty, closed and convex subset of C, consider a map $T: X \to E^*$ and define $f(x,y) = \langle T(x), y-x\rangle$, where $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$ denotes the duality pair, that is, $\langle z, x\rangle = z(x)$. Then $QEP(f,K)$ is equivalent to the quasi-variational inequality problem $QVIP(T,K)$, consisting of finding a point $x^* \in K(x^*)$ such that $\langle T(x^*), x-x^*\rangle \geq 0$ for all $x \in K(x^*)$.

The theory of equilibrium problems (and, in particular, VIP) has permitted the general and unified study of a large number of problems in mathematical economics, optimization and operational research. Some important problems such as scalar and vector optimization problems, saddle-point (minimax) problems, variational inequalities, Nash equilibria problems, complementarity problems and fixed point problems can be formulated in the form of EP. Equilibrium problems have been studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors e.g.([5, 7, 8, 14, 17, 18, 19, 20, 28, 29, 31]).

The quasi-equilibrium problem (and, in particular, quasi-variational inequality) have a number of important applications, for example, in economics, engineering, and operations research, see ([15, 16, 27, 37]). Quasi-quilibrium problems with monotone and pseudo-monotone bifunctions were studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors (for example, [11, 34]).
Very recently, Khatibzadeh and Ranjbar [24] generalized monotone operators and their resolvents to Hadamard spaces by using the duality theory introduced in [22]. Reich and Salinas [32] established metric convergence theorems for infinite products of possibly discontinuous operators defined on Hadamard spaces (see also [31, 35]).

Inspired by the [34], in this paper we perform some modifications on the Halpern regularization method in order to introduce and analyze the Halpern regularization method with linesearch for finding a common element of the set of solutions of a quasi-equilibrium problem and a common zero of a finite family of monotone operators in Hadamard spaces under rather mild assumptions (see also [1, 25]).

The paper is organized as follows. In Section 2, we give a brief introduction of Hadamard spaces and some some preliminary material related to the geometry of Hadamard spaces that we need to prove the main result. In Section 3, we introduce and analyze the Halpern regularization method with linesearch for finding a common element of the set of solutions of a quasi-equilibrium problem and a common zero of a finite family of monotone operators in Hadamard spaces, and we prove the strong convergence of the generated sequence to a common element of the set of solutions of a quasi-equilibrium problem and a common zero of a finite family of monotone operators. Also, we will illustrate some applications for our algorithm. Finally, in Section 4, a numerical example for validity our main theorem will be exposed.

2. Preliminaries

Let us present some concepts and facts regarding Hadamard spaces (for more detail, see [2, 36]).

Let (X, d) be a metric space and $x, y \in X$. A geodesic path connecting points x and y is an isometry $c : [0, d(x, y)] \to X$ such that $c(0)=x, c(l)=y$, where $l := d(x, y)$ and $d(c(t), c(t'))=|t-t'|$ for all $t, t' \in [0, l]$. The set $c([0, l]) \subset X$ is denoted by $[x, y]$ and is called a geodesic segment (or geodesic) with the ends x and y. Metric space (X, d) is called a geodesic space if any two points of X can be connected by a geodesic, and it is called a uniquely geodesic space if for any two points from X there exists exactly one geodesic connecting them. Let X be a uniquely geodesic metric space, for each $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z)=(1-t)d(x, y)$ and $d(y, z)=td(x, y)$. We will use the notation $tx \oplus (1-t)y$ for denoting the unique point z satisfying the above statement.

Definition 2.1. ([10]) A geodesic space X is called $CAT(0)$ space if for all $x, y, z \in X$ and $t \in [0, 1]$ it holds that

$$d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y). \tag{2.1}$$

A complete $CAT(0)$ space is called an Hadamard space.

Remark 2.2. Important examples of $CAT(0)$ spaces are Euclidean spaces, $\mathbb{R}-$trees, Hadamard manifolds (complete connected Riemannian manifolds of nonpositive curvature) and Hilbert ball with hyperbolic metrics [2, 6, 36].

Berg and Nikolaev in [3, 4] introduced the concept of quasi-linearization as follows. Let us formally denote a pair $(a, b) \in X \times X$ as $\overrightarrow{ab}$ and call it a vector.

Then quasi-linearization is characterized as a map $\langle\cdot,\cdot\rangle:(X\times X)\times(X\times X)\to\mathbb{R}$ defined by

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\frac{1}{2}\Big(d^2(a,d)+d^2(b,c)-d^2(a,c)-d^2(b,d)\Big),\quad a,b,c,d\in X. \tag{2.2}$$

It is easy to see that $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\langle\overrightarrow{cd},\overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=-\langle\overrightarrow{ba},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ax},\overrightarrow{cd}\rangle+\langle\overrightarrow{xb},\overrightarrow{cd}\rangle=\langle\overrightarrow{ab},\overrightarrow{cd}\rangle$ for all $a,b,c,d,x\in X$. We say that X satisfies the Cauchy-Schwarz inequality if $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle\le d(a,b)d(c,d)$ for all $a,b,c,d\in X$. It is known (Corollary 3 of [4]) that a geodesically connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy-Schwarz inequality.

Let (X,d) be an Hadamard space and $\{x_n\}$ be a bounded sequence in X. Take $x\in X$. Let $r(x,\{x_n\})=\limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius of $\{x_n\}$ is given by:

$$r(\{x_n\})=\inf\{r(x,\{x_n\}):x\in X\},$$

and the asymptotic center of $\{x_n\}$ is the set

$$AC(\{x_n\})=\{x\in X:r(x,\{x_n\})=r(\{x_n\})\}.$$

It is known that in an Hadamard space, $AC(\{x_n\})$ consists exactly one point.

Definition 2.3. ([26]) A sequence $\{x_n\}$ in an Hadamard space (X,d) is $\triangle$-convergent to $x\in X$ if $AC(\{x_{n_k}\})=\{x\}$, for each subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

We denote $\triangle$-convergence in X by $\xrightarrow{\triangle}$ and the metric convergence by $\to$. Now, we present a known result related to the notion of $\triangle$-convergence.

Lemma 2.4. ([26]) *Let X be an Hadamard space. Then, every bounded, closed and convex subset of X is $\triangle$-compact, that is, every bounded sequence in it has a $\triangle$-convergent subsequence.*

Lemma 2.5. ([10]) *Let (X,d) be a $CAT(0)$ space. Then, for all $x,y,z\in X$ and $t\in[0,1]$, it holds that*

$$d(tx\oplus(1-t)y,z)\le td(x,z)+(1-t)d(y,z).$$

Definition 2.6. Let (X,d) be an Hadamard space and $C\subset X$. The projection on C, $P_C:X\to C$, by taking $P_C(x)$ as the $u\in C$ such that

$$d(u,x)=\inf\{d(z,x):z\in C\}. \tag{2.3}$$

It is well known that if $C\subset X$ is nonempty, closed and convex, then for any $x\in X$ there exists a unique $u\in C$ which satisfies (2.3).

We give next a characterization of the projection.

Proposition 2.7. ([9]) *Let C be a nonempty convex subset of a $CAT(0)$ space X, $x{\in}X$ and $u{\in}C$. Then $u{=}P_C(x)$ if and only if*

$$\langle \overrightarrow{yu}, \overrightarrow{xu} \rangle \leq 0, \quad \forall\, y \in C.$$

Kakavandi and Amini [22] have introduced the concept of dual space of an Hadamard space X, based on a work of Berg and Nikolaev [4], as follows. Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ defined by:

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (a, b, x \in X, t \in \mathbb{R}),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on $\mathbb{R} \times X \times X$. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$, for all $t \in \mathbb{R}$ and $a, b \in X$, where $L(\varphi) = \sup\{\dfrac{\varphi(x) - \varphi(y)}{d(x, y)}; x, y \in X,\ x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by:

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (a, b, c, d \in X,\ t, s \in \mathbb{R}).$$

For an Hadamard space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$. By ([22], Lemma 2.1), $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$. Thus, D induces an equivalence relation on $\mathbb{R} \times X \times X$ where the equivalence class of (t, a, b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd};\ t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle,\ \forall\, x, y \in X\}.$$

The set $X^* := \{[t\overrightarrow{ab}]; (t, a, b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b), (s, c, d))$, which is called the dual space of (X, d). It is clear that $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a, b \in X$. Fix $o \in X$, we write $\mathbf{0} = [\overrightarrow{oo}]$ as the zero of the dual space. Note that X^* acts on $X \times X$ by:

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle,\ (x^* = [t\overrightarrow{ab}] \in X^*,\ x, y \in X).$$

Let X be an Hadamard space with dual X^* and let $A{:}X{\rightrightarrows}X^*$ be a multivalued operator with domain $D(A){:=}\{x{\in}X, Ax{\neq}\emptyset\}$, range $R(A){:=}\bigcup_{x\in X} Ax$, $A^{-1}(x^*){=}\{x{\in}X, x^*{\in}Ax\}$ and graph

$$grap(A) := \{(x, x^*){\in}X \times X^*, x{\in}D(A), x^*{\in}Ax\}.$$

Definition 2.8. Let X be an Hadamard space with dual X^*. The multivalued operator $A : X{\rightrightarrows}X^*$ is said to be monotone if the inequality $\langle x^* - y^*, \overrightarrow{yx} \rangle{\geq}0$ holds for every $(x, x^*), (y, y^*){\in}grap(A)$.

Definition 2.9. Let X be an Hadamard space with dual X^*, $\lambda>0$ and let $A:X\rightrightarrows X^*$ be a multivalued operator. The resolvent of A of order λ is the multivalued mapping $J_\lambda^A:X\rightrightarrows X$, defined by $J_\lambda^A(x):=\{z\in X, [\frac{1}{\lambda}\overrightarrow{zx}]\in Az\}$. Indeed

$$J_\lambda^A=(\overrightarrow{oI}+\lambda A)^{-1}\circ\overrightarrow{oI},$$

where o is an arbitrary member of X and $\overrightarrow{oI}(x):=[\overrightarrow{ox}]$. It is obvious that this definition is independent of the choice of o.

Let C be a nonempty subset of an Hadamard space X and $T:C\to C$ be a mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T)=\{x\in C: x=Tx\}$.

Definition 2.10. The mapping $T:C\to C$ is called quasi-nonexpansive whenever $F(T)\neq\emptyset$ and $d(p,Tx)\leqslant d(p,x)$ for all $p\in F(T), x\in C$.

Definition 2.11. Suppose that $K:C\to 2^C$ is a multivalued mapping such that for every $x\in C$, $K(x)$ is nonempty, closed and convex. K is called quasi-nonexpansive whenever the mapping $T(\cdot):=P_{K(\cdot)}(\cdot)$ is quasi-nonexpansive where P is the projection mapping.

Theorem 2.12. ([24]) *Let X be a $CAT(0)$ space with dual X^* and let $A: X\rightrightarrows X^*$ be a multivalued mapping. Then*

(i) *for any $\lambda>0$, $R(J_\lambda^A)\subset D(A), F(J_\lambda^A)=A^{-1}(0)$,*

(ii) *if A is monotone, then J_λ^A is a single-valued on its domain and*

$$d^2(J_\lambda^A x, J_\lambda^A y)\leq\langle\overrightarrow{J_\lambda^A xJ_\lambda^A y},\overrightarrow{xy}\rangle,\quad\forall\, x,y\in D(J_\lambda^A),$$

in particular J_λ^A is a nonexpansive mapping.

(iii) *if A is monotone and $0<\lambda\leq\mu$, then $d^2(J_\lambda^A x, J_\mu^A x)\leq\frac{\mu-\lambda}{\mu+\lambda}d^2(x,J_\mu^A x)$, which implies that $d(x,J_\lambda^A x)\leq 2d(x,J_\mu^A x)$.*

It is well known that if T is a nonexpansive mapping on a subset C of a $CAT(0)$ space X, then $F(T)$ is closed and convex. Thus, if A is a monotone operator on a $CAT(0)$ space X, then, by parts (i) and (ii) of Theorem 2.12, $A^{-1}(0)$ is closed and convex. Also using part (ii) of this theorem for all $u\in F(J_\lambda^A)$ and $x\in D(J_\lambda^A)$, we have

$$d^2(J_\lambda^A x,x)\leq d^2(u,x)-d^2(u,J_\lambda^A x). \tag{2.4}$$

Definition 2.13. The mapping $K:C\to 2^C$ is called demiclosed, whenever we have $x_k\xrightarrow{\triangle}\overline{x}$ and $\lim\limits_{k\to\infty}d(x_k,K(x_k))=0$, then $\overline{x}\in F(K)$.

Lemma 2.14. ([10]) *Let C be a closed and convex subset of an Hadamard space X, $T : C \to C$ be a nonexpansive mapping and $\{x_n\}$ be a bounded sequence in C such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $x_n \overset{\triangle}{\to} x$. *Then* $x = Tx$.

Lemma 2.15. ([30]) *Let (X, d) be an Hadamard space and $\{x_n\}$ be a sequence in X. If there exists a nonemty subset C of X satisfying:*

(i) *for every $z \in C$, $\lim_{n\to\infty} d(x_n, z)$ exists,*

(ii) *if a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ is $\triangle$-convergent to $x \in X$, then $x \in C$.*

Then, there exists $p \in C$ such that $\{x_n\}$ is $\triangle$-convergent to p in X.

In order to prove the strong convergence result by algorithm in the next section, we need an intermediate result which establishes an elementary property of real sequences.

Lemma 2.16. ([33]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n, \quad \textit{for all } n \geq 1.$$

If $\limsup_{k\to\infty} t_{n_k} \leq 0$, then for every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ satisfying

$$\liminf_{k\to\infty} (s_{n_{k+1}} - s_{n_k}) \geq 0,$$

it holds $\lim_{k\to\infty} s_n = 0$.

We introduce now some conditions on the bifunction f and the multivalued mapping K which are needed in the convergence analysis.

B_1: $f(x, x) = 0$ for all $x \in X$.

B_2: $f(x, \cdot) : X \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in X$.

B_3: $f(\cdot, y)$ is $\triangle$-upper semicontinuous for all $y \in X$.

B_4: f is Lipschitz-type continuous, that is, there exist two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 d^2(x, y) - c_2 d^2(y, z), \quad \forall\, x, y, z \in X.$$

B_5: f is pseudo-monotone, that is, whenever $f(x, y) \geq 0$ with $x, y \in X$, it holds that $f(y, x) \leq 0$.

B_6: $K_j : C \to 2^C, (1 \leq j \leq M)$ are quasi-nonexpansive and demiclosed mappings with nonempty, closed and convex values.

In order to well definedness and boundedness of the generated sequences by our algorithm in this paper, we assume that

$$S^* = \Big\{x \in \bigcap_{j=1}^{M} K_j(x) \cap \bigcap_{i=1}^{N} A_i^{-1}(0) : f(x,y) \geq 0, \ \ \forall\, y \in C\Big\} \neq \emptyset.$$

Note that under $B_1 - B_6$, S^* is closed and convex. It is easy to see that $S^* \subset \Omega$, where $\Omega := \bigcap_{i=1}^{N} A_i^{-1}(0) \cap S(f, \bigcap_{j=1}^{M} K_j)$.

3. Halperns regularization method

In this section, we assume that $C \subset X$ is a nonempty, closed and convex set of an Hadamard space X. Let the bifunction $f{:}X \times X \to \mathbb{R}$ satisfies $B1 - B5$ and let $K_j{:}C \to 2^C$ for all $1 \leq j \leq M$ be multivalued quasi-nonexpansive mapping and satisfy in $B6$. Let $A_i{:}X \rightrightarrows X^*$ for all $1 \leq i \leq N$ be multi-valued monotone operators. We introduce the following algorithm for finding a common element of the set of solutions of a quasi-equilibrium problem and a common zero of a finite family of monotone operators.

Algorithm 3.1. Initialization: Choose $v_0, u \in C$ and for sequences $\{\lambda_n\}$, $\{\beta_n\}$, $\{\gamma_n^i\}$ and $\{\alpha_n\}$ such that

(H_1) $\{\lambda_n\} \subset [a,b] \subset \Big(0, \Big\{ \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\Big\}\Big)$,
(H_2) $\{\beta_n\} \subset [c,d] \subset (0,1)$,
(H_3) $\{\gamma_n^i\} \subset (0,\infty)$ and $\liminf_{n\to\infty} \gamma_n^i > 0$ for $i = 1, 2, \cdots, N$,
(H_4) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Step 0: Set $n = 0$ and go to Step 1.

Step 1: Compute $t_n = J_{\gamma_n^N}^{A_N} o \cdots o J_{\gamma_n^1}^{A_1}(v_n)$.

Step 2: Compute $w_n = P_{K_M(t_n)} o \cdots o P_{K_1(t_n)}(t_n)$.

Step 3: Compute $x_n = \beta_n v_n \oplus (1-\beta_n) w_n$.

Step 4: Solve the following minimization problem and let y_n be the solution of it, that is, $y_n = argmin_{y \in C}\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}$.

Step 5: Solve the following minimization problem and let z_n be the solution of it, that is, $z_n = argmin_{y \in C}\{f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}$.

Step 6: Compute $v_{n+1} = \alpha_n u \oplus (1-\alpha_n) z_n$.

Step 7: Put $n := n+1$ and go to Step 1.

Remark 3.2. The sequences generated by the Algorithm 3.1 are well defined.

The following lemma will be useful in the proof of the main theorem.

Lemma 3.3. *Let $\{t_n\}$, $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$ be sequences generated by Algorithm* 3.1 *and $x^*{\in}S^*$. Then*

$$\begin{aligned}d^2(z_n,x^*) \leq d^2(v_n,x^*) - \beta_n(1-\beta_n)d^2(v_n,w_n)\\ - (1-2c_1\lambda_n)d^2(x_n,y_n) - (1-2c_2\lambda_n)d^2(y_n,z_n).\end{aligned}$$

Proof. By the definition of z_n, we have

$$f(y_n,z_n){+}\frac{1}{2\lambda_n}d^2(x_n,z_n){\leq}f(y_n,y){+}\frac{1}{2\lambda_n}d^2(x_n,y), \quad \forall\, y\in C.$$

Substituting $y=tz_n{\oplus}(1{-}t)x^*$ into the last inequality and using B_2, we have

$$\begin{aligned}&f(y_n,z_n){+}\frac{1}{2\lambda_n}d^2(x_n,z_n)\\ &\leq f(y_n,tz_n{\oplus}(1{-}t)x^*){+}\frac{1}{2\lambda_n}d^2(x_n,tz_n{\oplus}(1{-}t)x^*)\\ &\leq tf(y_n,z_n){+}(1-t)f(y_n,x^*)\\ &\quad+\frac{1}{2\lambda_n}\big\{td^2(x_n,z_n){+}(1-t)d^2(x_n,x^*){-}t(1-t)d^2(z_n,x^*)\big\}.\end{aligned}$$

Since f is pseudo-monotone, we have $f(y_n,x^*){\leq}0$. Hence, we get

$$\begin{aligned}&f(y_n,z_n){+}\frac{1}{2\lambda_n}d^2(x_n,z_n)\\ &\leq tf(y_n,z_n){+}\frac{1}{2\lambda_n}\big\{td^2(x_n,z_n){+}(1-t)d^2(x_n,x^*){-}t(1-t)d^2(z_n,x^*)\big\},\end{aligned}$$

therefore,

$$(1-t)f(y_n,z_n){\leq}\frac{1}{2\lambda_n}\big\{(1{-}t)d^2(x_n,x^*){-}t(1{-}t)d^2(z_n),x^*)-(1{-}t)d^2(x_n,z_n)\big\}.$$

Then, we obtain

$$f(y_n,z_n)\leq\frac{1}{2\lambda_n}\big\{d^2(x_n,x^*){-}d^2(x_n,z_n){-}td^2(z_n,x^*)\big\}.$$

Taking the limit in the last inequality as $t{\to}1^-$, we obtain

$$f(y_n,z_n){\leq}\frac{1}{2\lambda_n}\big\{d^2(x_n,x^*){-}d^2(x_n,z_n){-}d^2(z_n,x^*)\big\}. \tag{3.1}$$

By the definition of y_n, we have

$$f(x_n,y_n){+}\frac{1}{2\lambda_n}d^2(x_n,y_n){\leq}f(x_n,y){+}\frac{1}{2\lambda_n}d^2(x_n,y), \quad \forall\, y\in C.$$

Substituting $y = ty_n \oplus (1-t)z_n$ into the last inequality and using B_2, we have

$$\begin{aligned} &f(x_n, y_n) + \frac{1}{2\lambda_n} d^2(x_n, y_n) \\ &\le f(x_n, ty_n \oplus (1-t)z_n) + \frac{1}{2\lambda_n} d^2(x_n, ty_n \oplus (1-t)z_n) \\ &\le tf(x_n, y_n) + (1-t)f(x_n, z_n) \\ &\quad + \frac{1}{2\lambda_n}\big\{td^2(x_n, y_n) + (1-t)d^2(x_n, z_n) - t(1-t)d^2(y_n, z_n)\big\}. \end{aligned}$$

Taking the limit in the last inequality as $t \to 1^-$, we obtain

$$f(x_n, y_n) - f(x_n, z_n) \le \frac{1}{2\lambda_n}\big\{d^2(x_n, z_n) - d^2(x_n, y_n) - d^2(y_n, z_n)\big\}. \tag{3.2}$$

Since f is Lipschitz-type continuous with constants c_1 and c_2, we have

$$f(x_n, y_n) + f(y_n, z_n) \ge f(x_n, z_n) - c_1 d^2(x_n, y_n) - c_2 d^2(y_n, z_n),$$

therefore, we obtain

$$f(y_n, z_n) \ge f(x_n, z_n) - f(x_n, y_n) - c_1 d^2(x_n, y_n) - c_2 d^2(y_n, z_n). \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we obtain

$$d^2(x_n, x^*) - d^2(z_n, x^*) \ge (1 - 2c_1\lambda_n) d^2(x_n, y_n) + (1 - 2c_2\lambda_n) d^2(y_n, z_n). \tag{3.4}$$

Note that $x_n := \beta_n v_n \oplus (1-\beta_n) w_n$, then we have

$$\begin{aligned} d^2(x_n, x^*) &= d^2(\beta_n v_n \oplus (1-\beta_n) w_n, x^*) \\ &\le \beta_n d^2(v_n, x^*) + (1-\beta_n) d^2(w_n, x^*) - \beta_n(1-\beta_n) d^2(v_n, w_n). \end{aligned} \tag{3.5}$$

Since $x^* \in S^*$, this implies that $x^* \in \bigcap_{j=1}^{M} K_j(x^*)$.

Note that $w_n = P_{K_M(t_n)} o \cdots o P_{K_1(t_n)}(t_n)$ and K_j for all $1 \le j \le M$ are quasi nonexpansive mappings, therefore we have

$$d(w_n, x^*) \le d(t_n, x^*). \tag{3.6}$$

From (3.5) and (3.6), we get

$$d^2(x_n, x^*) \le \beta_n d^2(v_n, x^*) + (1-\beta_n) d^2(t_n, x^*) - \beta_n(1-\beta_n) d^2(v_n, w_n). \tag{3.7}$$

Combining (3.4) and (3.7), we obtain

$$\begin{aligned} d^2(z_n, x^*) \le\ & \beta_n d^2(v_n, x^*) + (1-\beta_n) d^2(t_n, x^*) - \beta_n(1-\beta_n) d^2(v_n, w_n) \\ & - (1 - 2c_1\lambda_n) d^2(x_n, y_n) - (1 - 2c_2\lambda_n) d^2(y_n, z_n). \end{aligned} \tag{3.8}$$

Since $x^* \in S^*$, this implies that $x^* \in \bigcap_{i=1}^{N} A_i^{-1}(0)$.

Note that $t_n = J^{A_N}_{\gamma^N_n} o \cdots o J^{A_1}_{\gamma^1_n}(v_n)$, A_i are monotone mappings and $J^{A_i}_{\gamma^i_n}$ for all $1 \leq i \leq N$ are nonexpansive mappings, therefore we have

$$d(t_n, x^*) \leq d(v_n, x^*). \tag{3.9}$$

This together with (3.8) completes the proof. □

Now, we are ready for the main result of this section.

Theorem 3.4. *Suppose that Conditions $B_1 - B_6$ hold and $S^* \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges strongly to $P_{S^*}u$.*

Proof. Let $x^* = P_{S^*}u$. First we show that the sequence $\{v_n\}$ generated by Algorithm 3.1 is bounded. From the definition of v_{n+1} and Lemmas 2.5, 3.3, we have

$$\begin{aligned} d(v_{n+1}, x^*) =& d\left(\alpha_n u \oplus (1-\alpha_n) z_n, x^*\right) \\ \leq& \alpha_n d(u, x^*) + (1-\alpha_n) d(z_n, x^*) \\ \leq& \alpha_n d(u, x^*) + (1-\alpha_n) d(v_n, x^*) \\ \leq& \max\left\{d(u, x^*), d(v_n, x^*)\right\} \\ & \vdots \\ \leq& \max\left\{d(u, x^*), d(v_0, x^*)\right\}. \end{aligned} \tag{3.10}$$

Hence the sequence $\{v_n\}$ is bounded and by Lemma 3.3, the sequence $\{z_n\}$ is bounded too. On the other hand, from (2.1) and Lemma 3.3, we obtain

$$\begin{aligned} d^2(v_{n+1}, x^*) &\leq \alpha_n d^2(u, x^*) + (1-\alpha_n) d^2(z_n, x^*) - \alpha_n(1-\alpha_n) d^2(u, z_n) \\ &\leq \alpha_n d^2(u, x^*) + (1-\alpha_n) d^2(v_n, x^*) - \alpha_n(1-\alpha_n) d^2(u, z_n). \end{aligned}$$

Now we show that $d^2(v_n, x^*) \to 0$. To do this using Lemma 2.16, it is sufficient to show that

$$\limsup \left(d^2(u, x^*) - (1-\alpha_{n_k}) d^2(u, z_{n_k})\right) \leq 0,$$

for every subsequence $\{d^2(v_{n_k}, x^*)\}$ of $\{d^2(v_n, x^*)\}$ that satisfies,

$$\liminf \left(d^2(v_{n_{k+1}}, x^*) - d^2(v_{n_k}, x^*)\right) \geq 0, \tag{3.11}$$

therefore, we have

$$\begin{aligned}
0 &\leqslant \liminf \left(d^2(v_{n_{k+1}}, x^*) - d^2(v_{n_k}, x^*)\right) \\
&\leq \liminf \left(\alpha_{n_k} d^2(u, x^*) + (1-\alpha_{n_k}) d^2(z_{n_k}, x^*) - d^2(v_{n_k}, x^*)\right) \\
&= \liminf \Big(\alpha_{n_k}\left(d^2(u, x^*) - d^2(z_{n_k}, x^*)\right) + d^2(z_{n_k}, x^*) - d^2(v_{n_k}, x^*)\Big) \\
&\leq \limsup \left(\alpha_{n_k}\left(d^2(u, x^*) - d^2(z_{n_k}, x^*)\right)\right) \\
&\quad + \liminf \left(d^2(z_{n_k}, x^*) - d^2(v_{n_k}, x^*)\right) \\
&= \liminf \left(d^2(z_{n_k}, x^*) - d^2(v_{n_k}, x^*)\right) \\
&\leq \limsup \left(d^2(z_{n_k}, x^*) - d^2(v_{n_k}, x^*)\right) \\
&\leq 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty} d^2(z_{n_k}, x^*) - d^2(v_{n_k}, x^*) = 0. \tag{3.12}$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty}(1-2c_i\lambda_n) > 0$ for $i = 1, 2$, replacing n by n_k in Lemma 3.3, we have

$$\begin{aligned}
\liminf d^2(z_{n_k}, x^*) \leq \liminf d^2(v_{n_k}, x^*) &- \beta_{n_k}(1-\beta_{n_k}) d^2(v_{n_k}, w_{n_k}) \\
&- (1-2c_1\lambda_{n_k}) d^2(x_{n_k}, y_{n_k}) - (1-2c_2\lambda_{n_k}) d^2(y_{n_k}, z_{n_k}),
\end{aligned}$$

this implies that

$$\lim d^2(x_{n_k}, y_{n_k}) = \lim d^2(y_{n_k}, z_{n_k}) = \lim d^2(v_{n_k}, w_{n_k}) = 0. \tag{3.13}$$

Replacing n by n_k in (3.1) and (3.4), taking limit and using (3.13), we get

$$\lim_{k\to\infty} f(y_{n_k}, z_{n_k}) = 0. \tag{3.14}$$

There exists subsequence $\{z_{n_{k_t}}\}$ of $\{z_{n_k}\}$ such that $z_{n_{k_t}} \stackrel{\Delta}{\to} p \in C$ and since $d^2(u, .)$ is $\triangle$-lower semicontinuous, we have

$$\begin{aligned}
\limsup(d^2(u, x^*) - (1-\alpha_{n_k}) d^2(u, z_{n_k})) &= \limsup(d^2(u, x^*) - (1-\alpha_{n_{k_t}}) d^2(u, z_{n_{k_t}})) \\
&\leq d^2(u, x^*) - d^2(u, p).
\end{aligned} \tag{3.15}$$

By the definition of z_n, we have

$$f(y_n, z_n) + \frac{1}{2\lambda_n} d^2(x_n, z_n) \leq f(y_n, z) + \frac{1}{2\lambda_n} d^2(x_n, z), \quad \forall\, z \in C.$$

Substituting $z := tz_n \oplus (1-t)y$ into the last inequality and using B_2, we have

$$\begin{aligned}
&f(y_n, z_n) + \frac{1}{2\lambda_n} d^2(x_n, z_n) \\
&\leq f(y_n, tz_n \oplus (1-t)y) + \frac{1}{2\lambda_n} d^2(x_n, tz_n \oplus (1-t)y) \\
&\leq tf(y_n, z_n) + (1-t)f(y_n, y) \\
&\quad + \frac{1}{2\lambda_n}\Big(td^2(x_n, z_n) + (1-t)d^2(x_n, y) - t(1-t)d^2(z_n, y)\Big),
\end{aligned}$$

therefore,

$$f(y_n, z_n) - f(y_n, y) \leq \frac{1}{2\lambda_n}\Big(d^2(x_n, y) - d^2(x_n, z_n) - td^2(z_n, y)\Big).$$

Taking the limit in the last inequality as $t \to 1^-$, we obtain

$$\begin{aligned}
f(y_n, y) - f(y_n, z_n) &\geq \frac{1}{2\lambda_n}\Big(d^2(x_n, z_n) + d^2(z_n, y) - d^2(x_n, y)\Big) \\
&\geq \frac{1}{2\lambda_n}\Big(d^2(z_n, y) - d^2(x_n, y)\Big) \\
&= \frac{1}{2\lambda_n}\left(d(z_n, y) - d(x_n, y)\right)\left(d(z_n, y) + d(x_n, y)\right) \\
&\geq \frac{-1}{2\lambda_n} d(x_n, z_n)\Big(d(z_n, y) + d(x_n, y)\Big). \qquad (3.16)
\end{aligned}$$

Therefore, we have

$$f(y_n, y) - f(y_n, z_n) \geq \frac{-1}{2\lambda_n} d(x_n, z_n)\Big(d(z_n, y) + d(x_n, y)\Big). \qquad (3.17)$$

On the other hand, since $z_{n_k} \overset{\triangle}{\to} p$ and using (3.13) we obtain that $y_{n_k} \overset{\triangle}{\to} p$. Replacing n_k with n_{k_t} in (3.17), considering the limsup and using (3.14), we obtain that

$$0 \leq \limsup f(y_{n_{k_t}}, y), \quad \forall\, y \in C.$$

Now, since $f(\cdot, y)$ is $\triangle$-upper semicontinuous, we get

$$f(p, y) \geq 0, \quad \forall\, y \in C. \qquad (3.18)$$

Note that $x_{n_{k_t}} \overset{\triangle}{\to} p$ therefore from (3.13), we have $v_{n_{k_t}} \overset{\triangle}{\to} p$ and $w_{n_{k_t}} \overset{\triangle}{\to} p$. Since K_j for $1 \leq j \leq M$ are demiclosed, thus $p \in \bigcap_{j=1}^M K_j(p)$. (3.18) implies that $p \in S(f, \bigcap_{j=1}^M K_j)$ hence $d(u, x^*) \leq d(u, p)$. Therefore from (3.17) we have

$$\limsup_{k\to\infty}\left(d^2(u, x^*) - (1-\alpha_{n_k})d^2(u, z_{n_k})\right) \leq 0. \qquad (3.19)$$

This together with Lemma 2.16 implies that

$$d^2(v_n, x^*) \to 0. \qquad (3.20)$$

Next, we have to show that $p\in\bigcap_{i=1}^{N} A_i^{-1}(0)$. From Lemma 3.3, we have $d^2(z_n,x)\leq d^2(v_n,x^*)$.

Note that $\Omega=\bigcap_{i=1}^{N} A_i^{-1}(0)\cap S(f,\bigcap_{j=1}^{M} K_j)\neq\emptyset$ and suppose that $x^*\in P_\Omega u$. From nonexpansivity of $J_{\gamma_n^i}^{A_i}$ for all $i=1,2,\cdots,N$, we have

$$\begin{aligned}
d(t_n,x^*) &=d(J_{\gamma_n^N}^{A_N} o\cdots oJ_{\gamma_n^1}^{A_1}(v_n),x^*)\\
&\leq d(J_{\gamma_n^{N-1}}^{A_{N-1}} o\cdots oJ_{\gamma_n^1}^{A_1}(v_n),x^*)\\
&\ \ \vdots\\
&\leq d(v_n,x^*).
\end{aligned}$$

Using Lemma 3.3, we have

$$d(w_n,x^*)\leq d(t_n,x^*)\leq d(v_n,x^*). \tag{3.21}$$

This together with (3.10) implies that $\{t_n\}$ and $\{w_n\}$ are bounded.

On the other hand, from definition of v_{n+1} and Lemma 3.3 we have

$$\begin{aligned}
d^2(v_{n+1},x^*) &= d^2\left(\alpha_n u\oplus(1-\alpha_n)z_n,x^*\right)\\
&\leq \alpha_n d^2(u,x^*)+(1-\alpha_n)d^2(z_n,x^*)-\alpha_n(1-\alpha_n)d^2(u,z_n)\\
&\leq \alpha_n d^2(u,x^*)+(1-\alpha_n)d^2(v_n,x^*)-\alpha_n(1-\alpha_n)d^2(u,z_n)\\
&= (1-\alpha_n)d^2(v_n,x^*)+\alpha_n\left(d^2(u,x^*)-(1-\alpha_n)d^2(u,z_n)\right).
\end{aligned}$$

Now we show that $d^2(v_n,x^*)\to 0$. To do this using Lemma 2.16, it is sufficient to show that

$$\limsup\left(d^2(u,x^*)-(1-\alpha_n)d^2(u,z_n)\right)\leq 0,$$

for every subsequence $\left\{d^2(v_{n_k},x^*)\right\}$ of $\left\{d^2(v_n,x^*)\right\}$ that satisfies,

$$\liminf\left\{d^2(v_{n_{k+1}},x^*)-d^2(v_{n_k},x^*)\right\}\geq 0. \tag{3.22}$$

We define for all $1\leq i\leq N$,

$$S_n^i:=J_{\gamma_n^i}^{A_i} o\ldots oJ_{\gamma_n^1}^{A_1}.$$

So $t_n=S_n^N z_n$ and assume that $S_0=I$ where I is the identity operator. Since $J_{\gamma_n^i}^{A_i}$ for all $1\leq i\leq N$ are nonexpansive mappings and using (3.21), we have

$$\begin{aligned}
d^2(t_n,x^*)-d^2(z_n,x^*) &= d^2(S_n^N v_n,x^*)-d^2(v_n,x^*)\\
&\leq d^2(S_n^i v_n,x^*)-d^2(v_n,x^*)\\
&\leq d^2(v_n,x^*)-d^2(v_n,x^*)\\
&= 0.
\end{aligned}$$

We deduce that

$$\limsup_{n\to\infty}\left(d^2(S_n^i v_n, x^*) - d^2(v_n, x^*)\right) \le 0, \quad i = 1, 2, \ldots N. \tag{3.23}$$

By definition of v_{n+1}, we have

$$\begin{aligned} d^2(v_{n+1}, x^*) &= d^2\left(\alpha_n u \oplus (1-\alpha_n) z_n, x^*\right) \\ &\le \alpha_n d^2(u, x^*) + (1-\alpha_n) d^2(z_n, x^*) - \alpha_n(1-\alpha_n) d^2(u, z_n) \\ &\le \alpha_n d^2(u, x^*) + d^2(z_n, x^*) - \alpha_n d^2(z_n, x^*) - \alpha_n(1-\alpha_n) d^2(u, z_n). \end{aligned}$$

Therefore

$$\begin{aligned} d^2(v_{n+1}, x^*) - d^2(v_n, x^*) \le & \alpha_n\left(d^2(u, x^*) - d^2(z_n, x^*) - (1-\alpha_n) d^2(u, z_n)\right) \\ & + d^2(z_n, x^*) - d^2(v_n, x^*). \end{aligned}$$

Note that $\lim_{n\to\infty}\alpha_n = 0$. Replacing n with n_k into the last inequality and from (3.11), we have

$$0 \le \liminf\left(d^2(S_{n_k}^i v_{n_k}, x^*) - d^2(v_{n_k}, x^*)\right).$$

This together with (3.23) implies that

$$\lim_{n\to\infty}\left(d^2(S_{n_k}^i v_{n_k}, x^*) - d^2(v_{n_k}, x^*)\right) = 0. \tag{3.24}$$

Applying (2.4), we obtain

$$\begin{aligned} d^2(S_{n_k}^i v_{n_k}, S_{n_k}^{i-1} v_{n_k}) &= d^2\left(J_{\gamma_n^i}^{A_i}(S_{n_k}^{i-1} v_{n_k}), S_{n_k}^i v_{n_k}\right) \\ &\le d^2(S_{n_k}^{i-1} x_{n_k}, x^*) - d^2(S_{n_k}^{i-1} x_{n_k}, x^*) \\ &\le d^2(x_{n_k}, x^*) - d^2(S_{n_k}^{i-1} x_{n_k}), x^*), \end{aligned}$$

using (3.24), we get

$$\lim d^2(S_{n_k}^i v_{n_k}, S_{n_k}^{i-1} v_{n_k}) = 0. \tag{3.25}$$

Indeed, we have

$$d(v_{n_k}, S_{n_k}^i v_{n_k}) \le d(v_{n_k}, S_{n_k}^1 x_{n_k}) + \cdots + d(S_{n_k}^{i-1} x_{n_k}, S_{n_k}^i x_{n_k}),$$

taking the limit in the last inequality as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} d(v_{n_k}, S_{n_k}^i v_{n_k}) = 0. \tag{3.26}$$

Since $\liminf_{n\to\infty}\gamma_n^i > 0$, there exists $\gamma \in \mathbb{R}$ such that $\gamma_n^i \ge \gamma > 0$ for all $n \in \mathbb{N}$ and $1 \le i \le N$. Now using inequality (2.4), we have

$$\begin{aligned} d\left(J_\gamma^{A_i}(S_{n_k}^{i-1} v_{n_k}), S_{n_k}^i v_{n_k}\right) &\le d\left(J_\gamma^{A_i}(S_{n_k}^{i-1} v_{n_k}), S_{n_k}^{i-1} v_{n_k}\right) + d\left(S_{n_k}^{i-1} v_{n_k}, S_{n_k}^i v_{n_k}\right) \\ &\le 2d\left(J_{\gamma_{n_k}^i}^{A_i}(S_{n_k}^{i-1} v_{n_k}), S_{n_k}^{i-1} v_n\right) + d\left(S_{n_k}^{i-1} v_{n_k}, S_{n_k}^i v_{n_k}\right) \\ &\le 3d\left(S_{n_k}^i v_{n_k}, S_{n_k}^{i-1} v_{n_k}\right). \end{aligned}$$

Taking the limit in the last inequality as $k \to \infty$ and using (3.26), we obtain that

$$\lim_{k\to\infty} d\left(J_\gamma^{A_i}(S_{n_k}^{i-1}v_{n_k}), S_{n_k}^i v_{n_k}\right) = 0. \tag{3.27}$$

Now for every $1 \le i \le N$, we have

$$\begin{aligned} d\left(J_\gamma^{A_i}v_{n_k}, v_{n_k}\right) &\le d\left(J_\gamma^{A_i}v_{n_k}, J_\gamma^{A_i}(S_{n_k}^{i-1}v_{n_k})\right) + d\left(J_\gamma^{A_i}(S_{n_k}^{i-1}v_{n_k}), S_{n_k}^i v_{n_k}\right) \\ &\quad + d(S_{n_k}^i v_{n_k}, v_{n_k}) \\ &\le d(S_{n_k}^i v_{n_k}, v_{nk}) + d\left(J_\gamma^{A_i}(S_{n_k}^{i-1}v_{n_k}), S_{n_k}^i v_{n_k}\right) \\ &\quad + d(S_{n_k}^i v_{n_k}, v_{n_k}). \end{aligned}$$

Taking the limit in the last inequality as $k \to \infty$ and using (3.27), we obtain that

$$\lim_{k\to\infty} d\left(J_\gamma^{A_i}v_{n_k}, v_{n_k}\right) = 0. \tag{3.28}$$

Let $\{v_{n_{k_t}}\}$ be a subsequence of $\{v_{n_k}\}$ such that $v_{n_{k_t}} \xrightarrow{\Delta} p$. By Lemma 2.14 and (3.28), we get $p \in A_i^{-1}(0)$. Therefore $p \in \bigcap_{i=1}^N A_i^{-1}(0)$. □

4. Application

We will start with definition and some properties of subdifferential mapping.

Definition 4.1. Let X be an Hadamard space with dual X^* and let $g{:}X \to (-\infty, +\infty]$ be a proper function with effective domain $D(g) := \{x : g(x) < +\infty\}$. Then, the subdifferential of g is the multivalued mapping $\partial g{:}X \rightrightarrows X^*$ defined by:

$$\partial g(x) = \left\{x^* \in X^* : \; g(z) - g(x) \ge \langle x^*, \overrightarrow{xz}\rangle, \;\; z \in X\right\},$$

when $x \in D(g)$ and $\partial g(x) = \emptyset$, otherwise.

Theorem 4.2. ([22]) *Let $g{:}X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function on an Hadamard space X with dual X^*. Then*

(i) *g attains its minimum at $x \in X$ if and only if $\mathbf{0} \in \partial g(x)$,*
(ii) *$\partial g{:}X \rightrightarrows X^*$ is a monotone operator,*
(iii) *for any $y \in X$ and $\alpha > 0$, there exists a unique point $x \in X$ such that $[\alpha\overrightarrow{xy}] \in \partial g(x)$.*

Part (iii) of Theorem 4.2 shows that the subdifferential of a convex, proper and lower semicontinuous function satisfies the range condition.

Lemma 4.3. ([24]) *Let $g{:}X{\to}(-\infty,+\infty]$ be a proper, convex and lower semi-continuous function on an Hadamard space X with dual X^*. Then*

$$J_{\lambda}^{\partial g}(x) = argmin_{y\in X}\{g(y) + \frac{1}{2\lambda}d^2(y,x)\}$$

for all $\lambda{>}0$ and $x{\in}X$.

Note that, $C{\subset}X$ is a nonempty, closed and convex set of an Hadamard space X. Let the bifunction $f{:}X{\times}X{\to}\mathbb{R}$ satisfies $B_1 - B_5$ and let $K_j : C{\to}2^C$ for all $1 \leq j \leq M$ be multivalued quasi-nonexpansive mapping and satisfy in B_6. Let $g_i : X{\rightrightarrows}X^*$ for all $1 \leq i \leq N$ be proper functions.

We introduce the following algorithm and corollary.

Algorithm 4.4. Initialization: Choose $z_0 \in C$, $0 < \alpha \leq \lambda_n \leq \beta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $0 < \gamma < \beta_n < \sigma < 1$, $\gamma_n^i \subset (0,\infty)$ and $\liminf_{n\to\infty} \gamma_n^i > 0$.

Step 0: Set $n = 0$ and go to Step 1.

Step 1: Compute $t_n = J_{\gamma_n^N}^{\partial g_N} o \cdots o J_{\gamma_n^1}^{\partial g_1}(v_n)$.

Step 2: Compute $w_n = P_{K_M(t_n)} o \cdots o P_{K_1(t_n)}(t_n)$.

Step 3: Compute $x_n = \beta_n v_n \oplus (1-\beta_n)w_n$.

Step 4: Solve the following minimization problem and let y_n be the solution of it, that is. $y_n = argmin_{y\in C}\{f(x_n,y) + \frac{1}{2\lambda_n}d^2(x_n,y)\}$.

Step 5: Solve the following minimization problem and let z_{n+1} be the solution of it, that is, $z_n = argmin_{y\in C}\{f(y_n,y) + \frac{1}{2\lambda_n}d^2(x_n,y)\}$.

Step 6: Compute $v_{n+1} = \alpha_n u \oplus (1-\alpha_n)z_n$. If $v_{n+1}{=}v_n$, then stop and x_n is solution. Otherwise, put $n := n+1$ and go to Step 1.

Therefore, we have the following corollary.

Corollary 4.5. *Suppose that Conditions $B_1 - B_6$ hold. Then, the sequence $\{x_n\}$ generated by the Algorithm 4.4, is convergent to a point of*

$$\Omega{:=}\bigcap_{i=1}^{N} argmin_{y\in C} g_i(y) {\cap} S(f, \bigcap_{j=1}^{M} K_j).$$

5. Numerical examples

Now, we provide a numerical experiment to validate our obtained results in an Hadamard space.

Example 5.1. Let $X = \mathbb{R}^2$ be endowed with a metric defined by

$$d_H(x, y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. So $(\mathbb{R}^2, d_H)$ is an Hadamard space (see [13, Example 5.2]). with the geodesic joining x to y given by

$$\gamma_{x,y}(t) = ((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)).$$

Let $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ be two functions defined by

$$g_1(x_1, x_2) = 100\big((x_2 + 1) - (x_1 + 1)^2\big)^2 + x_1^2, \quad g_2(x_1, x_2) = 100x_1^2.$$

It follows from [13, Example 5.2] that g_1 is a proper, convex and lower semi-continuous function in $(\mathbb{R}^2, d_H)$ but not convex in the classical sense. Let $f : X \times X \to \mathbb{R}$ be a function defined by

$$f(x, y) = d_H^2(y, 0) - d_H^2(x, 0).$$

It is obvious that f satisfies $B_1 - B_5$ and $c_1 = c_2 = \frac{1}{4}$. Letting $N = 2$, $A_1 = \partial g_1$ and $A_2 = \partial g_2$. Now, let $C = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$, $M = 1$ and $K(\cdot) : C \to 2^C$ be defined by

$$K(x) = \Big\{x \in C : d(0, z) \leq 2 + \frac{1}{2}\sqrt{x_1^2 + (x_1^2 - x_2)^2}\Big\}, \quad \forall\, x \in C.$$

It can be shown that $K(\cdot) : C \to 2^C$ is a multivalued mapping with nonempty, closed and convex values, which is quasi-nonexpansive and demiclosed. Hence B_6 is satisfied and we have $S \neq \emptyset$. In this case, Algorithm 4.4 takes the following form

$$\begin{cases} h_n = argmin_{y \in C}\{g_1(y) + \frac{1}{2\gamma_n^1} d_H^2(y, v_n)\}, \\ t_n = argmin_{y \in C}\{g_2(y) + \frac{1}{2\gamma_n^2} d_H^2(y, h_n)\}, \\ w_n = P_{K(t_n)}(t_n), \\ x_n = \beta_n v_n \oplus (1 - \beta_n) w_n, \\ y_n = argmin_{y \in C}\{f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}, \\ z_n = argmin_{y \in C}\{f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y)\}, \\ v_{n+1} = \alpha_n u \oplus (1 - \alpha_n) z_n. \end{cases}$$

Now, take $\alpha_n = \beta_n = \frac{1}{2}$, $\lambda_n = \frac{1}{2} + \frac{1}{n+2}$, $\gamma_n^1 = \gamma_n^2 = 2n$ for every $n \in \mathbb{N}$ and the initial point $v_1 = (1, 2), u = (1, 1)$. Now, we have numerical results in Fig 1 and Fig 2.

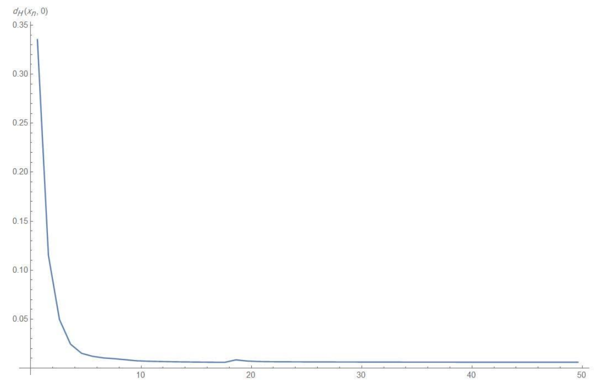

FIGURE 1. Plotting of $d_H(x_n, 0)$

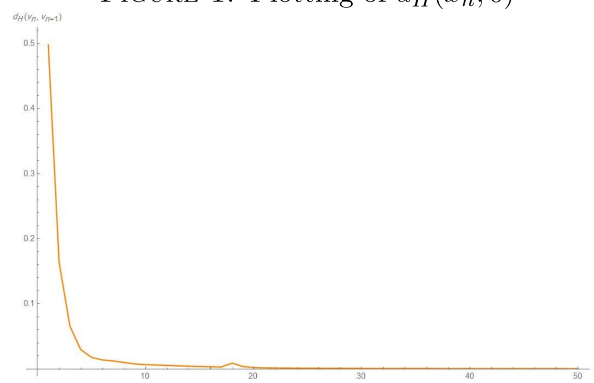

FIGURE 2. Plotting of $d_H(v_n, v_{n-1})$

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