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BOUNDARY CONTROL FOR HEAT EQUATION UNDER CONJUGATION CONDITIONS

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Abstract. In this paper, new problems concerned with boundary control for cooperative Dirichlet or Neumann parabolic systems with conjugation conditions in the presence of concentrated heat capacity are considered. Such systems with observation under conjugation conditions are described. First, the existence and uniqueness of the state process for 2×2 systems is proved,then the set of equations and inequalities that characterizes the boundary control is obtained. The case of $n \times n$ cooperative systems is also established.

1. Introduction

The optimal control problem is one of the important topics in applied mathematics and in several areas related to it, such as biology, economics, ecology, engineering, finance, management, medicine and many others [5, 6, 8, 16]. Various optimal control problems of systems governed by finite order elliptic, parabolic and hyperbolic operators with finite number of variables have been introduced by Lions [7].

Serag et al. have been extended to non-cooperative systems [10, 11] or cooperative systems [2, 3, 4, 9], [12]-[15].

New optimal control problems have been introduced by Sergienko and Deineka [17]-[19] for distributed parameter systems with conjugation conditions and by

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a quadratic cost functional.The considered systems in these problems are in the scalar case(system of one equation).

In this paper we discuss the optimal control of boundary type for cooperative parabolic systems under conjugation conditions. Our paper is organized as follows: In section two, we first prove the existence and uniqueness of the state for 2×2 Dirichlet cooperative system under conjugation conditions, then we study the optimal boundary control of this system. Section three is devoted to discuss the boundary control for 2×2 Neumann cooperative parabolic system under conjugation conditions. Finally, in section four, we generalize the discussion to $n \times n$ cooperative systems.

2. Boundary control for 2×2 cooperative parabolic systems with Dirichlet and conjugation conditions

In this section, we consider the following 2×2 cooperative Dirichlet parabolic system:

$$\begin{bmatrix} \frac{\partial G_1}{\partial t} \\ \frac{\partial G_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \nabla\cdot(\tau\nabla)+\eta_{11} & \eta_{12} \\ \eta_{21} & \nabla\cdot(\tau\nabla)+\eta_{22} \end{bmatrix} \begin{bmatrix} G_1(x,t) \\ G_2(x,t) \end{bmatrix} + \begin{bmatrix} \rho_1(x,t) \\ \rho_2(x,t) \end{bmatrix} \text{ in } \Omega_T, \tag{2.1}$$

$$\begin{bmatrix} G_1(x,0) \\ G_2(x,0) \end{bmatrix} = \begin{bmatrix} G_{1,0}(x) \\ G_{2,0}(x) \end{bmatrix}, G_{1,0}(x), G_{2,0}(x) \in L^2(\Omega) \quad \text{in } \Omega, \tag{2.2}$$

$$\begin{bmatrix} G_1(x,t) \\ G_2(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{on } \Gamma_T, \tag{2.3}$$

is specified in the domain Ω_T, where Ω is a domain that consists of two open, non-intersecting and bounded, continuous, strictly Lipschitz domains Ω_1 and Ω_2 of R^n such that

$$\Omega = (\Omega_1 \cup \gamma \cup \Omega_2), (\Omega_1 \cap \Omega_2) = \phi \quad \text{and} \quad \bar{\Omega} = (\bar{\Omega}_1 \cup \bar{\Omega}_2).$$

Furthermore, let$\Gamma = (\partial\Omega_1 \cup \partial\Omega_2)\backslash\gamma$ be the boundary of the domain $\bar{\Omega}$, $\partial\Omega_k$ be the boundary of a domain Ω_k, $k = 1, 2$, $\Omega_T = \Omega \times (0, T)$ be a complicated cylinder and $\Gamma_T = \Gamma \times (0, T)$,$\rho_i \in L^2(\Omega_T)$ be given function, $\tau = \tau(x)$ be a positive function having discontinuity along γ.

On γ_T, the conjugation conditions are

$$\begin{bmatrix} \left[\sum_{i,j=1}^{n} \tau \frac{\partial G_1}{\partial x_j} \cos(\nu, x_i) \right] \\ \left[\sum_{i,j=1}^{n} \tau \frac{\partial G_2}{\partial x_j} \cos(\nu, x_i) \right] \end{bmatrix} = \begin{bmatrix} c_1 \frac{\partial G_1}{\partial t} \\ c_2 \frac{\partial G_2}{\partial t} \end{bmatrix} \quad \text{on } \gamma_T \tag{2.4}$$

and

$$\begin{bmatrix} [G_1] \\ [G_2] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{on} \quad \gamma_T, \tag{2.5}$$

where $c \in L^2(\gamma)$, $\gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi$, $\gamma_T = \gamma \times (0,T)$ and ν is an ort of a normal to γ and such normal is directed into the domain Ω_2. In addition,

$$[\theta] = \theta^+ - \theta^-,$$

$$\begin{aligned} \theta^+ &= \{\theta\}^+ = \theta(x,t) \quad \text{under } (x,t) \in \gamma_T^+, \\ \theta^- &= \{\theta\}^- = \theta(x,t) \quad \text{under } (x,t) \in \gamma_T^-, \\ \gamma_T^+ &= (\partial\Omega_2 \cap \gamma) \times (0,T), \\ \gamma_T^- &= (\partial\Omega_1 \cap \gamma) \times (0,T). \end{aligned}$$

Definition 2.1. System (2.1) is called cooperative if

$$\eta_{ij} > 0, \ \forall\, i \neq j, \tag{2.6}$$

otherwise is called non-cooperative system [7].

Definition 2.2. ([19, Friedrich inequality]) For any bounded Lipschitz domain Ω, there is a constant $m(\Omega) > 0$, which depends only on Ω, such that

$$\int_\Omega |G|^2\, dx \leq m(\Omega) \int_\Omega |\nabla G|^2\, dx. \tag{2.7}$$

First,we prove the existence and uniqueness for system (2.1)-(2.5), then we prove the existence of boundary control for this system and we find the set of equations and inequalities that characterizes this boundary control.

2.1. **Existence and uniqueness of the state.** Now we define the space

$$V = \{G(x,t) = (G_1, G_2)\mid_{\Omega_i} \in (H^1(\Omega_i)), i=1,2, \ \forall\, t \in [0,T], [G] = 0, G\mid_{\Gamma_T} = 0\}.$$

Analogously, we can define the spaces

$$L^2(0,T;L^2(\Omega)) = L^2(\Omega_T)$$

and

$$V_0 = \{G(x) = (G_1,G_2)\mid_{\Omega_i} \in (H^1(\Omega_i)),\ i\ = 1\ ,\ 2, [G] = 0, G\mid_{\Gamma_T} = 0\},$$

then we have a chain in the form

$$(L^2(0,T;V))^2 \subseteq (L^2(0,T;L^2(\Omega)))^2 \subseteq (L^2(0,T;V'))^2.$$

We introduce the Hilbert space

$$(W(0,T))^2 = \left\{ G : G \in (L^2(0,T;V))^2, \frac{\partial G}{\partial t} \in (L^2(0,T;V')^2 \right\},$$

$(L^2(0,T;V'))^2$ is the dual space of $(L^2(0,T;V))^2$, that being supplied with the norm:

$$\|G(t)\|^2_{W(0,T)} = \left(\int_{]0,T[} \|G(t)\|^2_V dt + \int_{]0,T[} \|\frac{dG}{dt}\|^2_{V'} dt \right).$$

We can then introduce the Sobolev space $(W(0,T))^2$ by Cartesian product:

$$(W(0,T))^2 = \Pi_{i=1}^n (W(0,T))_i$$

with norm defined by

$$\|G\|_{(W(0,T))^2} = \sum_{i=1}^{2} \|G_i\|_{W(0,T)}.$$

$(W(0,T))^2$ may be verified to be a Hilbert space and its dual is denoted by $(W(0,T)')^2$.

The considered spaces in this paper are assumed to be real. Now, let us define on $(L^2(0;T,V))^2$, a bilinear form

$$a(G,\theta) : (W(0,T))^2 \times (W(0,T))^2 \to R$$

by for all $\theta = (\theta_1, \theta_2) \in (V_0)^2$,

$$\begin{aligned} a(t,G;\theta) = a(G,\theta) = & \int_\Omega (\tau(x)\nabla G_1 \nabla \theta_1 + \tau \nabla G_2 \nabla \theta_2)\, dx \\ & - \int_\Omega (\eta_{11} G_1 \theta_1 + \eta_{12} G_2 \theta_1 + \eta_{21} G_1 \theta_2 + \eta_{22} G_2 \theta_2) dx. \end{aligned} \tag{2.8}$$

This bilinear form is continuous, since

$$|a(G,\theta)| \le \alpha_1 \|G\| \|\theta\|. \tag{2.9}$$

Lemma 2.3. *The bilinear form* (2.8) *is coercive on* $(L^2(0;T,V))^2$*, that is, there exists a positive constants* K *and* α *such that:*

$$a(G,G) + K\|G\|^2_{(L^2(\Omega))^2} \ge \alpha \|G\|^2_{(H_0^1(\Omega))^2}, \quad \forall\ G = (G_1, G_2) \in (W(0,T))^2. \tag{2.10}$$

Proof. Since,

$$a(G,G)=\frac{\tau(x)}{2(\eta_{12}+\eta_{21})}\int_\Omega(|\nabla G_1|^2+|\nabla G_1|^2)dx$$
$$+\frac{\tau(x)}{2(\eta_{12}+\eta_{21})}\int_\Omega(|\nabla G_2|^2+|\nabla G_2|^2)dx$$
$$-\int_\Omega G_1G_2dx-\frac{\eta_{11}}{(\eta_{12}+\eta_{21})}\int_\Omega|G_1|^2\,dx$$
$$-\frac{\eta_{22}}{(\eta_{12}+\eta_{21})}\int_\Omega|G_2|^2\,dx$$

and

$$\|G\|^2_{L^2(\Omega)}=\int_\Omega|G|^2\,dx \text{ and } \|G\|^2_{H^1_0(\Omega)}$$
$$=\int_\Omega|\nabla G|^2\,dx,\|G\|^2_{H^1(\Omega)}$$
$$=\int_\Omega(|G|^2+|\nabla G|^2)dx,$$

we get

$$a(G,G)+\max(\frac{(\eta_{11},\eta_{22})}{(\eta_{12}+\eta_{21})})[\|G_1\|^2_{L^2(\Omega)}+\|G_2\|^2_{L^2(\Omega)}]$$
$$=\frac{\tau(x)}{2(\eta_{12}+\eta_{21})}\int_\Omega(|\nabla G_1|^2+|\nabla G_1|^2)dx$$
$$+\frac{\tau(x)}{2(\eta_{12}+\eta_{21})}\int_\Omega(|\nabla G_2|^2+|\nabla G_2|^2)dx-\int_\Omega G_1G_2dx.$$

By Cauchy Schwartz inequality and from Friedrichs inequality, we deduce

$$a(G,G)+\max(\frac{(\eta_{11},\eta_{22})}{(\eta_{12}+\eta_{21})})[\|G_1\|^2_{L^2(\Omega)}+\|G_2\|^2_{L^2(\Omega)}]$$
$$\geq\frac{\tau(x)}{2(\eta_{12}+\eta_{21})}\int_\Omega(|\nabla G_1|^2+(m(\Omega))^{-1}|G_1|^2)dx$$
$$+\frac{\tau(x)}{2(\eta_{12}+\eta_{21})}\int_\Omega(|\nabla G_2|^2+(m(\Omega))^{-1}|G_2|^2)dx-(\int_\Omega|G_1|^2\,dx)^{\frac{1}{2}}(\int_\Omega|G_2|^2\,dx)^{\frac{1}{2}}.$$

This inequality is equivalent to

$$\begin{aligned}
a(G,G) &+ \max(\frac{(\eta_{11},\eta_{22})}{(\eta_{12}+\eta_{21})},\frac{1}{2})[\|G_1\|^2_{L^2(\Omega)}+\|G_2\|^2_{L^2(\Omega)}] \\
&\geq \frac{\tau(x)}{2(\eta_{12}+\eta_{21})}\min(1,(m(\Omega))^{-1})[\|G_1\|^2_{H^1(\Omega)}+\|G_2\|^2_{H^1(\Omega)}] \\
&\quad + (\frac{1}{\sqrt{2}}\|G_1\|_{L^2(\Omega)}-\frac{1}{\sqrt{2}}\|G_2\|_{L^2(\Omega)})^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a(G,G)+K\|G\|^2_{(L^2(\Omega))^2} &\geq \alpha[{G_1}^2_{H^1_0(\Omega)}+{G_2}^2_{H^1_0(\Omega)}] \\
&\geq \alpha\|G\|^2_{(H^1_0(\Omega))^2}, \quad \forall\, G\in(H^1_0(\Omega))^2,
\end{aligned}$$

where

$$K=\max\left(\frac{(\eta_{11},\eta_{22})}{(\eta_{12}+\eta_{21})},\frac{1}{2}\right),\ \ \alpha=\frac{\tau(x)}{2(\eta_{12}+\eta_{21})},$$

which proves the coerciveness condition. □

The model of our systems $A(t)$ in this case is given by

$$\begin{aligned}
AG(x,t) &= A(G_1,G_2) \\
&= (-\nabla\cdot(\tau\nabla G_1)-\eta_{11}G_1-\eta_{12}G_2,-\nabla\cdot(\tau\nabla G_2)-\eta_{21}G_1-\eta_{22}G_2).
\end{aligned}$$

Let $\theta\to f_m(\theta)$ be a linear defined on $(L^2(0,T;V))^2$ by

$$f_m(\theta)=\int_\Omega \rho_1(x,t)\theta_1(x)dx+\int_\Omega \rho_2(x,t)\theta_2(x)dx.$$

Then it is continuous, since

$$|f_m(\theta)|\leq c_3(\|\theta_1\|_{(H^1_0(\Omega))}+\|\theta_2\|_{(H^1_0(\Omega))})\leq c_3\|\theta\|_{(H^1_0(\Omega))^2}, \tag{2.11}$$

where, c_3 is a constant.

Based on (2.9), (2.10), (2.11) and Lax- Milgram Lemma [1], we can get the following theorem.

Theorem 2.4. *For a given* $\rho=(\rho_1,\rho_2)\in(L^2(\Omega_T))^2$, *the boundary-value problems* (2.1)-(2.5) *has a unique solution* $G(x,t)\in(W(0,T))^2$.

2.2. **Formulation of the control problem.** Let $U = (L^2(\Gamma_T))^2$ be a control Hilbert space. For every control $u = \{u_1, u_2\} \in U$, determine a system state $\mathrm{G} = G(x,t;u) = \{G_1(u), G_2(u)\}$ as a generalized solution to the boundary-value problem specified by the equations (2.1), boundary condition:

$$\begin{bmatrix} G(x,t;u) \\ G(x,t;u) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{2.12}$$

initial condition (2.2) and the conjugation conditions (2.4), (2.5).

According to the Lax- Milgramm Lemma [1], a unique state, namely, a function $G(x,t;u) \in (W(0,T))^2$ corresponds to every control $u \in U$, minimizes the energy functional

$$\Phi(w) = a(w,w) - 2f_m(w), \ \ \forall\, w(x) \in \ (V_0)^2 \tag{2.13}$$

on $(W(0,T))^2$ and it is the unique solution in $(W(0,T))^2$ to the variational boundary value problem which stated that:

Find an element $G(u) \in (L^2(0,T;V))^2$ that meets the equation

$$\int_\Omega \frac{\partial G}{\partial t} w dx + \int_\gamma c \frac{\partial G}{\partial t} w d\gamma + a(G,w) = f_m(u,w) \tag{2.14}$$

and

$$\begin{aligned} &\int_\Omega G_1(x,0;u) w_1(x) dx + \int_\Omega G_2(x,0;u) w_2(x) dx \\ &\quad + \int_\gamma c_1 G_1(x,0;u) w_1(x) d\gamma + \int_\gamma c_2 G_2(x,0;u) w_2(x) d\gamma \\ &= \int_\Omega G_{1,0}(x) w_1(x) dx + \int_\Omega G_{2,0}(x) w_2(x) dx \\ &\quad + \int_\gamma c_1 G_{1,0}(x) w_1(x) d\gamma + \int_\gamma G_{2,0}(x) w_2(x) d\gamma, \end{aligned} \tag{2.15}$$

where the bilinear form $a(t;G,w)$ has the form of expression (2.8) and the linear functional is

$$\begin{aligned} f_m(w) = f_m(u,w) = &\int_\Omega \rho_1(x,t) w_1(x) dx + \int_\Omega \rho_2(x,t) w_2(x) dx \\ &+ \int_\Gamma u_1 w_1 d\Gamma + \int_\Gamma u_2 w_2 d\Gamma. \end{aligned} \tag{2.16}$$

Specify the observation as

$$\begin{aligned} Z(u) &= \{Z_1(u), Z_2(u)\} = C_1 G(u) \\ &= \{G_1(u), G_2(u)\}\ , C_1 \in \pounds((W(0,T))^2; (L^2(\gamma_T))^2). \end{aligned} \tag{2.17}$$

Bring a value of the cost functional

$$J(u) = \|G_1(u) - z_{1g}\|^2_{L^2(\gamma_T)} + \|G_2(u) - z_{2g}\|^2_{L^2(\gamma_T)} + (Nu,u)_{(L^2(\Gamma_T))^2}, \quad (2.18)$$

in this case, z_g is a known element of the space $(L^2(\gamma_T))^2$ and $N \in £(U;U)$ is a hermitian positive definite operator such that

$$(Nu,u)_{(L^2(\Gamma_T))^2} \geq M\|u\|^2_{(L^2(\Gamma_T))^2}, \ \ M > 0, \ \ \forall \ u \in U. \quad (2.19)$$

The control problem then is to find:

$$\begin{cases} u = \{u_1, u_2\} \in U_{ad}, \\ J(u) = \inf \ J(v), \quad \forall \ v \in U_{ad}, \end{cases} \quad (2.20)$$

where U_{ad} is closed convex subset of $(L^2(\Gamma_T))^2$.

Definition 2.5. If an element $u \in U_{ad}$ meets condition (2.20), it is called an optimal control [7].

Rewrite the cost functional (2.18) as

$$J(u) = \pi(u,u) - 2h(u) + \|z_{1g} - G_1(0))\|^2_{L^2(\gamma_T)} + \|z_{2g} - G_2(0))\|^2_{L^2(\gamma_T)}, \quad (2.21)$$

where the bilinear form $\pi(u,v)$ and linear functional $h(v)$ are expressed as

$$\begin{aligned} \pi(u,v) =& (G_1(u) - G_1(0), G_1(v) - G_1(0))_{L^2(\gamma_T)} \\ &+ (G_2(u) - G_2(0), G_2(v) - G_2(0))_{L^2(\gamma_T)} + (Nu,v)_{(L^2(\Gamma_T))^2} \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} h(v) = & (z_{1g} - G_1(0), G_1(v) - G_1(0))_{L^2(\gamma_T)} \\ &+ (z_{2g} - G_2(0), G_2(v) - G_2(0))_{L^2(\gamma_T)}. \end{aligned} \quad (2.23)$$

The form $\pi(u,v)$ is a continuous bilinear form and from (2.19), it is coercive on U, that is,

$$\pi(v,v) \geq M\|v\|^2_{L^2(\Gamma_T)^2}, \quad (2.24)$$

also $h(v)$ a continuous linear form on $(L^2(\gamma_T))^2$. On the basis of the theory of Lions [7], the validity of the following statement is proved.

Theorem 2.6. *Let a system state be determined as a solution to equivalent problems* (2.13)-(2.15)*. Then, there exists a unique element u of a convex set in U_{ad} that is closed in U, and relation*

$$J(u) = \inf_{v \in U_{ad}} J(v),$$

takes place for u, where the cost functional is specified by expression (2.18)*.*

Proof. A control $u = \{u_1, u_2\} \in L^2((\Gamma_T))^2$ is optimal if and only if the following inequality is true:

$$\begin{aligned}(G_1(u) - z_{1g}, G_1(v) - G_1(u))_{L^2(\gamma_T)} \\ + (G_2(u) - z_{2g}, G_2(v) - G_2(u))_{L^2(\gamma_T)} \\ + (Nu, v - u)_{L^2(\Gamma_T)^2} \geq 0, \quad \forall\, v \in U_{ad}.\end{aligned} \tag{2.25}$$

As for the control $u \in U$, the adjoint state

$$p(u) = \{p_1(u), p_2(u)\} \in (L^2(0,T;V^*))^2 = (L^2(0,T;V))^2$$

is specified by the relations

$$\begin{bmatrix} -\frac{\partial p_1(u)}{\partial t} \\ -\frac{\partial p_2(u)}{\partial t} \end{bmatrix} + \begin{bmatrix} -\nabla\cdot(\tau\nabla) - \eta_{11} & -\eta_{21} \\ -\eta_{12} & -\nabla\cdot(\tau\nabla) - \eta_{22} \end{bmatrix} \begin{bmatrix} p_1(u) \\ p_2(u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } \Omega_T, \tag{2.26}$$

$$\begin{bmatrix} p_1(x,T;u) \\ p_2(x,T;u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{in} \quad \bar{\Omega}, \tag{2.27}$$

$$\begin{bmatrix} p_1(x,t;u) \\ p_2(x,t;u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{2.28}$$

$$\left\{ \left[\begin{bmatrix} \sum\limits_{i,j=1}^{n} \tau \frac{\partial p_1(u)}{\partial x_j} \cos(\nu, x_i) \\ \sum\limits_{i,j=1}^{n} \tau \frac{\partial p_2(u)}{\partial x_j} \cos(\nu, x_i) \end{bmatrix} \right] = \begin{bmatrix} -c_1 \frac{\partial p_1(u)}{\partial t} + G_1(v) - z_{1g} \\ -c_2 \frac{\partial p_2(u)}{\partial t} + G_2(v) - z_{2g} \end{bmatrix} \right. \quad \text{on} \ \gamma_T \tag{2.29}$$

and

$$\left\{ \begin{bmatrix} [p_1(u)] \\ [p_2(u)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right. \quad \text{on} \quad \gamma_T. \tag{2.30}$$

Problem (2.26)-(2.30) has the unique generalized solution $p(u) = \{p_1(u), p_2(u)\}$ as the unique one to the following equality system:

$$-\int_\Omega \frac{\partial p(u)}{\partial t} w dx - \int_\gamma c \frac{\partial p(u)}{\partial t} w d\gamma + a(p(u), w) = -\int_\gamma (G(u) - z_g) w d\gamma \tag{2.31}$$

and

$$\int_\Omega p(x,T,u) w dx + \int_\gamma c p(x,T,u) w d\gamma = 0. \tag{2.32}$$

Choose the difference $G(v)-G(u)$ instead of w in equality (2.31), and integrate from 0 to T, we obtain,

$$\begin{aligned}
&-\int_0^T (G(u)-z_g, G(v)-G(u))_{(L^2(\gamma))^2}dt \\
&= -\{(p(u), G(v)-G(u))_{(L^2(\Omega))^2} + (cp(u), G(v)-G(u))_{(L^2(\gamma))^2}\}|_0^T \\
&\quad + \int_0^T (p(u), \frac{\partial}{\partial t}(G(v)-G(u)))_{(L^2(\Omega))^2}dt \\
&\quad + \int_0^T (cp(u), \frac{\partial}{\partial t}(G(v)-G(u)))_{(L^2(\gamma))^2}dt \\
&\quad + \int_0^T a(p(u), G(v)-G(u))dt,
\end{aligned} \tag{2.33}$$

we get from (2.32), (2.14) and (2.15), the following

$$-\int_0^T (G(u)-z_g, G(v)-G(u))_{(L^2(\gamma))^2}dt = \int_0^T (p(u),(v-u))_{(L^2(\Gamma))^2}dt. \tag{2.34}$$

Therefore, the necessary and sufficient condition for the optimality of the control u may be written as follows:

$$(-p_1(u)+Nu_1, v_1-u_1)_{L^2(\Gamma_T)} + (-p_2(u)+Nu_2, v_2-u_2)_{L^2(\Gamma_T)} \geq 0, \quad \forall\, v \in U_{ad}. \tag{2.35}$$

Thus, the optimal control $u \in U_{ad}$ is specified by relations (2.14), (2.15), (2.31), (2.32) and (2.35). □

3. Boundary control for 2 × 2 Neumann parabolic systems

In this section we discuss the boundary control for the following 2×2 cooperative parabolic system with non-homogenous Neumann conditions and with observation under conjugation conditions.

Assume that equations (2.1) and (2.2) are specified in the domain Ω_T. Conjugation conditions (2.4) and (2.5) are specified, in their turn on γ_T and on Γ_T, the boundary conditions are

$$\begin{bmatrix} \frac{\partial G_1}{\partial \nu_A} \\ \frac{\partial G_2}{\partial \nu_A} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{3.1}$$

where $(g_1, g_2) \in (L^2(\Gamma_T))^2$ are given functions.

Let us define

$$(W_1(0,T))^2 = \left\{ G : G \in (L^2(0,T;V_1))^2, \frac{\partial G}{\partial t} \in (L^2(0,T;V_1')^2 \right\},$$

where

$$V_1 = \{G(x,t) = (G_1, G_2) \mid_{\Omega_i} \in (H^1(\Omega_i)),\ i\ = 1\ ,\ 2,\ \ \forall\, t \in [0,T], [G] = 0\}$$

and

$$V_2 = \{G(x) = (G_1, G_2) \mid_{\Omega_i} \in (H^1(\Omega_i)),\ i\ = 1, 2, [G] = 0\}.$$

We introduce again the bilinear form (2.8) which is coercive on $(V)^2$, since

$$(L^2(0,T,V))^2 \subseteq (L^2(0,T,V_1))^2.$$

Then by Lax- Milgram Lemma [1], there exists a unique solution $G \in (W_1(0,T))^2$ for system (2.1), (2.2), (2.4), (2.5) and (3.1) according the equation

$$(\frac{\partial G}{\partial t}, \theta) + a(t; G, \theta) = f_g(\theta),\ \ \forall\, \theta = (\theta_1, \theta_2) \in (V_2)^2,$$

where

$$f_g(\theta) = \int_\Omega \rho_1(x)\theta_1(x)dx + \int_\Omega \rho_2(x)\theta_2(x)dx + \int_\Gamma g_1(x)\theta_1(x)d\Gamma + \int_\Gamma g_2(x)\theta_2(x)d\Gamma$$

is a continuous linear form defined on $(W_1(0,T))^2$. For every control $u = \{u_1, u_2\} \in (L^2(\Gamma_T))^2$, determine a system state $G = G(x,t;u) = \{G_1(u), G_2(u)\}$ as a generalized solution to the initial boundary- value problems specified by the equations (2.1), initial condition (2.2), the conjugation conditions (2.4), (2.5) and the boundary conditions:

$$\begin{bmatrix} \frac{\partial G_1(u)}{\partial \nu_A} \\ \frac{\partial G_2(u)}{\partial \nu_A} \end{bmatrix} = \begin{bmatrix} g_1 + u_1 \\ g_2 + u_2 \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{3.2}$$

it is easy to state the equivalence of problems (2.13) and (2.14).

For a given $z_g = (z_{1g}, z_{2g})$ represent the observation by expression like (2.17) and the value of the cost functional is again given by (2.18). Then, using the theory of Lions [7], there exists a unique optimal control $u \in U_{ad}$ for (2.20) and we deduce:

Theorem 3.1. *If the cost functional is given by* (2.18)*, then there exists a unique boundary control* $u \in U_{ad}$ *such that:*

$$J(u) = \inf_{v \in U_{ad}} J(v).$$

Moreover, it is characterized by the following equations and inequalities

$$\begin{bmatrix} -\frac{\partial p_1(u)}{\partial t} \\ -\frac{\partial p_2(u)}{\partial t} \end{bmatrix} + \begin{bmatrix} -\nabla \cdot (\tau \nabla) - \eta_{11} & -\eta_{21} \\ -\eta_{12} & -\nabla \cdot (\tau \nabla) - \eta_{22} \end{bmatrix} \begin{bmatrix} p_1(u) \\ p_2(u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } \Omega_T, \tag{3.3}$$

$$\begin{bmatrix} p_1(x, T; u) \\ p_2(x, T; u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{in} \quad \bar{\Omega}, \tag{3.4}$$

$$\left\{ \left[\begin{bmatrix} \sum_{i,j=1}^{n} \tau \frac{\partial p_1(u)}{\partial x_j} \cos(\nu, x_i) \\ \sum_{i,j=1}^{n} \tau \frac{\partial p_2(u)}{\partial x_j} \cos(\nu, x_i) \end{bmatrix} \right] \right. = \begin{bmatrix} -c_1 \frac{\partial p_1(u)}{\partial t} + G_1(v) - z_{1g} \\ -c_2 \frac{\partial p_2(u)}{\partial t} + G_2(v) - z_{2g} \end{bmatrix} \quad \text{on } \gamma_T, \tag{3.5}$$

$$\left\{ \begin{bmatrix} [p_1(u)] \\ [p_2(u)] \end{bmatrix} \right. = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{on} \quad \gamma_T \tag{3.6}$$

and

$$\begin{bmatrix} \frac{\partial G_1(u)}{\partial \nu_A} \\ \frac{\partial G_2(u)}{\partial \nu_A} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{3.7}$$

$$(-p_1(u) + Nu_1, v_1 - u_1)_{L^2(\Gamma_T)} + (-p_2(u) + Nu_2, v_2 - u_2)_{L^2(\Gamma_T)} \geq 0 \tag{3.8}$$

for all $v \in U_{ad}$, together with (2.1), (2.2), (3.2).

4. Boundary control for $n \times n$ systems described by cooperative parabolic equation under conjugation conditions

4.1. The $n \times n$ Dirichlet systems under conjugation conditions. In this section,we generalize the discussion which has been introduced in section two to the following $n \times n$ cooperative parabolic Dirichlet systems

$$\frac{\partial}{\partial t} \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix} + A(t) \begin{bmatrix} G_1(x,t) \\ G_2(x,t) \\ \vdots \\ G_n(x,t) \end{bmatrix} = \begin{bmatrix} \rho_1(x,t) \\ \rho_2(x,t) \\ \vdots \\ \rho_n(x,t) \end{bmatrix} \quad \text{in} \quad \Omega_T, \tag{4.1}$$

$$\begin{bmatrix} G_1(x,0) \\ G_2(x,0) \\ \vdots \\ G_n(x,0) \end{bmatrix} = \begin{bmatrix} G_{1,0}(x) \\ G_{2,0}(x) \\ \vdots \\ G_{n,0}(x) \end{bmatrix}, \quad G_{i,0}(x) \in L^2(\Omega) \quad \text{in } \Omega, \tag{4.2}$$

$$\begin{bmatrix} G_1(x,t) \\ G_2(x,t) \\ \vdots \\ G_n(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{4.3}$$

under conjugation conditions:

$$\begin{bmatrix} \left[\tau \frac{\partial G_1}{\partial \nu_A}\right] \\ \left[\tau \frac{\partial G_2}{\partial \nu_A}\right] \\ \vdots \\ \left[\tau \frac{\partial G_n}{\partial \nu_A}\right] \end{bmatrix} = \begin{bmatrix} c_1 \frac{\partial G_1}{\partial t} \\ c_2 \frac{\partial G_2}{\partial t} \\ \vdots \\ c_n \frac{\partial G_n}{\partial t} \end{bmatrix} \quad \text{on} \quad \gamma_T, \tag{4.4}$$

$$\begin{bmatrix} [G_1] \\ [G_2] \\ \vdots \\ [G_n] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{on} \quad \gamma_T. \tag{4.5}$$

To study our problem, we introduce By Cartesian product the following chain of Sobolev spaces:

$$(L^2(0,T;V))^n \subseteq (L^2(0,T;L^2(\Omega)))^n \subseteq (L^2(0,T;V'))^n.$$

On $(L^2(0,T;V))^n$, we define the bilinear form by:

$$\begin{aligned} a(G,\theta) = & \sum_{i=1}^{n} \int_\Omega \tau(x) \nabla G_i \nabla \theta_i dx - \sum_{i \neq j}^{n} \int_\Omega \eta_{ij} G_j \theta_i dx \\ & - \sum_{i=j=1}^{n} \int_\Omega \eta_{ij} G_i \theta_i dx. \end{aligned} \tag{4.6}$$

Lemma 4.1. *The bilinear form* (4.6) *is coercive on* $(L^2(0;T,V))^n$, *that is, there exists a positive constants* k *and* α *such that*

$$a(G,G)+k\|G\|^2_{(L^2(\Omega))^n}\geq\alpha\|G\|^2_{(H^1_0(\Omega))^n},\ \ \forall\, G=(G_i)^n_{i=1}\in(W(0,T))^n. \quad (4.7)$$

Proof. Since

$$a(G,G)=\frac{\tau(x)}{2\sum_{i\neq j}^n(\eta_{ij}+\eta_{ji})}\sum_{i=1}^n\int_\Omega|\nabla G_i|^2\,dx-\sum_{i\neq j}^n\int_\Omega G_iG_jdx\\ -\frac{\sum_{i=1}^n\eta_{ii}}{\sum_{i\neq j}^n(\eta_{ij}+\eta_{ji})}\int_\Omega|G_i|^2\,dx,$$

we get

$$a(G,G)+\max(\frac{\sum_{i=1}^n(\eta_{ii},\eta_{ii})}{\sum_{i\neq j}^n(\eta_{ij}+\eta_{ji})})\sum_{i=1}^n\|G_i\|^2_{L^2(\Omega)}\\ =\frac{\tau(x)}{2\sum_{i\neq j}^n(\eta_{ij}+\eta_{ji})}\sum_{i=1}^n\int_\Omega|\nabla G_i|^2\,dx-\sum_{i\neq j}^n\int_\Omega G_iG_jdx.$$

By Cauchy Schwartz inequality and from Friedrichs inequality, we deduce

$$a(G,G)+\max(\frac{\sum_{i=1}^n(\eta_{ii},\eta_{ii})}{\sum_{i\neq j}^n(\eta_{ij}+\eta_{ji})})\|G_i\|^2_{L^2(\Omega)}\\ \geq\frac{\tau(x)}{2\sum_{i\neq j}^n(\eta_{ij}+\eta_{ji})}\sum_{i=1}^n\int_\Omega(|\nabla G_i|^2+(m(\Omega))^{-1}|G_i|^2)dx\\ -\sum_{i\neq j}^n\|G_i\|_{L^2(\Omega)}\|G_j\|_{L^2(\Omega)}.$$

This inequality is equivalent to

$$a(G,G)+\max(\frac{\sum_{i=1}^n(\eta_{ii},\eta_{ii})}{(\eta_{ij}+\eta_{ji})},\frac{1}{2})\|G_i\|^2_{L^2(\Omega)}\\ \geq\frac{\tau(x)}{2\sum_{i\neq j}^n(\eta_{ij}+\eta_{ji})}\min(1,(m(\Omega))^{-1})\|G_i\|^2_{H^1(\Omega)}\\ +\sum_{i\neq j}^n(\frac{1}{\sqrt{2}}\|G_i\|_{L^2(\Omega)}-\frac{1}{\sqrt{2}}\|G_j\|_{L^2(\Omega)})^2,$$

therefore,

$$a(G,G)+K\|G\|^2_{(L^2(\Omega))^n} \geq \alpha \sum_{i=1}^{n} G_i{}^2_{H_0^1(\Omega)}$$
$$\geq \alpha \|G\|^2_{(H_0^1(\Omega))^n}, \ \ \forall\, G \in (H_0^1(\Omega))^n,$$

where

$$K = \max\left(\frac{\sum_{i=1}^{n}(\eta_{ii},\eta_{ii})}{\sum_{i\neq j}^{n}(\eta_{ij}+\eta_{ji})}, \frac{1}{2}\right), \ \ \alpha = \frac{\tau(x)}{2\sum_{i\neq j}^{n}(\eta_{ij}+\eta_{ji})} min(1,(\mu(\Omega))^{-1}).$$

□

Now, let $\Phi \to L(\Phi)$ be a linear form defined on $(L^2(0,T;V))^n$ by

$$L(\Phi) = \sum_{i=1}^{n} \int_{\Omega} \rho_i(x,t)\varphi_i(x)\ dx, \ \forall\, \Phi = (\phi_i)_{i=1}^n \in (V_0))^n.$$

Then by Lax-Milgram Lemma, there exists a unique solution $G \in (W(0,T))^n$ such that:

$$(\frac{\partial G}{\partial t}, \Phi) + a(t;G,\Phi) = L(\Phi), \quad \forall\, \Phi = (\varphi_i)_{i=1}^n \in (V_0))^n. \tag{4.8}$$

Then, we have proved the following theorem

Theorem 4.2. *For a given $f = (f_i)_{i=1}^n \in (L^2(\Omega_T))^n$, there exists a unique solution $G = (G_i)_{i=1}^n \in (W(0,T))^n$ for cooperative Dirichlet system* (4.1)-(4.3) *with conjugation conditions* (4.4), (4.5).

4.2. **Formulation of the control problem.** The space $(L^2(\Gamma_T))^n$ being the space of controls.

For each control $u = (u_i)_{i=1}^n \in (L^2(\Gamma_T))^n$, let us define the state $G = (G_i)_{i=1}^n = G(x,t;u)$ of the system as a generalized solution to the boundary-value problem specified by the equations (4.1), boundary condition:

$$\begin{bmatrix} G_1(x,t;u) \\ G_2(x,t;u) \\ \vdots \\ G_n(x,t,;u) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{4.9}$$

initial condition (4.2) and the conjugation conditions (4.4), (4.5).

Specify the observation as

$$Z(u) = (Z_i(u))_{i=1}^n = C_1 G(u) = (G_i(u))_{i=1}^n\ , C_1 \in \pounds((W(0,T))^n;(L^2(\gamma_T))^n). \tag{4.10}$$

Bring a value of the cost functional

$$J(u) = \sum_{i=1}^{n} \|G_i(u) - z_{ig}\|^2_{L^2(\gamma_T)} + (Nu, u)_{(L^2(\Gamma_T))^n}, \tag{4.11}$$

in this case, $Z_g = (z_{ig})_{i=1}^n$ is a known element of the space $(L^2(\gamma_T))^n$ and $N \in £(U; U)$ is a Hermitian positive definite operator such that:

$$(Nu, u)_{(L^2(\Gamma_T))^n} \geq M\|u\|^2_{(L^2(\Gamma_T))^n}, \qquad M > 0, \quad \forall\, u \in U. \tag{4.12}$$

The control problem then is to find:

$$\begin{cases} u = (u_1, u_2, ...u_n) \in U_{ad}, \\ J(u) = inf\ J(v), \quad \forall\, v \in U_{ad}, \end{cases} \tag{4.13}$$

where U_{ad} is closed convex subset of $(L^2(\Gamma_T))^n$.

Definition 4.3. If an element $u \in U_{ad}$ meets condition (4.13), it is called an optimal control [7].

Rewrite the cost functional (4.11) as

$$J(v) = \pi(v, v) - 2f(v) + \sum_{i=1}^{n} \|z_{ig} - G_i(0))\|^2_{L^2(\gamma_T)}, \tag{4.14}$$

where the bilinear form $\pi(u, v)$ and linear functional $f(v)$ are expressed as

$$\pi(u, v) = \sum_{i=1}^{n} (G_i(u) - G_i(0), G_i(v) - G_i(0))_{L^2(\gamma_T)} + \sum_{i=1}^{n} (Nu_i, v_i)_{(L^2(\Gamma_T))} \tag{4.15}$$

and

$$f(v) = \sum_{i=1}^{n} (z_{ig} - G_i(0), G_i(v) - G_i(0))_{L^2(\gamma_T)}. \tag{4.16}$$

The form $\pi(u, v)$ is a continuous bilinear form and from (4.12), it is coercive on U, that is,

$$\pi(v, v) \geq M\|v\|^2_{(L^2(\Gamma_T))^n}, \tag{4.17}$$

also $f(v)$ a continuous linear form on $(L^2(\gamma_T))^n$. On the basis of the theory of Lions [7], there exists a unique optimal control of problem (4.13). Moreover it is characterized by

Theorem 4.4. *Let us suppose that* (4.7) *holds and the cost functional is given by* (4.11). *Then the boundary control* u *is characterized by*

$$\begin{cases} -\frac{\partial}{\partial t}\begin{bmatrix} p_1(u) \\ p_2(u) \\ \vdots \\ p_n(u) \end{bmatrix} + A^*(t)\begin{bmatrix} p_1(u) \\ p_2(u) \\ \vdots \\ p_n(u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad in \ \Omega_T, \\ \begin{bmatrix} p_1(u)(x;T,u) \\ p_2(u)(x;T,u) \\ \vdots \\ p_n(u)(x;T,u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ in \ \Omega, \\ \begin{bmatrix} p_1(u) \\ p_2(u) \\ \vdots \\ p_n(u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ on \ \Gamma_T, \end{cases} \tag{4.18}$$

$$\begin{bmatrix} \left[\tau\frac{\partial p_1(u)}{\partial \nu_A^*}\right] \\ \left[\tau\frac{\partial p_2(u)}{\partial \nu_A^*}\right] \\ \vdots \\ \left[\tau\frac{\partial p_n(u)}{\partial \nu_A^*}\right] \end{bmatrix} = \begin{bmatrix} -c_1\frac{\partial p_1}{\partial t} + G_1(v) - z_{1g} \\ -c_2\frac{\partial p_2}{\partial t} + G_2(v) - z_{2g} \\ \vdots \\ -c_n\frac{\partial p_n}{\partial t} + G_n(v) - z_{ng} \end{bmatrix} \quad \text{on} \quad \gamma_T, \tag{4.19}$$

$$\begin{bmatrix} [p_1(u)] \\ [p_2(u)] \\ \vdots \\ [p_n(u)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{on} \quad \gamma_T \tag{4.20}$$

and

$$\sum_{i=1}^{n}\int_0^T\int_\Gamma(-p_i(u)+Nu_i)(v_i-u_i)\ d\Gamma dt \ \geq 0, \ \forall \ v=(v_i)_{i=1}^n \in U_{ad}, \tag{4.21}$$

where $A^*(t)$ is the transpose of $A(t)$ such that

$$A^*(t) = \begin{bmatrix} -\nabla\cdot(\tau\nabla)-\eta_{11} & -\eta_{21} & \cdots & -\eta_{n1} \\ -\eta_{12} & -\nabla\cdot(\tau\nabla)-\eta_{22} & \cdots & -\eta_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_{1n} & -\eta_{2n} & \cdots & -\nabla\cdot(\tau\nabla)-\eta_{nn} \end{bmatrix}, \tag{4.22}$$

and $p(u) = (p_i(u))_{i=1}^n$ is the adjoint state. Therefore, the necessary and sufficient condition for the optimality of the control u may be written as (4.21), together with (4.1), (4.2) and (4.9).

4.3. **The $n \times n$ Neumann systems under conjugation conditions.** We study the $n \times n$ cooperative Neumann parabolic system of the form

$$\begin{bmatrix} \frac{\partial G_1}{\partial \nu_A} \\ \frac{\partial G_2}{\partial \nu_A} \\ \vdots \\ \frac{\partial G_n}{\partial \nu_A} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \quad \text{on} \quad \Gamma_T, \tag{4.23}$$

with (4.1), (4.2) and conjugation conditions (4.4), (4.5), we introduce again the bilinear form (2.8) which is coercive on $(W_1(0,T))^n$, since

$$(L^2(0,T,V))^n \subseteq (L^2(0,T,V_1))^n,$$

then based on (2.10) and Lax- Milgram Lemma there exists a unique solution $G = (G_i)_{i=1}^n \in (W_1(0,T))^n$ for system (4.1), (4.2), (4.23) and conjugation conditions (4.4), (4.5) such that

$$(\frac{\partial G}{\partial t}, \theta) + a(t; G, \theta) = h_g(\theta), \quad \forall\, \theta = (\theta_i)_{i=1}^n \in (V_2)^n, \tag{4.24}$$

where

$$h_g(\theta) = \sum_{i=1}^n \int_\Omega \rho_i(x,t)\theta_i(x)dx + \sum_{i=1}^n \int_\Gamma g_i(x,t)\theta_i(x)d\Gamma, \tag{4.25}$$

is a continuous linear form defined on $(V_1)^n$. Let us multiply both sides of first equation of (4.1) by $\theta \in (V_2)^n$ and integrating over Ω_T, we get

$$\int_{\Omega_T} \frac{\partial G_i}{\partial t}\theta_i(x)dxdt + \int_{\Omega_T} (-\nabla\cdot(\eta\nabla G_i)\theta_i(x)dxdt - \sum_{j=1}^n \int_{\Omega_T} \eta_{ij} G_j \theta_i dxdt$$
$$= \sum_{i=1}^n \int_{\Omega_T} \rho_i(x,t)\theta_i dxdt.$$

Using Green's formula we obtain

$$\pi(t;G,\theta)-\sum_{i=1}^{n}\int_{\Gamma_T}\tau\frac{\partial G_i}{\partial\nu}\theta_i d\Gamma=\sum_{i=1}^{n}\int_{\Omega_T}\rho_i(x,t)\theta_i dxdt.$$

From (4.24), we get

$$\sum_{i=1}^{n}\int_{\Omega_T}\rho_i(x,t)\theta_i(x)dxdt+\sum_{i=1}^{n}\int_{\Gamma_T}g_i(x,t)\theta_i(x)d\Gamma dt-\sum_{i=1}^{n}\int_{\Gamma_T}\tau\frac{\partial G_i}{\partial\nu}\theta_i d\Gamma dt$$
$$=\sum_{i=1}^{n}\int_{\Omega_T}\rho_i(x,t)\theta_i dxdt.$$

Hence we obtain the Neumann conditions

$$\tau\frac{\partial G_i}{\partial\nu_A}=g_i \quad on \;\; \Gamma_T,$$

so we can formulate the corresponding the control problem:

$(L^2(\Gamma_T))^n$ is the space of controls. The state $G(u)\in(W_1(0,T))^n$ of the system is given by the solution of

$$\left\{\begin{array}{ll}\frac{\partial}{\partial t}\begin{bmatrix}G_1(u)\\G_2(u)\\\vdots\\G_n(u)\end{bmatrix}+A(t)\begin{bmatrix}G_1(u)\\G_2(u)\\\vdots\\G_n(u)\end{bmatrix}=\begin{bmatrix}\rho_1\\\rho_2\\\vdots\\\rho_n\end{bmatrix} & in\;\;\Omega_T,\\ \begin{bmatrix}G_1(u)(x;t,u)\\G_2(u)(x;t,u)\\\vdots\\G_n(u)(x;t,u)\end{bmatrix}=\begin{bmatrix}G_{1,0}(x)\\G_{2,0}(x)\\\vdots\\G_{n,0}(x)\end{bmatrix} & in\;\;\Omega,\\ \begin{bmatrix}\frac{\partial G_1}{\partial\nu_A}\\\frac{\partial G_2}{\partial\nu_A}\\\vdots\\\frac{\partial G_n}{\partial\nu_A}\end{bmatrix}=\begin{bmatrix}g_1+u_1\\g_2+u_2\\\vdots\\g_n+u_n\end{bmatrix} & on\;\;\Gamma_T,\end{array}\right. \tag{4.26}$$

under conjugation conditions (4.4), (4.5).

For a given $z_d\in(L^2(\gamma_T))^n$, the cost functional is again given by (4.11), then there exists a unique optimal control $u=(u_i)_{i=1}^n\in U_{ad}$ for problem (4.13).

Moreover it is characterized by the following equations and inequalities

$$\left\{\begin{array}{ll} -\frac{\partial}{\partial t}\begin{bmatrix} p_1(u) \\ p_2(u) \\ \vdots \\ p_n(u) \end{bmatrix} + A^*(t)\begin{bmatrix} p_1(u) \\ p_2(u) \\ \vdots \\ p_n(u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & in \ \Omega_T, \\ \begin{bmatrix} p_1(u)(x;T,u) \\ p_2(u)(x;T,u) \\ \vdots \\ p_n(u)(x;T,u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & in \ \Omega, \\ \begin{bmatrix} \frac{\partial p_1(u)}{\partial \nu_A} \\ \frac{\partial p_2(u)}{\partial \nu_A} \\ \vdots \\ \frac{\partial p_n(u)}{\partial \nu_A} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & on \ \Gamma_T, \end{array}\right. \tag{4.27}$$

$$\begin{bmatrix} \left[\tau \frac{\partial p_1(u)}{\partial \nu_A^*}\right] \\ \left[\tau \frac{\partial p_2(u)}{\partial \nu_A^*}\right] \\ \vdots \\ \left[\tau \frac{\partial p_n(u)}{\partial \nu_A^*}\right] \end{bmatrix} = \begin{bmatrix} -c_1\frac{\partial p_1}{\partial t} + G_1(v) - z_{1g} \\ -c_2\frac{\partial p_2}{\partial t} + G_2(v) - z_{2g} \\ \vdots \\ -c_n\frac{\partial p_n}{\partial t} + G_n(v) - z_{ng} \end{bmatrix} \quad on \ \gamma_T, \tag{4.28}$$

$$\begin{bmatrix} [p_1(u)] \\ [p_2(u)] \\ \vdots \\ [p_n(u)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad on \ \gamma_T. \tag{4.29}$$

Therefore, the necessary condition for the optimality of the control u is

$$\sum_{i=1}^{n}\int_0^T\int_\Gamma(-p_i(u) + Nu_i)(v_i - u_i)\ d\Gamma dt \ \geq 0, \ \ \forall\ v = (v_i)_{i=1}^n \in U_{ad}. \tag{4.30}$$

Thus, the optimal control $u = (u_i)_{i=1}^n \in U_{ad}$ is specified by relations (2.14), (2.15), (2.31), (2.32) and (4.30), where $p(u) = (p_i(u))_{i=1}^n$ is the adjoint state.

Remark 4.5. If the constraints are absent, that is, when $U_{ad} = U$, then the equality

$$-p_1(u) + Nu_1 - p_2(u) + Nu_2 = 0$$

is obtained from inequality (3.8).

The control

$$\sum_{i=1}^{2} u_i = \frac{p_i}{N}, \quad (x,t) \in \ \Omega_T \tag{4.31}$$

is found from the latter equality.

Conclusions

In this paper, we focused on boundary control problems for cooperative parabolic systems under conjugation conditions.Under some conditions on the coefficients, we proved the existence and uniqueness of the state for 2×2 Dirichlet cooperative elliptic system under conjugation conditions. Then we demonstrated the existence and uniqueness of the optimal control of boundary type for this system. We gave the set of equations and inequalities that characterizes this control.Also, we studied the problem with Neumann condition. Finally, we generalized the discussion to $n \times n$ cooperative parabolic systems under conjugation conditions.

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