Nonlinear Functional Analysis and Applications
Vol. 29, No. 4 (2024), pp. 991-1006
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2024.29.04.04
http://nfaa.kyungnam.ac.kr/journal-nfaa

STRONG b-METRIC SPACES AND FIXED POINT THEOREMS

Eslam Qudah[1] and Abdallah Talafhah[2]

[1]Department of Mathematics, University of Jordan, Amman, Jordan
e-mail: eslamqudah3@gmail.com

[2]Department of Mathematics, University of Jordan, Amman, Jordan
e-mail: a.tallafha@ju.edu.jo

Abstract. In this paper, we shall give an example of a strong b-metric space which is not a $b-$metric space. Besides some fixed point result is proved in such spaces.

1. Introduction

There are a number of generalizations of metric spaces and Banach contraction principle. In this sequel, Bakhtin [4] and Czerwik [9] introduced $b-$metric spaces as a generalization of metric spaces. They proved the contraction mapping principle in $b-$metric space that generalized the famous Banach contraction principle in such spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in $b-$metric space (see e.g., [2, 7, 8, 11–13]) and the references therein.

In [10] Doan define strong $b-$metric space which is clearly a $b-$metric space, but he did not give an example of a strong $b-$metric which is not a $b-$metric, the purpose of this paper is to give an example, besides proving some fixed point theorems in strong $b-$metric space, also we shall generalize a theorem given by Agrawal and it all [14]. For more studies see [1, 5, 6, 15–20, 22, 23].

First, we recall some definitions from metric and b-metric spaces [9].

[0]Received October 30, 2023. Revised May 19, 2024. Accepted May 31, 2024.
[0]2020 Mathematics Subject Classification: 47H09, 47H10, 54E50.
[0]Keywords: Monotone operator, b-metric space, strong b-metric space, fixed point, contraction.
[0]Corresponding author: E. Qudah(eslamqudah3@gmail.com).

Definition 1.1. ([9]) Let X be a nonempty set and the mapping $d : X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for non-negative reals) satisfies the following conditions,

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a metric on X and (X, d) is called a metric space.

Definition 1.2. ([9]) Let X be a nonempty set and the mapping $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions,

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(3) there exists a real number $s \geq 1$ such that for all $x, y, z \in X$,

$$d(x, y) \leq s[d(x, z) + d(z, y)].$$

Then d is called a b-metric on X and (X, d) is called a b-metric space with coefficient s.

Definition 1.3. ([10]) A strong b-metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$such that for all $x, y, z \in X$:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) there exists a real number $s \geq 1$ such that

$$d(x, y) \leq d(x, z) + sd(z, y).$$

Then d is called a strong $b-$metric on X and (X, d) is called a strong $b-$metric space with coefficient s.

Every metric space is a strong b-metric space with coefficient $s = 1$ and every strong $b-$metric space with coefficient s is a $b-$metric space with coefficient s but the converse of these facts need not be true.

Example 1.4. Let $X = \{1, 2, 3\}$, define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = d(y, x) = \begin{cases} 0, & if \quad x = y, \\ 5, & if \quad x = 1, y = 2, \\ 1, & if \quad x \in \{1, 2\} \text{ and } y \in \{3\}. \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2} > 1$ and (X, d) is a strong $b-$metric space with coefficient $s = 4$, but (X, d) is not a metric space as

$$d(1, 2) = 5 > 2 = d(1, 3) + d(3, 2).$$

Example 1.5. Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$ define $d : X \times X \to \mathbb{R}$ by:

$$d(x,y) = \begin{cases} 0, & x = y, \\ \frac{n}{2}, & \text{if one is 0 and} \\ & \text{the other is } \frac{1}{n}, \\ d\left(\frac{1}{n}, \frac{1}{m}\right) = n + m, & n \neq m. \end{cases}$$

Then (X, d) is a b-metric space with constant 2 , which is not a strong b-metric space.

Definition 1.6. ([10]) Let $\{x_n\}$ be a sequence in a strong $b-$metric space (X, d).

(1) A sequence $\{x_n\}$ is called convergent if and only if there is $x \in X$ such that $\lim\limits_{n\to\infty} d(x, x_n) = 0$.

(2) $\{x_n\}$ is a Cauchy sequence if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.

(3) A strong $b-$metric space is said to be complete if and only if each Cauchy sequence in this space is convergent.

Regarding the properties of a strong $b-$metric space, we recall that if the limit of a convergent sequence exists, then it is unique. Also, each convergent sequence is a Cauchy sequence.

2. Fixed point theorems

Since the strong $b-$metric space is a $b-$metric, then we have the following theorem which is an analog to Banach contraction principle in strong $b-$metric space.

Theorem 2.1. *Let (X, d) be a complete strong $b-$metric space with coefficient $s \geq 1$ and $f : X \to X$ be a mapping satisfying the following condition:*

$$d(fx, fy) \leq \lambda d(x, y) \quad \textit{for all} \quad x, y \in X, \tag{2.1}$$

where $\lambda \in \left[0, \frac{1}{s}\right)$. Then f has a unique fixed point $u \in X$.

Theorem 2.2. *Let (X, d) be a complete strong b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a mapping satisfying the following condition:*

$$d(fx, fy) \leq \lambda[d(x, fx) + d(y, fy)] \ \forall\ x, y \in X, \tag{2.2}$$

where $\lambda \in \left[0, \frac{1}{2}\right) \setminus \left\{\frac{1}{s}\right\}$. Then f has a unique fixed point $u \in X$.

Proof. Let us first show that if f has a fixed point, then it is unique. Let $u, v \in X$ be two fixed points of f, that is, $fu = u, fv = v$. It follows from

(2.2) that

$$\begin{aligned} d(u,v) = d(fu,fv) &\leq \lambda[d(u,fu)+d(v,fv)] \\ &= \lambda[d(u,u)+d(v,v)] = 0. \end{aligned}$$

Therefore, we must have $d(u,v)=0$, that is, $u=v$. Thus, if fixed point of f exists then it is unique. For existence of fixed point, let $x_0 \in X$ be arbitrary; set $x_n = f^n x_0$ and $d_n = d\left(x_n, x_{n+1}\right)$. we can assume $d_n > 0$ for all $n \geq 0$, otherwise x_n is a fixed point of f for at least one $n \geq 0$.

For any $n \in \mathbb{N}$, it follows from (2.2) that

$$\begin{aligned} d_n = d\left(x_n, x_{n+1}\right) &= d\left(fx_{n-1}, fx_n\right) \\ &\leq \lambda\left[d\left(x_{n-1}, fx_{n-1}\right) + d\left(x_n, fx_n\right)\right] \\ &= \lambda\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right] \\ &= \lambda\left[d_{n-1} + d_n\right], \end{aligned}$$

it implies that

$$(1-\lambda)d_n \leq \lambda d_{n-1}.$$

Therefore, $d_n \leq \mu d_{n-1}$, where $\mu = \frac{\lambda}{1-\lambda} \in [0,1)$. On repeating this process, we obtain

$$d_n \leq \mu^n d_0.$$

Therefore, $\lim\limits_{n\to\infty} d_n = 0$.

Now we shall show that $\{x_n\}$ is a Cauchy sequence, it follows from (2.2) that for $m,n \in \mathbb{N}$

$$\begin{aligned} d\left(x_n, x_m\right) = d\left(f^n x_0, f^m x_0\right) &= d\left(fx_{n-1}, fx_{m-1}\right) \\ &\leq \lambda\left[d\left(x_{n-1}, fx_{n-1}\right) + d\left(x_{m-1}, fx_{m-1}\right)\right] \\ &= \left[d\left(x_{n-1}, x_n\right) + d\left(x_{m-1}, x_m\right)\right] \\ &= \lambda\left[d_{n-1} + d_{m-1}\right]. \end{aligned}$$

This implies that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

By completeness of (X,d), there exists $u \in X$ such that

$$\lim_{n\to\infty} d\left(x_n, u\right) = 0. \tag{2.3}$$

We shall show that u is a fixed point of f. For any $n \in \mathbb{N}$, it follows from (2.4) that

$$\begin{aligned} d(u,fu) &\leq d\left(u, x_{n+1}\right) + sd\left(x_{n+1}, fu\right) \\ &= d\left(u, x_{n+1}\right) + sd\left(fx_n, fu\right) \\ &\leq d\left(u, x_{n+1}\right) + s\lambda\big[d\left(x_n, fx_n\right) + d(u,fu)\big], \end{aligned}$$

that is,

$$d(u, fu) \leq d\left(u, x_{n+1}\right) + s\lambda d\left(x_n, fx_n\right) + s\lambda d(u, fu),$$

it implies that

$$(1 - s\lambda) d(u, fu) \leq d\left(u, x_{n+1}\right) + s\lambda d\left(x_n, fx_n\right).$$

Hence, we have

$$d(u, fu) \leq \frac{1}{(1 - s\lambda)} d\left(u, x_{n+1}\right) + \frac{s\lambda}{(1 - s\lambda)} d\left(x_n, x_{n+1}\right).$$

Note that $\lambda \neq \frac{1}{s}$, therefore, it follows from (2.3) and the above inequality that $d(u, fu) = 0$, that is, $fu = u$. Thus u is a unique fixed point of f. □

Theorem 2.3. *Let (X, d) be a strong b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a mapping satisfying:*

$$d(fx, fy) \leq \lambda \max\{d(x, y), d(x, fx), d(y, fy)\} \tag{2.4}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{s}\right)$. Then f has a unique fixed point $u \in X$.

Proof. Let us first show that if fixed point of f exists, then it is unique. Let $u, v \in X$ be two fixed points of f, that is, $fu = u, fv = v$, if $d(u, v) \neq 0$. It follows from (2.4) that

$$\begin{aligned} d(u, v) &= d(fu, fv) \\ &\leq \lambda \max\{d(u, v), d(u, fu), d(v, fv)\} \\ &= \lambda \max\{d(u, v), d(u, u), d(v, v)\} \\ &= \lambda d(u, v), \end{aligned}$$

which implies $\lambda \geq 1$, which is a contradiction. Therefore, we must have $d(u, v) = 0$, that is, $u = v$. Thus, if fixed point of f exists then it is unique.

For the existence of fixed point, let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for all $n \geq 0$. Then, we may assume $d(x_{n+1}, x_n) > 0$, $\forall n$, otherwise x_n is a fixed point of f.

Now, for any n we obtain from (2.6) that

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(fx_n, fx_{n-1}\right) \\ &\leq \lambda \max\left\{d\left(x_n, x_{n-1}\right), d\left(x_n, fx_n\right), d\left(x_{n-1}, fx_{n-1}\right)\right\} \\ &= \lambda \max\left\{d\left(x_n, x_{n-1}\right), d\left(x_n, x_{n+1}\right), d\left(x_{n-1}, x_n\right)\right\} \\ &= \lambda \max\left\{d\left(x_n, x_{n-1}\right), d\left(x_n, x_{n+1}\right)\right\}. \end{aligned}$$

If $\max\left\{d\left(x_n, x_{n-1}\right), d\left(x_n, x_{n+1}\right)\right\} = d\left(x_n, x_{n+1}\right)$, then we obtain from the above inequality

$$d\left(x_{n+1}, x_n\right) \leq \lambda d\left(x_n, x_{n+1}\right) < d\left(x_n, x_{n+1}\right),$$

which is a contradiction. Therefore, we must have

$$\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\}=d(x_n,x_{n-1}),$$

and then from the above inequality we obtain

$$d(x_{n+1},x_n)\le \lambda d(x_n,x_{n-1}).$$

By repeating this process, we obtain

$$d(x_{n+1},x_n)\le \lambda^n d(x_1,x_0)\ \text{ for all } n\ge 0. \tag{2.5}$$

For $m,n\in\mathbb{N}$ with $m>n$, we obtain

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1})+sd(x_{n+1},x_m)\\ &\le d(x_n,x_{n+1})+s\big[d(x_{n+1},x_{n+2})+sd(x_{n+2},x_m)\big]. \end{aligned}$$

So we have,

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1})+sd(x_{n+1},x_{n+2})\\ &\quad + s^2d(x_{n+2},x_{n+3})+s^3d(x_{n+3},x_{n+4})\\ &\quad +\cdots+s^{m-n-1}d(x_{m-1},x_m)\\ &\le d(x_n,x_{n+1})+sd(x_{n+1},x_{n+2})\\ &\quad + s^2d(x_{n+2},x_{n+3})+s^3d(x_{n+3},x_{n+4})\\ &\quad +\cdots+s^{m-1}d(x_{m-1},x_m). \end{aligned}$$

Using (2.5) in the above inequality, we have

$$\begin{aligned} d(x_n,x_m) &\le \lambda^n d(x_1,x_0)+s\lambda^{n+1}d(x_1,x_0)\\ &\quad + s^2\lambda^{n+2}d(x_1,x_0)+s^3\lambda^{n+3}d(x_1,x_0)\\ &\quad +\cdots+s^{m-1}\lambda^{m-1}d(x_1,x_0)\\ &\le \lambda^n d(x_1,x_0)+s\lambda^{n+1}\big[1+s\lambda\\ &\quad + s^2\lambda^2+s^3\lambda^3+\cdots\big]d(x_1,x_0)\\ &\le \lambda^n d(x_1,x_0)+\frac{s\lambda^{n+1}}{1-s\lambda}d(x_1,x_0)\\ &= \left(\lambda^n+\frac{s\lambda^{n+1}}{1-s\lambda}\right)d(x_1,x_0)\\ &= \frac{\lambda^n}{1-s\lambda}d(x_1,x_0). \end{aligned}$$

As $\lambda\in\big[0,\frac{1}{s}\big)$ and $s>0$, it follows from the above inequality

$$\lim_{n,m\to\infty} d(x_n,x_m)=0.$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $u \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} d(x_n, u) &= \lim_{n,m\to\infty} d(x_n, x_m) \\ &= d(u, u) \\ &= 0. \end{aligned} \tag{2.6}$$

So, we have $\lim_{n\to\infty} x_n = u$.

We shall show that u is a fixed point of f.
For any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(u, fu) &\leq d(u, x_{n+1}) + sd(x_{n+1}, fu) \\ &= d(u, x_{n+1}) + sd(fx_n, fu) \\ &\leq d(u, x_{n+1}) + s\Big[\lambda \max\big\{d(x_n, u), d(x_n, fx_n), d(u, fu)\big\}\Big]. \end{aligned}$$

Using (2.8) this implies

$$d(u, f(u)) \leq d(u, u) + s\lambda d(u, fu).$$

Hence, we obtain $d(u, fu) = 0$, that is, $fu = u$. Thus u is a fixed point of f, and it is a unique fixed point of f. □

The following theorem is given by Reich [21].

Theorem 2.4. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping with the following property:*

$$d(fx, fy) \leq ad(x, fx) + bd(y, fy) + cd(x, y)$$

for all $x, y \in X$, where a, b, c are non-negative and satisfy $a + b + c < 1$. Then f has a unique fixed point.

We have extended the a bove theorem to the strong $b-$metric space.

Theorem 2.5. *Let (X, d) be a complete strong $b-$metric space with coefficient $s \geq 1$ and $f : X \to X$ be a mapping with the following:*

$$d(fx, fy) \leq ad(x, fy) + bd(y, fx) + cd(x, y)$$

for all $x, y \in X$, where a, b, c are non-negative real numbers and satisfy $a + c + bs < 1$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X such that

$$x_n = fx_{n-1} = f^n x_0.$$

Now

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(f x_n, f x_{n-1}\right) \\ &\leq a d\left(x_n, f x_{n-1}\right) + b d\left(x_{n-1}, f x_n\right) + c d\left(x_n, x_{n-1}\right) \\ &= a d\left(x_n, x_n\right) + b d\left(x_{n-1}, x_{n+1}\right) + c d\left(x_n, x_{n-1}\right). \end{aligned}$$

So, we have

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &\leq b d\left(x_{n-1}, x_{n+1}\right) + c d\left(x_n, x_{n-1}\right) \\ &\leq b\Big[d\left(x_{n-1}, x_n\right) + s d\left(x_n, x_{n+1}\right)\Big] + c d\left(x_n, x_{n-1}\right). \end{aligned}$$

Hence

$$d\left(x_{n+1}, x_n\right) \leq b d\left(x_{n-1}, x_n\right) + s b d\left(x_n, x_{n+1}\right) + c d\left(x_n, x_{n-1}\right),$$

it implies that

$$(1 - bs) d\left(x_{n+1}, x_n\right) \leq (b + c) d\left(x_n, x_{n-1}\right).$$

Therefore, we have

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &\leq \frac{(b+c)}{(1-bs)} d\left(x_n, x_{n-1}\right) \\ &= \lambda d\left(x_n, x_{n-1}\right), \end{aligned}$$

that is,

$$d\left(x_{n+1}, x_n\right) < \lambda d\left(x_n, x_{n-1}\right).$$

Continuing this process we can easily show that

$$d(x_{n+1}, x_n) \leq \lambda^n d\left(x_1, x_0\right). \tag{2.7}$$

For $m, n \in \mathbb{N}$ with $m > n$, we obtain

$$\begin{aligned} d\left(x_n, x_m\right) \leq & d\left(x_n, x_{n+1}\right) + s d\left(x_{n+1}, x_m\right) \\ \leq & d\left(x_n, x_{n+1}\right) + s\Big[d\left(x_{n+1}, x_{n+2}\right) + s d\left(x_{n+2}, x_m\right)\Big] \\ \leq & d\left(x_n, x_{n+1}\right) + s d\left(x_{n+1}, x_{n+2}\right) \\ & + s^2 d\left(x_{n+2}, x_{n+3}\right) + s^3 d\left(x_{n+3}, x_{n+4}\right) \\ & + \cdots + s^{m-n-1} d\left(x_{m-1}, x_m\right) \\ \leq & d\left(x_n, x_{n+1}\right) + s d\left(x_{n+1}, x_{n+2}\right) \\ & + s^2 d\left(x_{n+2}, x_{n+3}\right) + s^3 d\left(x_{n+3}, x_{n+4}\right) \\ & + \ldots + s^{m-1} d\left(x_{m-1}, x_m\right). \end{aligned}$$

Using (2.7) in the above inequality, we have

$$\begin{aligned} d(x_n, x_m) \leq & \lambda^n d(x_1, x_0) + s\lambda^{n+1} d(x_1, x_0) \\ & + s^2\lambda^{n+2} d(x_1, x_0) + s^3\lambda^{n+3} d(x_1, x_0) \\ & + \cdots + s^{m-1}\lambda^{m-1} d(x_1, x_0) \\ \leq & \lambda^n d(x_1, x_0) + s\lambda^{n+1}\Big[1 + s\lambda + s^2\lambda^2 + s^3\lambda^3 + \dots\Big] d(x_1, x_0) \\ \leq & \lambda^n d(x_1, x_0) + \frac{s\lambda^{n+1}}{1 - s\lambda} d(x_1, x_0) \\ = & \left(\lambda^n + \frac{s\lambda^{n+1}}{1 - s\lambda}\right) d(x_1, x_0). \end{aligned}$$

Hence

$$d(x_n, x_m) \leq \frac{\lambda^n}{1 - s\lambda} d(x_1, x_0).$$

Taking limit as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a cauchy sequence in X. Since X is complete, we consider that $\{x_n\}$ converges to u.

Now, we show that u is a fixed point of f. We have

$$\begin{aligned} d(u, fu) & \leq d(u, x_n) + sd(x_n, fu) \\ & = d(u, x_n) + sd(fx_{n-1}, fu) \\ & \leq d(u, x_n) + s\Big[ad(x_{n-1}, fu) + bd(u, fx_{n-1}) + cd(x_{n-1}, u)\Big], \end{aligned}$$

and so, we have

$$\begin{aligned} d(u, fu) \leq & d(u, x_n) + sad(x_{n-1}, fu) \\ & + sbd(u, x_n) + scd(x_{n-1}, u). \end{aligned}$$

Hence,

$$\begin{aligned} d(u, fu) \leq & (1 + sb) d(u, x_n) + sad(x_{n-1}, fu) \\ & + scd(x_{n-1}, u). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(u, fu) = 0,$$

that is $f(u) = u$. Thus, u is the fixed point of f.

Now, for the uniqueness of fixed point. Let u and v be two fixed point of f. Then

$$u = f(u), \quad v = f(v)$$

and

$$\begin{aligned} d(u,v) &= d\left(f(u), f(v)\right) \\ &\leq ad(u, f(v)) + bd(v, f(u)) + cd(u,v) \\ &= ad(u,v) + bd(v,u) + cd(u,v) \\ &= (a+b+c)d(u,v) = kd(u,v). \end{aligned}$$

So, we have

$$d(u,v) \leq kd(u,v),$$

which is a contradiction. The proof is complete. □

Now, we shall generalized the theorem given by Agrawal and et al. [3] in $b-$metric space.

Theorem 2.6. *Let (X,d) be a complete strong $b-$metric space with coefficient $s \geq 1$. Let $f : X \to X$ be a mapping such that*

$$\begin{aligned} d(fx, fy) \leq& a \max\left\{d(x, f(x)), d(y, f(y)), d(x,y)\right\} \\ &+ b\{d(x, fy) + d(y, fx)\}, \end{aligned} \tag{2.8}$$

where $a, b > 0$ such that $a + b + bs < 1$ for all $x, y \in X$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X such that

$$x_n = fx_{n-1} = f^n x_0, \quad n = 1, 2, 3, 4, \cdots . \tag{2.9}$$

By (2.8) and (2.9) we obtain that

$$\begin{aligned} d\left(x_{n+1}, x_n\right) =& d\left(fx_n, fx_{n-1}\right) \\ &\leq a \max\Big\{d\left(x_n, x_{n-1}\right), d\left(x_{n-1}, f\left(x_{n-1}\right)\right), d\left(x_n, f\left(x_n\right)\right)\Big\} \\ &\quad + b\Big\{d\left(x_{n-1}, f\left(x_n\right)\right) + d\left(x_n, f\left(x_{n-1}\right)\right)\Big\}, \end{aligned}$$

$$\begin{aligned} d\left(fx_n, fx_{n-1}\right) \leq& a \max\Big\{d\left(x_n, x_{n-1}\right), d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\Big\} \\ &+ b\Big\{d\left(x_{n-1}, x_{n+1}\right) + d\left(x_n, x_n\right)\Big\}, \end{aligned}$$

$$\begin{aligned} d\left(fx_n, fx_{n-1}\right) \leq& a \max\Big\{d\left(x_n, x_{n-1}\right), d\left(x_n, x_{n+1}\right)\Big\} \\ &+ b\left\{d\left(x_{n-1}, x_{n+1}\right)\right\} \end{aligned}$$

and

$$d(x_{n+1}, x_n) \leq a \max\Big\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\Big\} + b\Big\{d(x_{n-1}, x_n) + sd(x_n, x_{n+1})\Big\}.$$

Hence, we have

$$d(x_{n+1}, x_n) \leq aM_1 + b\Big\{d(x_{n-1}, x_n) + sd(x_n, x_{n+1})\Big\}, \tag{2.10}$$

where $M_1 = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$.

Now two cases arise:

Case 1: Suppose that $M_1 = d(x_n, x_{n+1})$, we have

$$d(x_{n+1}, x_n) \leq ad(x_n, x_{n+1}) + b\Big\{d(x_{n-1}, x_n) + sd(x_n, x_{n+1})\Big\},$$

this implies that

$$(1-a-bs)d(x_{n+1}, x_n) \leq bd(x_{n-1}, x_n).$$

Hence

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \left(\frac{b}{1-a-bs}\right) d(x_{n-1}, x_n) \\ &\leq \left(\frac{a+b}{1-bs}\right) d(x_{n-1}, x_n). \end{aligned}$$

Let $K = \frac{a+b}{1-bs} < 1$. Then we have

$$d(x_{n+1}, x_n) \leq Kd(x_{n-1}, x_n).$$

Therefore,

$$d(x_{n+1}, x_n) \leq K^2 d(x_{n-2}, x_{n-1}).$$

Continuing this process, we get

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1). \tag{2.11}$$

Case 2: Suppose that $M_1 = d(x_n, x_{n-1})$. Then we have

$$d(x_{n+1}, x_n) \leq ad(x_n, x_{n-1}) + b\Big\{d(x_{n-1}, x_n) + sd(x_n, x_{n+1})\Big\},$$

this implies that

$$(1-bs)d(x_{n+1}, x_n) \leq (a+b)d(x_{n-1}, x_n).$$

Hence

$$d(x_{n+1}, x_n) \leq \left(\frac{a+b}{1-bs}\right) d(x_{n-1}, x_n).$$

Let $k = \frac{a+b}{1-bs} < 1$. Then we have

$$d(x_{n+1}, x_n) \leq kd(x_{n-1}, x_n).$$

Therefore,

$$d(x_{n+1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}).$$

Continuing this process, we get

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1). \tag{2.12}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X.

For $m, n \in \mathbf{N}$, with $m > n$, we obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + s\big[d(x_{n+1}, x_{n+2}) + sd(x_{n+2}, x_m)\big] \\ &\leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) \\ &\quad + s^2 d(x_{n+2}, x_{n+3}) + s^3 d(x_{n+3}, x_{n+4}) \\ &\quad + \cdots + s^{m-n-1} d(x_{m-1}, x_m) \\ &\leq d(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) \\ &\quad + s^2 d(x_{n+2}, x_{n+3}) + s^3 d(x_{n+3}, x_{n+4}) \\ &\quad + \cdots + s^{m-1} d(x_{m-1}, x_m). \end{aligned}$$

Using (2.12) in the above inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq k^n d(x_1, x_0) + sk^{n+1} d(x_1, x_0) \\ &\quad + s^2 k^{n+2} d(x_1, x_0) + s^3 k^{n+3} d(x_1, x_0) \\ &\quad + \cdots + s^{m-1} k^{m-1} d(x_1, x_0) \\ &\leq k^n d(x_1, x_0) + sk^{n+1}\big[1 + sk + s^2k^2 + s^3k^3 + \cdots\big] d(x_1, x_0) \\ &\leq k^n d(x_1, x_0) + \frac{sk^{n+1}}{1 - sk} d(x_1, x_0) \\ &= \left(k^n + \frac{sk^{n+1}}{1 - sk}\right) d(x_1, x_0) \\ &= \frac{k^n}{1 - sk} d(x_1, x_0), \end{aligned}$$

this implies that,

$$d(x_n, x_m) \leq \left(\frac{k^n}{1 - sk}\right) d(x_1, x_0).$$

Then

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 \quad \text{as } n, m \to \infty,$$

since $k < 1$,

$$\lim_{n\to\infty} \frac{k^n}{1 - sk} d(x_1, x_0) = 0 \quad \text{as } n, m \to \infty.$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, we consider that $\{x_n\}$ converges to u.

Now, we show that u is fixed point of f. In fact,

$$\begin{aligned} d(u,f(u)) &\leq d\left(u,x_{n+1}\right)+sd\left(x_{n+1},f(u)\right) \\ &= d\left(u,x_{n+1}\right)+sd\left(f\left(x_n\right),f(u)\right), \end{aligned}$$

$$\begin{aligned} d(u,f(u)) \leq\ & d\left(u,x_{n+1}\right)+sa\max\Big\{d\left(x_n,fx_n\right),d(u,f(u)),d\left(x_n,u\right)\Big\} \\ &+sb\Big\{d\left(x_n,fu\right)+d\left(u,fx_n\right)\Big\}, \end{aligned}$$

$$\begin{aligned} d(u,f(u)) \leq\ & d\left(u,x_{n+1}\right)+sa\max\Big\{d\left(x_n,x_{n+1}\right),d(u,f(u)),d\left(x_n,u\right)\Big\} \\ &+sb\Big\{d\left(x_n,fu\right)+d\left(u,x_{n+1}\right)\Big\}, \end{aligned}$$

$$\begin{aligned} d(u,f(u)) \leq\ & d\left(u,x_{n+1}\right)+sa\max\Big\{d\left(x_n,x_{n+1}\right),d(u,f(u)),d\left(x_n,u\right)\Big\} \\ &+sb\Big\{d\left(x_n,u\right)+sd(u,fu)\Big\}+sbd\left(u,x_{n+1}\right) \\ =\ & d\left(u,x_{n+1}\right)+sa\max\Big\{d\left(x_n,x_{n+1}\right),d(u,f(u)),d\left(x_n,u\right)\Big\} \\ &+bd\left(x_n,u\right)+s^2bd(u,fu)+sbd\left(u,x_{n+1}\right). \end{aligned}$$

Let $M_2=\max\left\{d\left(x_n,x_{n+1}\right),d(u,f(u)),d\left(x_n,u\right)\right\}$. Then

$$\begin{aligned} d(u,f(u)) \leq\ & d\left(u,x_{n+1}\right)+saM_2+sbd(x_n,u) \\ &+s^2bd(u,fu)+sbd\left(u,x_{n+1}\right), \end{aligned}$$

this implies that

$$\begin{aligned} \left(1-s^2b\right)d(u,f(u)) \leq\ & (1+sb)d\left(u,x_{n+1}\right)+saM_2 \\ &+sbd\left(x_n,u\right). \end{aligned}$$

Case 1: Suppose that $M_2=d\left(x_n,x_{n+1}\right)$. Then we have

$$\begin{aligned} \left(1-s^2b\right)d(u,f(u)) &\leq (1+sb)d\left(u,x_{n+1}\right) \\ &\quad +sad\left(x_n,x_{n+1}\right)+sbd\left(x_n,u\right) \\ &\leq (1+sb)d\left(u,x_{n+1}\right) \\ &\quad +sa\Big\{d\left(x_n,u\right)+sd\left(u,x_{n+1}\right)\Big\}+sbd\left(x_n,u\right) \\ &= (1+sb)d\left(u,x_{n+1}\right)+sad\left(x_n,u\right) \\ &\quad +s^2ad\left(u,x_{n+1}\right)+sbd\left(x_n,u\right) \\ &= \left(1+sb+s^2a\right)d\left(u,x_{n+1}\right)+sad\left(x_n,u\right) \\ &\quad +sbd\left(x_n,u\right), \end{aligned}$$

this implies that,

$$\left(1-s^2b\right)d(u,f(u)) \leq \left(1+sb+s^2a\right)d\left(u,x_{n+1}\right) \\ +s(a+b)d\left(x_n,u\right).$$

Therefore,

$$d(u,f(u)) \leq \frac{\left(1+sb+s^2a\right)}{(1-s^2b)}d\left(u,x_{n+1}\right)+\frac{s(a+b)}{(1-s^2b)}d\left(x_n,u\right).$$

Case 2: Suppose that $M_2 = d\left(x_n,u\right)$. Then we have

$$\left(1-s^2b\right)d(u,f(u)) \leq (1+sb)d\left(u,x_{n+1}\right)+sad(x_n,u)+sbd\left(x_n,u\right).$$

Therefore,

$$d(u,f(u)) \leq \frac{(1+sb)}{(1-s^2b)}d\left(u,x_{n+1}\right)+\frac{s(a+b)}{(1-s^2b)}d\left(x_n,u\right).$$

Case 3: Suppose that $M_2 = d(u,fu)$. Then we have

$$\left(1-s^2b\right)d(u,f(u)) \leq (1+sb)d\left(u,x_{n+1}\right)+sad(u,fu)+sbd\left(x_n,u\right).$$

This implies that

$$\left(1-s^2b-sa\right)d(u,f(u)) \leq (1+sb)d\left(u,x_{n+1}\right)+sbd\left(x_n,u\right).$$

Hence, we have

$$d(u,f(u)) \leq \frac{(1+sb)}{(1-s^2b-sa)}d\left(u,x_{n+1}\right)+\frac{sb}{(1-s^2b-sa)}d\left(x_n,u\right).$$

So in both cases, we have

$$d(u,f(u)) \leq \max\left\{\frac{1+sb+s^2b,}{1-s^2b},\frac{1+sb}{1-s^2b-sa}\right\}d(u,x_{n+1}) \\ +\max\left\{\frac{s(a+b)}{(1-s^2b)},\frac{sb}{(1-s^2b-sa)}\right\}d\left(x_n,u\right).$$

Taking limit $n\to\infty$, we get

$$\lim_{n\to\infty} d(u,fu)=0,$$

that is $f(u)=u$. Therefore, u is the fixed point of f.

For uniqueness of fixed point, we have to show that u is unique fixed point of f.

Assume that x is another fixed point of f. Then we have

$$fx=x \quad \text{and} \quad d(u,x)=d(fu,fx).$$

So, we have

$$\begin{aligned} d(u,x) &= d(fu,fx) \\ &\leq a\max\{d(u,fu),d(x,fx),d(u,x)\} + b\{d(u,fx)+d(x,fu)\} \\ &\leq a\max\{d(u,u),d(x,x),d(u,x)\} + b\{d(u,x)+d(x,u)\} \\ &\leq ad(u,x) + b\{d(u,x)+d(x,u)\} \\ &\leq ad(u,x) + 2bd(u,x) \\ &= (a+2b)d(u,x). \end{aligned}$$

This is a contradiction. Therefore, $u = x$. Hence, u is the unique fixed point of f. This completes the proof. □

Clearly Agrawal and et al. theorem can be proved as a corollary of Theorem 2.6.

Corollary 2.7. *Let* (X,d) *be a complete* b*-metric space. Let* $f : X \to X$ *be a mapping such that*

$$\begin{aligned} d(fx,fy) \leq a\max\Big\{d(x,fx), d(y,fy), d(x,y)\Big\} \\ +b\Big\{d(x,fy)+d(y,fx)\Big\}, \end{aligned}$$

where $a,b>0$ *such that* $a+2bs \leq 1$ *for all* $x,y \in X$ *and* $s \geq 1$*. Then* f *has a unique fixed point.*

Proof. If $a,b>0$ such that $a+2bs<1$, then $a+b+bs<1$, then by Theorem 2.6 f has a unique fixed point. □

References

[1] Y. Achtoun, M.S. Lamarti and I. Tahiri, *Multi-valued Hicks contractions in b-Menger spaces*, Nonlinear Funct. Anal. Appl., **29**(2) (2024), 477-485.

[2] A. Aghajani, M. Abbas and J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces,* Math. Slovaca, **64** (2014), 941-960

[3] S. Agrawal and K. Qureshi and J. Nema, *A fixed Point Theorem for b-Metric Spaces,* Int. J. Pure Appl. Math. Sci., **9** (2016), 45-50.

[4] I. Bakhtin, *The contraction mapping principle in quasimetric spaces,* Functional Anal., **30** (1989), 26-37.

[5] A. Bataihah, T. Qawasmeh and M. Shatnawi, *Discussion on b-metric spaces and related results in metric and g-metric spaces,* Nonlinear Funct. Anal. Appl., **27**(2) (2022), 233-247.

[6] A. Bataihah, W.A. Shatanawi, T. Qawasmeh and R. Hatamleh, *On h-simulation functions and fixed point results in the setting of ωt-distance mappings with application on matrix equations.* Mathematics, **8**(5): 837, math8050837, (2020).

[7] M. Boriceanu, M. Bota and A. Petruşel, *Multivalued fractals in b-metric spaces,* Central Eur. J. Math., **8** (2010), 367-377.
[8] M. Bota, A. Molnar and C. Varga, *On ekeland's variational principle in b-metric spaces,* Fixed Point Theory, **12**(2) (2011), 21-28.
[9] S. Czerwik, *Contraction mappings in b-metric spaces,* Acta Math. et Informatica Universitatis Ostraviensis, **1**(1) (1993), 5-11.
[10] H. Doan, *A new type of kannan's fixed point theorem in strong b-metric spaces,* AIMS Mathematics, **6**(7) (2021), 7895-7908.
[11] M. Jovanovic, Z. Kadelburg and S. Radenovic, *Common fixed point results in metric type spaces,* Fixed Point Theory Appl., **2010** (2010), 1-15.
[12] R. Kannan, *Some results on fixed points-II,* The Amer. Math. Mont., **76**(4) (1969), 405-408.
[13] M. Khamsi and N. Hussain, *KKM mappings in metric type spaces,* Nonlinear Anal., **73**(9) (2010), 3123-3129.
[14] W.A. Kirk, *Fixed points of asymptotic contractions,* J. Math. Anal. Appl., **277**(2) (2003), 645-650.
[15] A. Malkawi, A. Tallafha and W. Shatanawi, *Coincidence and fixed point results for* $(\psi, 1)$*-m-weak contraction mapping on mr-metric spaces,* Italian J. Pure App. Math., **47** (2020), 751-768.
[16] A.A.-R.M. Malkawi, A. Talafhah and W. Shatanawi, *Coincidence and fixed point results for generalized weak contraction mapping on b-metric spaces.* Nonlinear Funct. Anal. Appl., **26**(1) (2021), 177-195.
[17] T. Qawasmeh, *(h, ω b)-interpolative contractions in ω b-distance mappings with application,* Europ. J. Pure Appl. Math., **16**(3) (2023), 1717-1730.
[18] T. Qawasmeh, *H-simulation functions and* Ωb*-distance mappings in the setting of gb-metric spaces and application,* Nonlinear Funct. Anal. Appl., **28**(2) (2023), 557-570.
[19] T. Qawasmeh, W. Shatanawi, A. Bataihah and A. Tallafha, *Fixed point results and* (α, β)*-triangular admissibility in the frame of complete extended b-metric spaces and application,* U.P.B. Sci. Bull., Series A, **83**(1) (2021), 113-124.
[20] A. Rabaiah, A.A.-R. Malkawi, A. Al-Rawabdeh, D. Mahmoud and M. Qousini, *Fixed point theorems in mr -metric space through semi-compatibility,* Adv. Math. Sci. J., **10**(6) (2021), 2831-2845.
[21] S. Reich, *Some remarks concerning contraction mappings,* Canadian Math. Bull., **14**(1) (1971), 121-124.
[22] W. Shatanawi, T. Qawasmeh, A. Bataihah and A. Tallafha, *New contractions and some fixed point results with application based on extended quasi b-metric spaces,* U.P.B. Sci. Bull., Series A, **83**(2) (2021), 39-48.
[23] P. Swapna, T. Phaneendra and M.N. Rajashekhar, *Fixed point theorems in b-metric and extended b-metric spaces*, Nonlinear Funct. Anal. Appl., **28**(4) (2023), 877-886.
[24] O. Yamaod and A. Wiriyapongsanon, *Fixed point theorem of* ψ_s*-rational type contractions along with altering distance functions in b-metric spaces*, Nonlinear Funct. Anal. Appl., **29**(3) (2024), 769-779.