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GENERALIZATIONS OF SUZUKI TYPE COMMON FIXED POINT THEOREMS

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Abstract. Chandra-Arya-Joshi [2] in 2017 obtained a common fixed point theorem for Suzuki type contractions [20] on complete metric spaces. Özkan [9] in 2023 extended it to partial metric spaces with some corollaries and an example. Moreover, he gave an application for a class of functional equations in dynamic programming. Our aim in this article is to show that the main results of [9] hold for quasi-metric spaces with simple proofs. So, we can eliminate the lengthy proofs in [2], [9], [20].

1. Introduction

Among hundreds of extensions of metric spaces, a quasi-metric is the one not necessarily symmetric. In fact, a quasi-metric d satisfies all axioms of a metric except the symmetry $d(x, y) = d(y, x)$ for all x, y in the space. Certain key results in Metric Fixed Point Theory hold for quasi-metric spaces from the beginning; for example, the Banach contraction principle, the Ekeland variational principle, the Caristi fixed point theorem, the Takahashi minimization principle, and their equivalents; see [13]–[16].

Let (X, d) be a quasi-metric space and a Rus-Hicks-Rhoades (RHR) map $f : X \to X$ is the one satisfying $d(fx, f^2x) \leq \alpha d(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. The fixed point theorems due to Rus [19] in 1973 and Hicks-Rhoades [5] in 1979 are origins of RHR maps. Recently, in [12], we noticed that it has an interesting long history. The RHR maps are closely related to

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the Banach contraction principle in 1922, but we found that it is more closer to its multi-valued versions due to Nadler [8] in 1969 and Covitz-Nadler [4] in 1970. The aim of [12] was to trace such history of the Rus-Hicks-Rhoades theorem, and to show its grown-up versions or equivalents or closely related theorems. Such theorems are too many and could be called its relatives.

One of the recent examples of RHR maps is the one called a Suzuki type map [20] in 2008, which is very popular and has a large number of followers; for some examples, see the references of [9]. Recently, we found an extension of Suzuki's theorem for quasi-metric spaces with a very simple proof in [15].

Recall that Chandra-Arya-Joshi [2] in 2017 obtained a common fixed point theorem for Suzuki type contractions on complete metric spaces. Later Özkan [9] in 2023 extended it to complete partial metric spaces with some corollaries and an example. Moreover, he gave an application showing existence and uniqueness of a common solution for a class of functional equations in dynamic programming.

Our aim in this article is to collect simple proofs of Suzuki's theorem [20] and to show that the main results of [9] hold for quasi-metric spaces with simple proofs by following our previous work [15]. So, we can eliminate the lengthy proofs in [20], [2] and [9].

This article organized as follows: Section 2 is for preliminaries for quasi-metric spaces and a basic fixed point theorem for RHR maps in our previous work [14]. In Section 3, we prove basic results of [9] for quasi-metric spaces with simple proofs. In Section 4, we introduce the contents of our previous works on RHR maps. In fact, we listed the abstracts of [12]–[15]. Section 5 deals with some comments on the results in [9]. Finally, in Section 6, we give some conclusion on the present article.

2. Preliminaries

We recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $d : X\times X \to \mathbb{R}^+ = [0,\infty)$ satisfying the following conditions for all $x, y, z \in X$:

(1) $d(x, y) = d(y, x) = 0 \iff x = y$ (self-distance);
(2) $d(x, z) \le d(x, y) + d(y, z)$ (triangle inequality).

A *metric* on a set X is a quasi-metric satisfying

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry).

The convergence and completeness in a quasi-metric space (X, d) are defined as follows:

Definition 2.2. ([1], [6])

(1) A sequence $\{x_n\}$ in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

(2) A sequence $\{x_n\}$ is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N$.
(3) A sequence $\{x_n\}$ is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m > n > N$.
(4) A sequence $\{x_n\}$ is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is $\{x_n\}$ is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3. ([1], [6])

(1) (X, d) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, d) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, d) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let (X, d) be a quasi-metric space and $f : X \to X$ a selfmap. The *orbit* of f at $x \in X$ is the set

$$O_f(x) = \{x, fx, \cdots, f^n x, \cdots\}.$$

The space X is said to be *f-orbitally complete* if every right-Cauchy sequence in $O_f(x)$ is convergent in X. A selfmap f of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} f^n x = x_0 \Longrightarrow \lim_{n\to\infty} f^{n+1} x = f x_0$$

for any $x \in X$.

Every quasi-metric induces a metric, that is, if (X, δ) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{\delta(x, y), \delta(y, x)\}$$

is a metric on X; see Jleli et al. [6].

In our previous works, we obtained the following:

Theorem 2.5. ([10]) *Let f be a selfmap of a quasi-metric space (X, d) satisfying:*

(i) *X is f-orbitally complete and $\delta(O_f(x)) < \infty$ for each $x \in X$, where δ denotes the diameter.*
(ii) *There exists a $u \in X$ such that $O_f(u)$ has a cluster point $p \in X$.*

(iii) *There exists a function* $\varphi : [0,\infty) \to [0,\infty)$ *which is nondecreasing, continuous from the right and satisfies* $\varphi(t) < t$ *for each* $t > 0$ *and the inequality*

$$d(fx, fy) \le \varphi(\delta(O_f(x) \cup O_f(y)) \quad \textit{for each} \quad x, y \in X.$$

Then p *is the unique fixed point of* f *and* $\lim_n f^n(u) = p$.

This extends works of Pal-Maiti, Park, Hegedüs, Daneš, and many others. The following is known as Theorem H(γ1) in 2022:

Theorem 2.6. ([11], [16]) *Let* X *be a quasi-metric space with a map* $f : X \to X$ *and* $0 < h < 1$ *satisfying* $d(fx, f^2x) \le hd(x, fx)$ *for all* $x \in X\backslash\{fx\}$*. If* X *is f-orbitally complete, then* f *has a fixed element* $v \in X$*, that is,* $v = fv$.

The following is a recent work:

Theorem 2.7. ([13], [15]) *Let* (X, d) *be a quasi-metric space and let* $f : X \to X$ *be an RHR map; that is,*

$$d(fx, f^2x) \le kd(x, fx) \textit{ for every } x \in X,$$

where $0 < k < 1$.

(1) *If* X *is* f*-orbitally complete, then, for each* $x \in X$*, there exists a point* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} f^n x = x_0$$

and

$$d(f^n x, x_0) \le \frac{k^n}{1-k} d(x, fx), \quad n = 1, 2, \cdots .$$

(2) $f : X \to X$ *is orbitally continuous at* $x_0 \in X$ *in* (1) *and* x_0 *is a fixed point of* T.

The following is the main result of our previous article [13]:

Theorem 2.8. ([13]) *Let* f *be a selfmap of a quasi-metric space* (X, d) *which is f-orbitally complete. Suppose* $\varphi : X \to [0,\infty)$ *is a function.*

(1) *If there exists a point* $x \in X$ *satisfying*

$$d(fx, f^2x) \le \varphi(x) - \varphi(fx),$$

then $\{f^n x\}$ *is a right-Cauchy sequence converging to some* $x_0 \in X$.

(2) f *is orbitally continuous at* x_0 *and* x_0 *is a fixed point of* f.

In view of Theorem 2.6, two equivalent statements in (2) of Theorems 2.7 and 2.8 are true.

3. Proofs of the Suzuki type theorems

A well-known fixed point theorem of Suzuki ([20], Theorem 2) for the so-called *Suzuki type map* on a complete metric space with a lengthy proof is very popular and has a large number of followers. In this section, we extend it and collect its simple proofs.

Theorem 3.1. ([20]) *Let* (X, d) *be a complete metric space, and* $T : X \to X$ *be a self-map. Define a nonincreasing function* $\theta : [0, 1) \to (1/2, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases} \tag{3.1}$$

Assume that there exists $r \in [0, 1)$ *such that*

$$\theta(r)d(x, Tx) \le d(x, y) \implies d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$*. Then there exists a unique fixed point* $z \in X$ *of* T*. Moreover* $\lim_n T^n x = z$ *for each* $x \in X$*.*

Proof. Since $\theta(r) \le 1$, $\theta(r)d(x, Tx) \le d(x, Tx)$ holds for every $x \in X$. By hypothesis, we have

$$d(Tx, T^2x) \le rd(x, Tx) \text{ for all } x \in X.$$

This clearly satisfies condition (iii) of Theorem 2.5 for $y = fx$. We now fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$. Then $d(u_n, u_{n+1}) \le r^n d(u, Tu)$, so $\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty$, and a standard argument shows $\{u_n\}$ is Cauchy and $\delta(O(u)) < \infty$. Then the conclusion follows from Theorem 2.5. □

The following extends Theorem 3.1 and the main theorems of Chandra-Arya-Joshi ([2], Theorem 2.1) for metric spaces and of Özkan ([9], Theorem 1) for partial metric spaces:

Theorem 3.2. *Let* (X, d) *be a complete quasi-metric space,* $T, S : X \to X$ *be two self-maps and a nonincreasing function* $\theta : [0, 1) \to (1/2, 1]$ *defined as in Theorem* 3.1*. If there exists* $r \in [0, 1/2)$ *such that*

$$\theta(r) \min\{d(x, Tx), d(x, Sx)\} \le d(x, y) \implies \tag{3.2}$$

$$\max\{d(Sx, Sy), d(Tx, Ty), \tfrac{1}{2}[d(Sx, Ty) + d(Sy, Tx)] \le r\, d(x, y)$$

for all $x, y \in X$*, then* T *and* S *have a unique common fixed point.*

Proof. We divide several steps to prove.

Step 1. Fixed point sets $\mathrm{Fix}(T) = \mathrm{Fix}(S)$:

Let $u = Tu$. Then

$$0 \le \theta(r) \min\{d(u, Tu), d(u, Su)\} \le d(u, Tu)$$

implies

$$d(Su,u) \leq \max\{d(Su,STu), d(Tu,T^2u), \frac{1}{2}[d(Su,T^2u) + d(Su,Tu)]\}$$
$$\leq r\, d(u,Tu).$$

Hence, we have $d(Su,u) \leq r\, d(u,u)$. Since d is a quasi-metric, $d(u,u) = 0$ and hence $Su = u$.

Similarly, $u = Su$ implies $Tu = u$.

Step 2. T is an RHR map:

Putting $y = Tx$, we have

$$\theta(r) \min\{d(x,Tx), d(x,Sx)\} \leq d(x,Tx)$$

implies

$$\max\{d(Sx,STx), d(Tx,T^2x), \frac{1}{2}[d(Sx,T^2x) + d(STx,Tx)]\} \leq rd(x,Tx)$$

for every $x \in X$. Hence, we get $d(Tx,T^2x) \leq r\, d(x,Tx)$.

Step 3. Existence and uniqueness of common fixed point:

Since T is an RHR map, we have $\mathrm{Fix}(T) = \mathrm{Fix}(S) \neq \emptyset$ by Theorem 2.5.

Now, to show the uniqueness of this common fixed point, we assume that u and v are common fixed points of T and S where $u \neq v$. Taking $x = u$ and $y = v$ in inequality (3.2), we have

$$0 = \theta(r) \min\{d(u,Tu), d(u,Su)\} \leq d(u,v).$$

Hence, we have

$$\max\{d(Su,Sv), d(Tu,Tv), \frac{1}{2}[d(Su,Tv) + d(Tu,Sv)] \leq rd(u,v),$$

it implies that

$$\max\{d(u,v), d(u,v), \frac{1}{2}[2d(u,v)]\} \leq rd(u,v).$$

Therefore, we have

$$d(u,v) \leq rd(u,v) < d(u,v).$$

This is a contradiction. Hence, $d(u,v) = 0$ and $u = v$. □

Remark 3.3. (1) We can add the statement (i) of Theorem 2.5 to Theorem 3.2.

(2) For $S = T$, Theorem 3.2 reduces to Theorem 3.1. Hence, the above proof gives the third proof of Theorem 3.1.

(3) Some particular cases of the RHR maps extending Suzuki type contractive conditions were appeared already as follows:

(3.1) Ćirić [3] in 1974:

$$\min d(Tx,Ty), d(x,Tx), d(y,Ty) - \min d(x,Ty), d(y,Tx) \leq r\, d(x,y).$$

(3.2) Suzuki [20] in 2008: $\theta(r)d(x,Tx) \leq d(x,y)$ implies

$$d(Tx,Ty) \leq r\, d(x,y).$$

(3.3) Kim-Sedghi-Shobkolaei [7] in 2015: $\theta(r)\min\{d(x,Tx), d(x,Sx)\} \leq d(x,y)$ implies

$$\max\{d(Sx,Sy), d(Tx,Ty), d(Sx,Ty), d(Sy,Tx) \leq r\, d(x,y).$$

(3.4) Rakočević-Samet [17] in 2017:
$\min\{||Tx-Ty||, ||x-Tx||, ||y-Ty||\} - \min\{||x-Ty||, ||y-Tx||\} \leq r||x-y||$.

The following is motivated by Chandra-Arya-Joshi ([2], Corollary 2.3) and Özkan ([6], Corollary 2):

Corollary 3.4. *Let (X,d) be a complete quasi-metric space, $f, S, T : X \to X$ be three self-maps and a nonincreasing function $\theta : [0,1) \to (1/2, 1]$ be defined as usual. If there exists $r \in [0, 1/2)$ such that*

$$\theta(r)\min\{d(x,fTx), d(x,fSx)\} \leq d(x,y)$$

implies

$$\max\{d(fSx,fSy), d(fTx,fTy), \frac{1}{2}[d(fSx,fTy) + d(fSy,fTx)]\} \leq r\, d(x,y), \tag{3.3}$$

also, if f is one to one, $fS = Sf$ and $fT = Tf$, then f, T and S have a common fixed point.

Proof. If we consider fS and fT as two maps with given contractive condition of Theorem 3.2, then fS and fT have a common fixed point $u \in X$. Namely, $fSu = fTu = u$. Since f is one to one, we get

$$fSu = fTu = u \implies Su = Tu.$$

Then, putting $x = u$ and $y = Tu$ in inequality (3.3)

$$\theta(r)\min\{d(u,fTu), d(u,fSu)\} \leq d(u,Tu),$$

it implies that

$$\max\{d(fSu,fSTu), d(fTu,fT^2u), \frac{1}{2}[d(fSu,fT^2u) + d(fSTu,fTu)]\}$$
$$\leq r\, d(u,Tu).$$

And also, we have

$$\max\{d(fSu, SfTu), d(fTu, TfTu), \frac{1}{2}[d(fSu, TfTu) + d(SfTu, fTu)]\}$$
$$\leq r\, d(u, Tu).$$

Hence,

$$\max\{d(u, Su), d(u, Tu), \frac{1}{2}[d(u, Tu) + d(Su, u)]\} \leq r\, d(u, Tu).$$

Therefore, we have

$$d(u, Tu) \leq r\, d(u, Tu).$$

Then, $d(u, Tu) = 0$. So, we get $Tu = u$ which implies $Tu = Su = u$ and also $fu = fTu = u$. So, f, T and S have a common fixed point. □

4. Previous works on RHR maps

The Banach contraction has thousands of related works. Recall that our RHR maps extend the Banach contraction and are anticipated much related works. We first introduced the RHR maps in [12] and obtained some related results in [13]–[16]. Here we introduce the abstracts of such articles for convenience of the readers or the possible workers in future study.

(1) [12] : Let (X, d) be a complete metric space and $f : X \to X$ a map satisfying $d(fx, f^2x) \leq \alpha d(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. The fixed point theorems due to Rus (1973) and Hicks-Rhoades (1979) on such maps were extended or improved by Park (1980), Harder-Hicks-Saliga (1993), Suzuki (2001), and Jachymski (2003). Moreover, fixed point theorems of Zermelo (1904), Banach (1922) and Caristi (1976), extended versions for multimaps due to Nadler (1969) and Covitz-Nadler (1970) are also closely related to the Rus-Hicks-Rhoades theorem. Finally, we unify these based on a particular form of our 2023 Metatheorem.

(2) [13] : Our aim in this article is to show that all metric fixed point theorems hold for quasi-metric spaces (without symmetry). In fact, we show some well-known theorems on metric spaces hold for quasi-metric spaces from the beginning. We check this fact for the Banach contraction principle, the Covitz-Nadler fixed point theorem, the Rus-Hicks-Rhoades fixed point theorem, and others. In these theorems the concepts of continuity and completeness can be replaced by orbital continuity and T-orbital completeness for a selfmap T, respectively. Consequently, we improve and generalize the basic known theorems in the metric fixed point theory.

(3) [14] : Our aim in this article is to show that certain well-known theorems on metric spaces hold for quasi-metric spaces (without symmetry) from the beginning. We check this claim for theorems of Banach, Ekeland, Caristi, Takahashi, Rus-Hicks-Rhoades, and others. Moreover our Brøndsted-Jachymski principle – on the relation among maximal elements, fixed elements, and periodic elements of partially ordered quasi-metric spaces – improves known fixed point theorems. Consequently, we extend many theorems in the metric or ordered fixed point theory by adopting quasi-metric instead of metric.

(4) [15] : Let (X, d) be a quasi-metric space. A Rus-Hicks-Rhoades (RHR) map $f : X \to X$ is the one satisfying $d(fx, f^2x) \leq \alpha d(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. In our previous work [12], we collected various fixed point theorems closely related to RHR maps. In the present article, we collect almost all things we know about the RHR maps and their examples. Moreover we derive new classes of generalized RHR maps and fixed point theorems on them. Consequently, many of known results in metric fixed point theory are improved and reproved in easy way.

5. The Özkan type

In the article [8] in 2023, Özkan proves a common fixed point theorem for Suzuki type contractions on complete partial metric spaces. Moreover, he states some corollaries related to Suzuki type common fixed point theorem. And he gives an example where he applies his main theorem on complete partial metric spaces. Finally, to show usability of our results, he gives an application showing existence and uniqueness of a common solution for a class of functional equations in dynamic programming.

Definition 5.1. Let $X \neq \emptyset$. A function $p : X \times X \to [0, \infty)$ is called a *partial metric*, if it holds the following properties for all $x, y, z \in X$

(p1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(p2) $p(x, x) \leq p(x, y)$,
(p3) $p(x, y) = p(y, x)$,
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Definition 5.2. Let (X, p) be a partial metric space (PMS).

(1) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(2) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.
(3) (X, p) is called complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

It is noted in [6] that, from (p1) and (p2), $p(x, y) = 0$ implies $x = y$. But the opposite may not be true. If we define partial metric as $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$, then the pair $(\mathbb{R}^+, p)$ is a PMS. This is a basic for PMS. This means that the quasi-metric and the partial metric are independent each other.

From now on, we compare the main results of Özkan [9] with our results in the present article:

(1) Theorem 1 of [9] : This is our Theorem 3.3 for PMS with more than five page proof.

(2) Corollary 1 of [9] : This is the Suzuki type result [13] and Theorem 3.1 for PMS.

(3) Corollary 2 of [9] : This is our Corollary 3.5 for PMS.

6. Conclusion

There are thousands of extensions, generalizations, modifications of complete metric spaces and hundreds of contractive type conditions on them. The Banach contraction principle originates such study in the last one hundred years.

Recently, we found the class of the Rus-Hicks-Rhoades (BHR) maps extending the Banach contractions and including a large number of proper examples; see [13], [14]. A typical example is the Suzuki type maps which were followed and modified by scores of authors. Recent studies on BHR maps in [12]–[16] show the importance of the RHR maps in the ordered fixed point theory initiated by the present author in [11].

In 2023, Kubra Özkan [6] extended the Suzuki type theorem to partial metric spaces which are different from quasi-metric spaces (without the symmetry). It would be interesting to check whether the symmetry (p3) was applied or not in [9].

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