Nonlinear Functional Analysis and Applications
Vol. 29, No. 4 (2024), pp. 1019-1029
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2024.29.04.06
http://nfaa.kyungnam.ac.kr/journal-nfaa

INNOVATIVE STOCHASTIC FINITE DIFFERENCE METHODS IN MEAN FIELD GAMES: A WIENER PROCESS APPROACH

Maan T. Alabdullah[1], Esam El–Siedy[2], Eliwa M. Roushdy[3] and Muner M. Abou Hasan[4]

[1]Department of Mathematics, Faculty of Science,
Ain Shams University, Egypt
e-mail: maenalabdala@gmail.com

[2]Professor of Pure Mathematics, Mathematics Department, Faculty of Science,
Ain Shams University, Egypt
e-mail: esam_ elsedy@hotmail.com

[3]Department of Basic and Applied Sciences, Arab Academy for Science,
Technology and Maritime Transport, Cairo, Egypt
e-mail: eroshdy64@gmail.com

[4]Faculty of Mathematics and Data Science Emirates Aviation University,
Dubai, UAE
e-mail: muneere@live.com, muner.abouhasan@emirates

Abstract. This study explores the integration of Wiener processes into stochastic extensions of mean field games (MFGs), employing nonstandard finite difference methods (NSFDMs) for enhanced accuracy and stability. We extend existing mathematical models in differential game theory, addressing the challenges of stochasticity and numerical approximation. The research highlights the adaptability and precision of NSFDM in complex stochastic environments, providing novel insights and methodologies for addressing planning issues in MFGs with Wiener processes.

[0]Received January 10, 2023. Revised February 7, 2023. Accepted February 15, 2023.
[0]2020 Mathematics Subject Classification: 91A23, 91A80, 65Q10.
[0]Keywords: Mean field games, Wiener process, nonstandard finite difference methods, numerical approximation.
[0]Corresponding author: Maan T. Alabdullah(maenalabdala@gmail.com).

1. Introduction

The introduction of mean field games (MFGs) has revolutionized the understanding of differential game problems, especially as participant numbers approach infinity. This research delves into the incorporation of Wiener processes in MFGs, aiming to enhance model accuracy and robustness. By adapting nonstandard finite difference methods (NSFDMs), we propose innovative solutions to address the stochastic complexities inherent in these models, paving the way for more accurate and reliable applications in various fields. Optimal control theory, however, offers a well-established framework for determining the optimal strategies of individual agents in dynamic systems. It allows for the identification of control policies that maximize certain objective functions while accounting for the system dynamics and constraints. The synergy between MFGs and optimal control lies in their shared objective of unraveling the optimal behaviors and strategies in complex systems ([6], [7], [8]).

In order to solve the differential game problems associated with MFGs, numerical methods are often required. Among these methods, the nonstandard finite difference method (NSFD) has emerged as a valuable tool. NSFD provides an innovative approach to discretizing the continuous-time dynamics of MFGs, enabling their formulation as discrete-time optimization problems. By incorporating nonstandard analysis principles, NSFD enhances the accuracy of the discretization and enables more precise approximations of the underlying dynamics ([9], [10]).

This paper explores the relationship between MFGs and optimal control and presents the use of the NSFD method to solve differential game problems. We will delve into the fundamental concepts of MFGs, highlighting their relevance in modeling large-scale systems with rational agents. We will also discuss optimal control theory and its role in determining optimal strategies for individual agents. Furthermore, we will introduce the NSFD method and demonstrate its application in solving differential game problems associated with MFGs.

Through this study, we seek to comprehensively understand the interplay between MFGs, optimal control, and the NSFD method. By elucidating the theoretical foundations and practical applications of these concepts, we aim to contribute to the growing body of knowledge in the field of differential game theory and its implications in various domains, including economics, finance, and social sciences.

Mean Field Games (MFGs) are a mathematical framework that studies the behavior of large populations of interacting agents in strategic settings. It combines concepts from game theory and optimal control theory to analyze the collective behavior of individuals. In MFGs, agents optimize their actions by considering the average behavior of others, referred to as the mean field.

MFGs provide insights into the equilibrium behavior, optimal strategies, and overall dynamics of large populations. By studying MFGs, researchers can analyze the collective behavior of individuals, make predictions about system-wide outcomes, and develop strategies for decision-making and policy design.

This paper is organized as follows: Section 2 introduces the stochastic extensions in mean field games and the integration of Wiener processes. Section 3 discusses the application of unique finite difference schemes with Wiener processes, followed by a detailed analysis of numerical experiments in Section 4. The paper concludes with Section 5, which summarizes our findings and discusses their implications.

2. Stochastic extensions in mean field games: Integrating Wiener processes for enhanced differential game models

Recently, Lasry and Lions ([6], [7], [8]) have introduced models based on mean field theory, effectively illustrating the behavior of differential game problems as the number of participants approaches infinity. This concept is embodied in a mathematical structure known as Mean Games (MFGs). Within this framework, the scalar functions $v = v(t, x)$ and $w = w(t, x)$ evolve over time and space, governed by the following stochastic differential equations:

$$\frac{\partial v}{\partial t} - c\Delta v + H(x, \nabla v) = C[w] + \sigma_v dW_t^v \text{ in } (0, T) \times T^d, \tag{2.1}$$

$$\frac{\partial w}{\partial t} + c\Delta w + \operatorname{div}\left(w \frac{\partial H}{\partial p}(x, \nabla v)\right) = \sigma_w dW_t^w \text{ in } (0, T) \times T^d. \tag{2.2}$$

The Wiener process terms $\sigma_v dW_t^v$ and $\sigma_w dW_t^w$ are introduced to represent random diffusion in the evolution of v and w over time. The initial and terminal conditions remain:

$$w(0, x) = w_0(x), \quad w(T, x) = w_T(x) \text{ in } T^d. \tag{2.3}$$

In this model, two probability densities, w_0 and w_T, are included, with w being a probability density. Consequently, the conditions are further expanded to:

$$\int_{T^d} w(t, x) dx = 1, \quad w > 0. \tag{2.4}$$

Here, $T^d = [0, 1]^d$ represents the d-dimensional unit torus. The constant c is nonnegative, and the operators Δ, ∇, and div refer to the Laplacian, gradient, and divergence operations applied to x. The system also includes a scalar Hamiltonian $H(x, p)$, typically convex in relation to the gradient variable, and an operator C, which transforms a probability density w on T^d into a real-valued function $C[w]$.

This research employs the nonstandard finite difference method (NSFDM), originally introduced by Mickens ([9]-[12]). NSFDM simplifies the construction and improves the discretization of specific elements in differential equations ([1], [16]). The choice of denominator function and discretization approach in NSFDM often results in greater accuracy and stability compared to traditional methods [3]. Its applications are extensive, encompassing areas such as physics, chemistry, engineering ([2], [4], [13], [17], [18]), mathematical biology, and ecology ([5], [15]). NSFDM is particularly beneficial in fractional-order systems, including fractional financial models and fractional-order neuron systems ([14]).

The focus of this paper is to outline the aforementioned system of equations and to numerically approximate its solution using a nonstandard finite difference scheme (NSFDS).

2.1. **Adaptation of the nonstandard finite difference approach with Wiener process.** Mickens ([10], [12]) pioneered the Nonstandard Finite Difference Method (NSFDM) for numerically solving differential equations, including both partial (PDEs) and ordinary (ODEs). This method involves several key processes:

(1) **Creation of Differential Equation in Discrete Form with Stochastic Element**: Convert continuous differential equations into discrete counterparts, incorporating a Wiener process term to represent random diffusion.
(2) **Establishment of an Unconventional Grid**: Implement a nonstandard grid, enhancing the reflection of the analytical solution's behavior, especially under stochastic influences.
(3) **Formulation of an Approximation Framework**: Develop an approximation strategy using the unconventional grid, considering the stochastic elements.
(4) **Utilization of a Distinctive Difference Operator**: Apply a specially designed difference operator to resolve the stochastic discrete equation.
(5) **Evaluation of Methodological Precision and Stability**: Assess the method's precision and stability, ensuring reliable outcomes under stochastic conditions.

In the context of the Euler method for $\frac{dy}{dt}$, this involves substituting $\frac{y(t+h)-y(t)}{h}$ with $\frac{y(t+h)-y(t)}{\phi(h)} + \sigma_y dW_t$, where $\phi(h)$ is a function of step size h, and $\sigma_y dW_t$ represents the Wiener process term.

Theorem 2.1. *Let* $dX_t = a(X_t, t)dt + b(X_t, t)dW_t$ *be a stochastic differential equation (SDE) with the following conditions:*

(1) *The functions* $a(X_t, t)$ *and* $b(X_t, t)$ *are continuous and locally Lipschitz in* X_t *for each* t.
(2) *The initial condition* $X_0 = x_0$ *is given.*

Then, there exists a unique strong solution X_t *to the SDE for* $t \geq 0$*, that is, a solution such that* X_t *is adapted to the filtration generated by the Brownian motion* W_t *and is continuous in both* t *and* X.

Proof. Consider the SDE:

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t.$$

We will construct a sequence of approximate solutions using the Euler-Maruyama method. Let $X_t^{(n)}$ be the approximate solution obtained by discretizing the time interval $[0, t]$ into n subintervals:

$$\Delta t = \frac{t}{n}.$$

The Euler-Maruyama method generates the approximate solution as:

$$X_{(k+1)\Delta t}^{(n)} = X_{k\Delta t}^{(n)} + a(X_{k\Delta t}^{(n)}, k\Delta t)\Delta t + b(X_{k\Delta t}^{(n)}, k\Delta t)\Delta W_k,$$

where ΔW_k is the increment of a Wiener process over the interval $[k\Delta t, (k+1)\Delta t]$, which is normally distributed with mean zero and variance Δt.

Now, for each n, $\{X_t^{(n)}\}$ is a sequence of continuous, adapted processes, and it can be shown that it converges to a limit X_t, in the space of continuous processes. Using the properties of Brownian motion, we can prove that X_t is a strong solution to the SDE.

Suppose there is another strong solution Y_t to the SDE with the same initial condition. We aim to show that X_t and Y_t are equal in distribution.

By the definition of strong solutions, X_t and Y_t are both adapted to the same filtration generated by W_t. We can then apply the uniqueness theorem for ordinary differential equations (ODEs) with Lipschitz coefficients to show that X_t and Y_t must be equal in distribution.

Therefore, there exists a unique strong solution X_t to the given SDE for $t \geq 0$. □

3. Application of unique finite difference schemes with Wiener process

In this section, we adapt our technique for discretizing systems (2.1), (2.2), (2.3), and (2.4) via finite differences to include the Wiener process. We establish a uniform grid on a two-dimensional torus with a mesh step h, represented as T_h^2. Points on this grid are labeled as $x_{i,j}$. A positive integer NT defines $\Delta t = \frac{T}{NT}$, and $t_n = n\Delta t$ for n ranging from 0 to NT.

Values of v and w at coordinates $(x_{i,j}, t_n)$ are estimated as $W_{i,j}^n$ and $V_{i,j}^n$, incorporating stochastic terms $\sigma_v dW_t^v$ and $\sigma_w dW_t^w$ for random diffusion. The operator $C[w](x_{i,j})$ is represented as $(C_h[W])_{i,j} = C[w_h](x_{i,j})$, where w_h is the grid approximation of w.

Discretizing these equations using finite differences now includes stochastic influences:

$$(D_1^+ V)_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{\phi(h)} + \sigma_{v,1} dW_{t,i+1,j}^v, \tag{3.1}$$

$$(D_2^+ V)_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{\phi(h)} + \sigma_{v,2} dW_{t,i,j+1}^v. \tag{3.2}$$

Define the differential operators with stochastic components:

$$[D_h V]_{i,j} = \begin{matrix} \big((D_1^+ V)_{i,j} + \sigma_{v,1} dW_{t,i,j}^v, (D_1^+ V)_{i-1,j} + \sigma_{v,2} dW_{t,i-1,j}^v, \\ (D_2^+ V)_{i,j} + \sigma_{v,3} dW_{t,i,j}^v, (D_2^+ V)_{i,j-1} + \sigma_{v,4} dW_{t,i,j-1}^v\big)^T, \end{matrix} \tag{3.3}$$

$$(\Delta_h V)_{i,j} = -\frac{1}{[\phi(h)]^2}(4V_{i,j} - V_{i+1,j} - V_{i-1,j} - V_{i,j+1} - V_{i,j-1}) + \sigma_\Delta dW_{t,i,j}^\Delta. \tag{3.4}$$

We redefine the function g as $g : T^2 \times \mathbb{R}^4 \to \mathbb{R}$, with $(x, r_1, r_2, r_3, r_4) \mapsto g(x, r_1, r_2, r_3, r_4 + \sigma_g dW_t^g)$, where $\sigma_g dW_t^g$ is the Wiener process term. The properties of g are:

(1) Monotonicity: g decreases with r_1 and r_3, and increases with r_2 and r_4.
(2) Consistency: $g(x, r_1, r_1, r_2, r_2) = H(x, q)$ for all $x \in T^2$ and $r = (r_1, r_2) \in \mathbb{R}^2$.
(3) Differentiability: g is differentiable and belongs to class C^1, and may possess properties like convexity and coercivity.

The coerciveness of g is given by:

$$\lim_{\|[D_h V + \sigma_{D_h} dW_t^{D_h}]\|_\infty \to \infty} \frac{\max_{i,j} g(x_{i,j}, [D_h V]_{i,j} + \sigma_{D_h} dW_t^{D_h})}{\|[D_h V + \sigma_{D_h} dW_t^{D_h}]\|_\infty} = +\infty. \tag{3.5}$$

Our discrete approach and the application of nonstandard finite differences allow for precise approximation of solutions for systems described by either partial or ordinary differential equations, now including stochastic elements.

3.1. **Strategic planning in stochastic environments: A finite difference approach.** The semi-implicit method provides a solution approximation for equation (1), formulated as:

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\phi(\Delta t)}-c(\Delta_h V^{n+1})_{i,j}+g(x_{i,j},[D_h V^{n+1}]_{i,j})+\sigma_V dW_t^V=(C_h[W^n])_{i,j}, \quad (3.6)$$

where, $\sigma_V dW_t^V$ represents the Wiener process term.

The weak formulation of equation (2.2) is approached by examining:

$$\int_{T^2} \operatorname{div}\left(w\frac{\partial h}{\partial p}(x,\nabla v)\right) z\,dx, \quad (3.7)$$

approximated as:

$$-h^2\sum_{i,j} W_{i,j}\nabla_r g(x_{i,j},[D_h V]_{i,j})\cdot[D_h Z]_{i,j}+h^2\sum_{i,j}\lambda_{i,j}(V,W)Z_{i,j}+\sigma_W dW_t^W. \quad (3.8)$$

The term $\sigma_W dW_t^W$ introduces an additional Wiener process element, and $\lambda_{i,j}(V,W)$ denotes a specific coefficient function in the context of the planning problem.

The coefficient function λ is redefined to include the Wiener Process, specified as follows:

$$\lambda_{i,j}(V,W) = \frac{1}{\phi(h)}\begin{pmatrix} \left(W_{i,j}\frac{\partial g}{\partial r_1}(x_{i,j},[D_h V]_{i,j})+\sigma_{\lambda,1}dW_t^\lambda-W_{i-1,j}\frac{\partial g}{\partial r_1}(x_{i-1,j},[D_h V]_{i-1,j})\right)\\ +\left(W_{i+1,j}\frac{\partial g}{\partial r_2}(x_{i+1,j},[D_h V]_{i+1,j})+\sigma_{\lambda,2}dW_t^\lambda-W_{i,j}\frac{\partial g}{\partial r_2}(x_{i,j},[D_h V]_{i,j})\right)\\ \left(W_{i,j}\frac{\partial g}{\partial r_3}(x_{i,j},[D_h V]_{i,j})+\sigma_{\lambda,3}dW_t^\lambda-W_{i,j-1}\frac{\partial g}{\partial r_3}(x_{i,j-1},[D_h V]_{i,j-1})\right)\\ +\left(W_{i,j+1}\frac{\partial g}{\partial r_4}(x_{i,j+1},[D_h V]_{i,j+1})+\sigma_{\lambda,4}dW_t^\lambda-W_{i,j}\frac{\partial g}{\partial r_4}(x_{i,j},[D_h V]_{i,j})\right)\end{pmatrix}. \quad (3.9)$$

The discrete analogue of equation (2.2) is implemented as:

$$\frac{W_{i,j}^{n+1}-W_{i,j}^{n}}{\phi(\Delta t)}+c(\Delta_h W^n)_{i,j}+\lambda_{i,j}(V^{n+1},W^n)+\sigma_W dW_t^W=0. \quad (3.10)$$

Understanding the linearization of the operator mapping V to $-c(\Delta_h V)_{i,j}+g(x_{i,j},[D_h V]_{i,j})$ is crucial. This linearization is the conjugate of the operator that maps W to $-c(\Delta_h W)_{i,j}-\lambda_{i,j}(V,W)+\sigma_\lambda dW_t^\lambda$.

Additionally, we define a compact and convex set κ:

$$\kappa=\left\{(W_{i,j})_{0\le i,j<NT}: W_{i,j}\ge 0,\ h^2\sum_{i,j}W_{i,j}=1\right\}. \quad (3.11)$$

In developing a discrete approach for a system, it involves discretizing the equations and expressing solutions as a series of numerical values. For the system governed by equations (2.1), (2.2), (2.3), and (2.4), the discrete strategy can be delineated by approximating solutions via nonstandard finite differences as such:

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\phi(\Delta t)}-c(\Delta_h U^{n+1})_{i,j}+g(x_{i,j},[D_h V^{n+1}]_{i,j})=(C_h[W^n])_{i,j}+\sigma_V dW_t^V, \tag{3.12}$$

$$\frac{W_{i,j}^{n+1}-W_{i,j}^{n}}{\phi(\Delta t)}-c(\Delta_h W^{n})_{i,j}+\lambda_{i,j}(V^{n+1},W^n)+\sigma_W dW_t^W=0, \tag{3.13}$$

where the conditions are

$$0\le i,j\le Nh,\ 0\le n\le NT,\ W^n\in\kappa,\ W_{i,j}^0=(w_0)_{i,j},\ W_{i,j}^{NT}=(w_T)_{i,j}, \tag{3.14}$$

$$(w_0)_{i,j}=\frac{1}{[\phi(h)]^2}\int_{|x-x_{i,j}|_\infty\le\frac{h}{2}}w_0 \quad\text{and}\quad (w_T)_{i,j}=\frac{1}{[\phi(h)]^2}\int_{|x-x_{i,j}|_\infty\le\frac{h}{2}}w_T. \tag{3.15}$$

The discretization of the Hamiltonian function, symbolized by $g(x,r1,r2,r3,r4)$, is formulated under the assumption that the Hamiltonian $H(x,p)$ is represented as $\psi(x,|p|)$ and subjected to a Godunov scheme. The discrete Hamiltonian, incorporating the Wiener Process, is expressed as:

$$g(x,r1,r2,r3,r4)$$
$$=\psi\left(x,\sqrt{(r_1+\sigma_{r1}dW_t^{r1})^2+(r_3+\sigma_{r3}dW_t^{r3})^2+(r_2+\sigma_{r2}dW_t^{r2})^2+(r_4+\sigma_{r4}dW_t^{r4})^2}\right). \tag{3.16}$$

When the Hamiltonian is defined as:

$$H(x,p)=\cos(4\pi x_1)+\sin(2\pi x_1)+\sin(2\pi x_2)+|p|^\alpha,\ C[W_{i,j}^n]=[W_{i,j}^n]^2, \tag{3.17}$$

with α assuming different values, the corresponding equations (3.6) and (3.8) include Wiener Process terms for stochastic modeling.

Equation (3.12) becomes:

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\phi(\Delta t)}-c\left(-\frac{1}{[\phi(h)]^2}(4V_{i,j}^{n+1}-V_{i+1,j}^{n+1}-V_{i-1,j}^{n+1}-V_{i,j+1}^{n+1}-V_{i,j-1}^{n+1})\right)$$
$$+[\sin(2\pi x_2)+\sin(2\pi x_1)+\cos(4\pi x_1)+|p|^\alpha]+\sigma_V dW_t^V=[W_{i,j}^n]^2. \tag{3.18}$$

Equation (3.13) is modified to:

$$\begin{aligned}\frac{W_{i,j}^{n+1}-W_{i,j}^{n}}{\phi(\Delta t)}&+c\left(-\frac{1}{[\phi(h)]^2}(4W_{i,j}^n-W_{i+1,j}^n-W_{i-1,j}^n-W_{i,j+1}^n-W_{i,j-1}^n)\right)\\&+\lambda_{i,j}(V^{n+1},W^n)+\sigma_W dW_t^W\\&=0.\end{aligned}\tag{3.19}$$

Further manipulations lead to Equation (3.14):

$$\begin{aligned}\left(1+\frac{4c}{\phi(\Delta t)[\phi(h)]^2}\right)V_{i,j}^{n+1}-\frac{c}{\phi(\Delta t)[\phi(h)]^2}(V_{i+1,j}^{n+1}+V_{i-1,j}^{n+1}+V_{i,j+1}^{n+1}+V_{i,j-1}^{n+1})-V_{i,j}^n\\=-\phi(\Delta t)[\sin(2\pi x_2)+\sin(2\pi x_1)+\cos(4\pi x_1)+|p|^\alpha]\\+\phi(\Delta t)[W_{i,j}^n]^2+\sigma_{V,18}dW_t^{V,18}.\end{aligned}\tag{3.20}$$

And Equation (3.15):

$$\begin{aligned}W_{i,j}^{n+1}-W_{i,j}^n-\frac{c}{\phi(\Delta t)[\phi(h)]^2}(4W_{i,j}^n-W_{i+1,j}^n-W_{i-1,j}^n-W_{i,j+1}^n-W_{i,j-1}^n)\\+\frac{\phi(\Delta t)}{\phi(h)}\begin{pmatrix}(W_{i,j}^n\frac{\partial g}{\partial r_1}-W_{i-1,j}^n\frac{\partial g}{\partial r_1}+W_{i+1,j}^n\frac{\partial g}{\partial r_2}-W_{i,j}^n\frac{\partial g}{\partial r_2})\\(W_{i,j}^n\frac{\partial g}{\partial r_3}-W_{i,j-1}^n\frac{\partial g}{\partial r_3}+W_{i,j+1}^n\frac{\partial g}{\partial r_4}-W_{i,j}^n\frac{\partial g}{\partial r_4})\end{pmatrix}\\+\sigma_{W,19}dW_t^{W,19}=0.\end{aligned}\tag{3.21}$$

4. Numerical experiments

- Parameter Values: The values of v, α, and T are set to 1, 2, and 1, respectively.
- Initial and Final Conditions for w:
  - At $t=0$: w is initialized to 1.5 for locations where $\max(|x-0.2|,|y-0.2|)\leq 0.75$.
  - At $t=T$: w is set to 1.5 for locations where $\max(|x|,|y|)\leq 0.75$.
- Function Values Visualization: The values of the functions u and w are visualized, as shown in a hypothetical Figure 1.

The functions v and w evolve over time according to the specified equations and stochastic influence.

5. Conclusion

The integration of Wiener processes into MFGs using NSFDMs marks a significant advancement in differential game theory. This approach not only addresses the stochasticity in MFGs but also enhances the accuracy and stability of numerical approximations. Our findings demonstrate the effectiveness of NSFDMs in dealing with complex stochastic environments, offering valuable

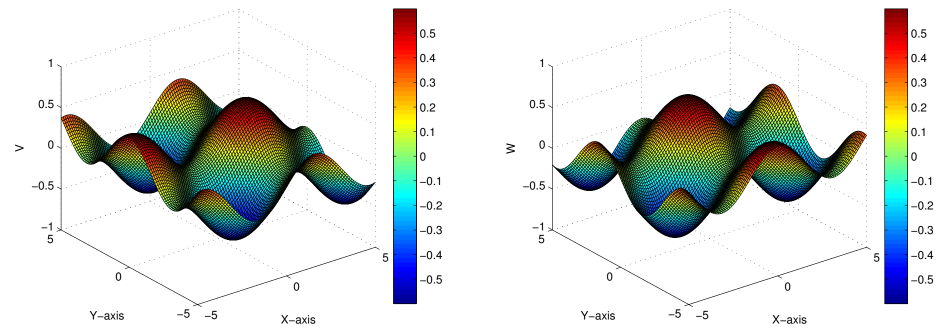

FIGURE 1. Visualization of function evolution in stochastic MFG models.

insights and methodologies for future research in this area. The proposed approach has the potential for widespread application in various fields requiring precise modeling of stochastic processes.

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