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EXISTENCE OF MIN-MAXIMAL SOLUTIONS TO $m-$POINT BOUNDARY VALUE PROBLEMS OF SINGULAR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we present new results on the existence of min-maximal solutions for the second-order singular impulsive differential equations $m-$point boundary value problem, where the nonlinearity is a a.e. continuous function. We also provide discussions to show the valid of our results. In particular, our results unify many known results.

1. Introduction

In this paper, we consider the existence of positive solutions for the following second-order singular impulsive differential equations $m-$point boundary value problem (BVP):

$$\begin{cases} x''(t) + g\,(t)\, f\,(t, x\,(t)) = 0, \quad t \in J, \\ \triangle x'|_{t=t_k} = \triangle x'\,(t_k) = I_k^*\,(x'\,(t_k))\,, \; k = 1,\ \ldots,\ m, \\ \triangle x|_{t=t_k} = \triangle x\,(t_k) = \bar{I}_k\,(x\,(t_k))\,, \quad k = 1,\ \ldots,\ m, \\ x\,(0) = d_1 x\,(1)\,, \quad x'\,(0) = d_2 x'\,(1) - \sum\limits_{i=1}^{m-2} a_i x\,(\eta_i)\,, \end{cases} \tag{1.1}$$

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where $\mathring{J} = (0,1),\ J = [0,1]\,, J_0 = [0,t_1)\,,\ J_1 = (t_1,t_2)\,, \ldots,\ J_m = (t_m,1]\,, \mathbb{R}^+ = [0,\,+\infty),\ 0 < t_1 < \cdots < t_m < 1,\ J' = J \setminus \{t_1,\ldots,t_m\},\ f \in C\left(J \times \mathbb{R}^+, \mathbb{R}^+\right)$, $d_1, d_2 > 0,\ g(t) \not\equiv 0$, and $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1,\ a_i \geq 0$ $(i = 1,\ldots,m-2)$, $I_k^*, \bar{I}_k \in C\left(\mathbb{R}^+, \mathbb{R}^+\right)$, $\triangle x' \mid_{t=t_k} = x'\left(t_k^+\right) - x'\left(t_k^-\right)$, $\triangle x \mid_{t=t_k} = x\left(t_k^+\right) - x\left(t_k^-\right)$, where $x'\left(t_k^+\right), x'\left(t_k^-\right), x\left(t_k^+\right), x\left(t_k^-\right)$ is the left and right limit of $x'(t)\,, x(t)$ at t_k. The theory of impulsive differential equations is an important area (see [1, 2, 4, 5, 6, 8, 11]).

Paper [8] discussed the existence of solutions for impulsive differential equations of the following:

$$\begin{cases} z'' = \Psi\left(t, z(t), z(\varepsilon(t)), Az(t), Bz(t), z(\gamma(t,\zeta(t)))\right), \quad t \in J', \\ \triangle z'|_{t=t_k} = \triangle z'(t_k) = I_k^*\left(z(t_k),\, z'(t_k)\right), \ k = 1,\ \ldots,\ m, \\ \triangle z|_{t=t_k} = \triangle z(t_k) = \bar{I}_k\left(z(t_k)\right), \qquad\quad k = 1,\ \ldots,\ m, \\ W_1(z(0), z(T)) = 0, \quad W_2(z'(0), z'(T)) = 0, \end{cases} \tag{1.2}$$

where $t \in J = [0,T](T > 0)$, $\Psi \in C(J \times \mathbb{R}^5, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, I_k^*, W_1, $W_2 \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\varepsilon \in C(J,\,J)$, $\zeta \in C(J, \mathbb{R})$, $\gamma \in C(J \times \mathbb{R}, J)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \backslash \{t_1, \cdots, t_m\}$, and

$$Az(t) = \int_0^t m(t,s) z(s) ds,\ Bz(t) = \int_0^T h(t,s) z(s) ds,$$

$m(t,s)$, $h(t,s) \in C(J \times J, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, by means of upper and lower solutions and the monotone iterative technique.

We shall consider, in this paper, the existence of solutions of boundary value problems (BVP, for short) for a second order impulsive differential equation in which impulses occur at fixed times t_k $(k = 1, \cdots, m)$. With the help of lower and upper solutions, we shall employ new comparison principle and the monotone iterative technique and establish the existence of min-maximal solutions, which are limits of monotone sequences of the impulsive differential equation boundary value problems.Then we shall obtain the existence theorem, utilizing a analysis technique with continuity and connectedness argument, for the $BVP(1.1)$. It follows therefore that assumptions list in this paper, which are sufficient to guarantee the existence of solutions to the impulsive differential equations boundary value problem (1.1).

Nonlinear differential equations with various nonlinear boundary conditions play an important role in both theory and applications. They have been attracted a lot of researcher's attention over the years, see ([9, 10, 12, 13, 14, 15]) and the references therein. They are often used to model various phenomena in chemistry, physics, biology, and infections diseases etc. in the positive energy problems. However, in many situations, including the cases just mentioned above, based on the fixed point theorems and fixed point index theories, the existence of solutions is easily obtained, one refers the reader to see ([13, 14, 15, 16]) for some references along this line.

In [9] (2012), by using the analytic techniques such as comparison principle, and maximum principle, Lee and $Chung$ presented the long time behaviors of nontrivial solutions for the p–Laplacian evolution $u_t = \Delta_p u$, with $p > 1$ and proved that the solution became extinct for $1 < p < 2$ and remained strictly positive for $p \geq 2$.

In [15](2022), $Wettstein$ studied the fractional harmonic gradient flow on $\mathfrak{F}^1$ getting value in $\mathfrak{F}^{m-1} \subset \mathbb{R}^m$ for $m \geq 2$, in particular constructing regularity and uniqueness of solutions through small enough energy for the weak fractional harmonic gradient flow $x_t + (-\Delta)^{\frac{1}{2}} x = x|d^{\frac{1}{2}} x|^2$, satisfying $x(0, \cdot) = x_0$ in the sense of $x(t, \cdot) \to x_0$ in L^2 as $t \to 0$, putting the existence of solutions. The author improved with extended and generalized many known results. Further, he contemplated a lot of convergence properties for solutions of the fractional gradient flow as $t \to \infty$.

Motivated by the results mentioned above, in the paper we establish the existence of min-maximal solutions for the problem (1.1). Usually, the problem (1.1) can be used to describe the numerical solutions. In the paper, however, we employ the analytic approaches, such as upper and lower solutions, new comparison principle and iterative of solution, instead of numerical ones. As far as we know, many nice of research works of the problem (1.1) are concerned with the numerical approach, but few works are constructed by the analytic method, such as upper and lower technique. We should also assert here that our results are new as well as extend and generalize together with improve the results in ([6, 7, 8, 9, 10, 11, 12, 13, 14, 15]).

The rest of the paper is organized as follows. In Section 2, we first introduce definitions and several lemmas with notations, and offer some key conditions, frequently exploited through the paper. In Section 3, we foremost derive the interesting properties of solutions of the problem (1.1). And then, we also present the main results as well as some their proofs. Finally, in Section 4, we give discussions and supply some examples to show the valid of the main results.

2. Preliminaries

Let $\mathfrak{X} = C[0,1]$ be a Banach space with the norm $\| z \| = \sup_{0 \leq t \leq 1} | z(t) |$, and let $\overline{K} = \{z \in \mathfrak{X} : z(t) \geq 0, \ \ 0 \leq t \leq 1\}$. Then $\overline{K}$ is a positive cone in $\mathfrak{X}$. Throughout the paper, the partial ordering is always given by $\overline{K}$. For the concepts and properties of $Krein - Kutmann$ theorems and fixed point index theory, one refers the reader to see [7].

Denote

$$PC(J,\mathbb{R})=\left\{\begin{array}{l} z \mid z \text{ is a map from } J \text{ onto } \mathbb{R} \text{ such that } z(t) \text{ is continuous at} \\ t \neq t_k, \text{ left continuous at } t=t_k, \text{ and its right limit exists at} \\ t=t_k (\text{denoted by})\, z(t_k^+), \text{ for } k=1,\cdots,m. \end{array}\right\}.$$

Evidently, $PC(J,\mathbb{R})$ is a Banach space with norm $\|z\|_{PC(J,\mathbb{R})}=\sup\limits_{t\in J}\|z(t)\|$.

$$PC^1(J,\mathbb{R})=\left\{\begin{array}{l} z\in PC(J,\mathbb{R}) \mid z'(t) \text{ is continuous at } t\neq t_k, \\ \text{and left continuous at } t=t_k, \text{ and } z(t_k^-),\ z(t_k^+),\ z'(t_k^-), \\ z'(t_k^+), z(t_k^-)=z(t_k^+)=z(t_k), \text{ exist for } k=1,\cdots,m. \end{array}\right\}.$$

Obviously, $PC^1[J,\mathbb{R}]$ is a Banach space with norm

$$\|z\|_{PC^1(J,\mathbb{R})}=\sup_{t\in J}\{\|z\|_{PC(J,\mathbb{R})},\ \|z'\|_{PC(J,\mathbb{R})}\}.$$

It is noticed that $\overline{K}\subset\mathfrak{X}$. Denote $P_r=\{z\in\overline{K}:\|z\|<r\}$, $\partial P_r=\{z\in\overline{K}:\|z\|=r\}$, $\overline{K}_{r,R}=\{z\in\overline{K}:r\leq\|z\|\leq R\}$, for any positive constants $0<r<R<+\infty$. Let $z'=\nu$, $z(0)=0$, $z(s)=\int_0^s\nu(t)dt+z(0)=\int_0^s\nu(t)dt$.

Definition 2.1. ([3, 6, 7, 8, 16]) $x_0(t)$ is call a lower solution of (1.1), if

$$\begin{cases} -x_0''(t)\leq g(t)\,f(t,x_0(t)), \\ \triangle x_0'(t_k)\geq I_k^*\left(x_0'(t_k)\right),\ \triangle x_0(t_k)=\bar{I}_k(x_0(t_k)),\ k=1,\cdots,m, \\ x_0(0)=d_1x_0(1),\quad x_0'(0)\geq d_2x_0'(1)-\displaystyle\sum_{i=1}^{m-2}a_ix_0(\eta_i). \end{cases}\tag{2.1}$$

Definition 2.2. ([3, 6, 7, 8, 16]) $y_0(t)$ is called a upper solution of (1.1), if

$$\begin{cases} -y_0''(t_k)\geq g(t)f(t,y_0(t)), \\ \triangle y_0(t_k)=\bar{I}_k(y_0(t_k)),\ \triangle y_0'(t_k)\leq I_k^*(y_0(t_k)),\ k=1,\ldots,m, \\ y_0(0)=d_1x_0(1),\ x_0'(0)\leq d_2x_0'(1)-\displaystyle\sum_{i=1}^{m-2}a_ix_0(\eta_i). \end{cases}\tag{2.2}$$

Throughout of this paper, we suppose that the following conditions hold:

(H_1) There exist $x_0(t),y_0(t)\in PC^1(J,\mathbb{R})\cap PC^2(J',\mathbb{R})$ such that $x_0(t)\leq y_0(t)$ satisfies (2.1) and (2.2), respectively.

(H_2) $g\in C((0,1),(0,+\infty))$, $f(t,x)\in C((0,1)\times(-\infty,+\infty),(0,+\infty))$. There exist function g^* and $\widetilde{g}\in C(J,\mathbb{R}^+)$, which satisfy (2.4) such that $g(t)f(t,x)\leq g^*(t)$ and for $t\in J$.

$$g(t)f(t,x)-g(t)f(t,\overline{x})\geq\widetilde{g}(t)(x-\overline{x}),\ x_0(t)\leq\overline{x}\leq x\leq y_0(t),$$

(H_3) $f(t,x)$ may be singular at $x=0$ and for any $0<r<R<+\infty$, we have

$$\lim_{j\to+\infty}\sup_{x\in\overline{P}_{r,R}}\int_{\varrho(j)} g(s)f(s,x(s))ds=0,$$

where $\varrho(j)=[0,\frac{1}{j}]\cup[\frac{j-1}{j},1]$, and $j>1$ is a certain positive integer number. There exist constants $0\le\mathfrak{Q}_k<1$, $0<\mathfrak{Q}_k^*<1\,(k=1,\cdots,m)$, such that $\overline{I}_k(x_k)\le x(t_k)\mathfrak{Q}_k$ and $\overline{I}_k^*(x_k)\le x(t_k)\mathfrak{Q}_k^*$,

$$I_k(x)-I_k(\overline{x})\ge-\mathfrak{Q}_k(x-\overline{x}),$$
$$I_k^*(x')-I_k^*(\overline{x}')\ge-\mathfrak{Q}_k^*(x'-\overline{x}'),$$

where $x_0(t_k)\le\overline{x}\le x\le y_0(t_k)$, $x_0(0)\le\overline{x}\le x\le y_0(0)$, $x_0(1)\le\overline{y}\le y\le y_0(1)$.

Lemma 2.3. ([2, 3]) *Let $\widetilde{C}_k>0$, $\widetilde{\phi}_k\,(k=1,\cdots,m)$ be constants and $s\in[0,1)$ be fixed. Assume that $\widehat{a}$, $\widehat{b}\in PC\,[J,\mathbb{R}]$, $z\in PC^1\,[J,\mathbb{R}]$. If*

$$\begin{cases} z'(t)\le\widehat{a}(t)\,z(t)+\widehat{b}(t), & t\in[s,1)\,,\ t\neq t_k,\\ z\left(t_k^+\right)\le\widetilde{C}_k z(t_k)+\widetilde{\phi}_k, & t_k\in[s,\,1)\,. \end{cases}$$

Then, for all $t\in[s,1]$,

$$\begin{aligned} z(t)\le & z\left(s^+\right)\left(\prod_{s<t_k<t}\widetilde{C}_k\right)\exp\left(\int_s^t\widehat{a}(r)dr\right)\\ &+\int_s^t\left(\prod_{r<t_k<t}\widetilde{C}_k\right)\exp\left(\int_r^t\widehat{a}(\tau)d\tau\right)\widehat{b}(r)dr\\ &+\sum_{s<t_k<t}\left(\prod_{t_k<t}\widetilde{C}_k\right)\exp\left(\int_{t_k}^t\widehat{a}(\tau)d\tau\right)\widetilde{\phi}_k. \end{aligned}$$

Lemma 2.4. (New comparison principle) *Let $u\in PC^1\,[J,\mathbb{R}]\cap C^2\,(J',\mathbb{R})$ satisfies the following:*

$$\begin{cases} x''(t)\le-g(t)\,f(t,x(t)),\ t\in J',\\ \triangle x(t_k)\le-\mathfrak{Q}_k x(t_k)\,,\quad \triangle x'(t_k)\le-\mathfrak{Q}_k^* x'(t_k)\,,\ k=1,\cdots,m,\\ x(0)\le d_1x(1)\,,\ x'(0)\le d_2x'(1)-\displaystyle\sum_{i=1}^{m-2}a_ix(\xi_i)\,, \end{cases}\tag{2.3}$$

where $\mathfrak{Q}_k \in [0,1]$, $d_1,\ d_2, \mathfrak{Q}_k^*, \xi_i \in (0,1)$, *for* $i = 1, \ldots, m-2$ *and* $k = 1, \ldots, m$,

$$\begin{aligned} d_1 \prod_{k=1}^{m} (1-\mathfrak{Q}_k)^2 \prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \left(1 - d_2 \prod_{k=1}^{m} (1-\mathfrak{Q}_k^*)\right) \\ \geq \int_0^1 g(s) f(s, x(s)) ds \int_0^1 \prod_{s<t_k<1} (1-\mathfrak{Q}_k)\, ds. \end{aligned} \tag{2.4}$$

Then $x(t) \leq 0$, $t \in J$.

Proof. Suppose to the contrary. Then there exists $t_0 \in J$, such that $x(t_0) > 0$, thus there are the following two cases:

Case I. There exists $\bar{t}_0 \in J$ such that $x(\bar{t}_0) > 0$, and $x(t) \geq 0$ for $t \in J$.

Case II. There exists $\underline{t}_*$, $\bar{t}^* \in J$ such that $x(\underline{t}_*) < 0$, and $x(\bar{t}^*) > 0$.

For **Case I.** It follows from (2.3) that $x''(t) \leq 0$ for $t \neq t_k$, and $x'\left(t_k^+\right) \leq (1-\mathfrak{Q}_k^*)\, x'(t_k)$. By making use of (2.3), we obtain $x'(t) \leq x'(0) \prod\limits_{0<t_k<t} (1-\mathfrak{Q}_k^*)$. Consequently, we get

$$x'(0) \leq d_2 x'(1) \leq d_2 x'(0) \prod_{k=1}^{m} (1-\mathfrak{Q}_k^*),$$

which implies that $x'(0) \leq 0$, thus $x'(t) \leq 0$. Then $x\left(t_k^+\right) \leq (1-\mathfrak{Q}_k)\, x(t_k) \leq x(t_k)$. Therefore, $x(t)$ is a non-increasing on J. So $x(0) \leq d_1 x(1) \leq d_1 x(0) < x(0)$, which is a contradiction.

For **Case II.** Let $l = -\inf\limits_{t\in J} x(t)$, then $l > 0$, and there exists $\underline{t}_* \in (t_i, t_{i+1}]$ such that $x(\underline{t}_*) = -l$ or $x(t_i^*) = -l$ for certain $i \in \{1, \ldots, m\}$. Without loss of generality, one only proves $x(t_i^*) = -l$, the proof of the case $x(\underline{t}_*) = -l$ is similar.

From (2.3), we get

$$\begin{cases} x''(t) \leq l g(t) f(t, x(t)), \\ x'(t_k^+) \leq (1-\mathfrak{Q}_k^*) x'(t_k). \end{cases}$$

Applying (2.3), we obtain

$$x'(t) \leq x'(0) \prod_{0<t_k<t} (1-\mathfrak{Q}_k^*) + \int_0^t l g(s) f(s, x(s)) \prod_{s<t_k<t} (1-\mathfrak{Q}_k^*)\, ds. \tag{2.5}$$

Let $t=1$, it follows from (2.5), we get

$$\begin{aligned} x'(0) &\le d_2 x'(1) \\ &\le d_2 x'(0)\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*) + d_2\int_0^1 l\prod_{s<t_k<1}(1-\mathfrak{Q}_k^*)\,g(s)f(s,x(s))ds \\ &\le d_2 x'(0)\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*) + d_2\int_0^1 lg(s)f(s,x(s))ds, \end{aligned}$$

which implies that

$$x'(0) \le d_2\int_0^1 lg(s)f(s,x(s))ds\left[1-d_2\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\right]^{-1}. \tag{2.6}$$

Using (2.5) and (2.6), we know that

$$\begin{aligned} x'(t) \le{}& d_2\int_0^1 lg(s)f(s,x(s))ds\left(1-d_2\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\right)^{-1}\prod_{0<t_k<t}(1-\mathfrak{Q}_k^*) \\ &+\int_0^t l\prod_{s<t_k<t}(1-\mathfrak{Q}_k^*)\,g(s)f(s,x(s))ds \\ \le{}& \int_0^1 lg(s)f(s,x(s))ds\left[d_2\prod_{0<t_k<t}(1-\mathfrak{Q}_k^*)\left(1-d_2\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\right)^{-1}+1\right] \\ \le{}& d_2\prod_{0<t_k<t}(1-\mathfrak{Q}_k^*)\int_0^1 lg(s)f(s,x(s))ds\times\left\{\left(1-d_2\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\right)^{-1}\right. \\ &\left.+\left[d_2\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\right]^{-1}\right\} \\ \le{}& \int_0^1 lg(s)f(s,x(s))ds\left[\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\left(1-d_2\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\right)\right]^{-1} \end{aligned}$$

for $t\in[\underline{t}_*,1]$, the above inequality and $x(t_k^+)\le(1-\mathfrak{Q}_k)x(t_k)$ implies that

$$\begin{aligned} x(t) \le{}& x(\underline{t}_*)\prod_{\underline{t}_*<t_k<t}(1-\mathfrak{Q}_k)+\int_{\underline{t}_*}^t\prod_{s<t_k<1}(1-\mathfrak{Q}_k)\,ds\int_0^1 lg(v)f(v,x(v))dv \\ &\times\left[\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\left(1-d_2\prod_{k=1}^{m}(1-\mathfrak{Q}_k^*)\right)\right]^{-1}ds \end{aligned} \tag{2.7}$$

$$\leq x\left(\underline{t}_*\right) \prod_{\underline{t}_*<t_k<t} \left(1-\mathfrak{Q}_k\right) + \int_0^1 lg(v)f(v,x(v))dv \int_{\underline{t}_*}^{t} \prod_{s<t_k<1} \left(1-\mathfrak{Q}_k\right) ds$$
$$\times \left[\prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right) \left(1-d_2 \prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right)\right)\right]^{-1}.$$

If $\overline{t}^* > \underline{t}_*$, let $t=\overline{t}^*$ in (2.7), we have

$$0 < -l \prod_{\underline{t}_*<t_k<\overline{t}^*} \left(1-\mathfrak{Q}_k\right) + \int_0^1 lg(v)f(v,x(v))dv \int_{\underline{t}_*}^{\overline{t}^*} \prod_{s<t_k<\overline{t}^*} \left(1-\mathfrak{Q}_k\right) ds$$
$$\times \left[\prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right) \left(1-d_2 \prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right)\right)\right]^{-1}.$$

Thus

$$\frac{\prod_{k=1}^{m} \left(1-\mathfrak{Q}_k\right)}{\int_0^1 \prod_{s<t_k<1} \left(1-\mathfrak{Q}_k\right) ds} \leq \frac{\prod_{\underline{t}_*<t_k<\overline{t}^*} \left(1-\mathfrak{Q}_k\right)}{\int_{\underline{t}_*}^{\overline{t}^*} \prod_{s<t_k<\overline{t}^*} \left(1-\mathfrak{Q}_k\right) ds}$$
$$< \int_0^1 lg(v)f(v,x(v))dv$$
$$\times \left[\prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right) \left(1-d_2 \prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right)\right)\right]^{-1},$$

which contradicts (2.4). Therefore, $x(t) \leq 0$ on J.

If $\overline{t}^* < \underline{t}_*$, without loss of generality, let $\underline{t}_* \in (t_{n-1}, t_n)$ and $\overline{t}^* \in (t_r, t_{r+1}]$, $0 \leq r \leq n-1$, n, $r \in \{1, \cdots, m\}$. From Lemma (2.3), we obtain

$$x\left(\overline{t}^*\right) \leq x(0) \prod_{0<t_k<\overline{t}^*} \left(1-\mathfrak{Q}_k\right) + \int_0^1 lg(v)f(v,x(v))dv$$
$$\times \left[\prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right) \left(1-d_2 \prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right)\right)\right]^{-1} \int_0^{\overline{t}^*} \prod_{s<t_k<\overline{t}^*} \left(1-\mathfrak{Q}_k\right) ds$$
$$= x(0) \prod_{k=1}^{r} \left(1-\mathfrak{Q}_k\right) + \int_0^1 lg(v)f(v,x(v))dv$$
$$\times \left[\prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right) \left(1-d_2 \prod_{k=1}^{m} \left(1-\mathfrak{Q}_k^*\right)\right)\right]^{-1} \int_0^{\overline{t}^*} \prod_{s<t_k<\overline{t}^*} \left(1-\mathfrak{Q}_k\right) ds. \tag{2.8}$$

On the other hand,

$$\begin{aligned}
x(0) &\le d_1 x(1) \\
&\le d_1 x(\underline{t}_*) \prod_{\underline{t}_* < t_k < 1} (1-\mathfrak{Q}_k) + d_1 \int_0^1 l g(v) f(v, x(v)) dv \\
&\quad \times \left[\prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \left(1 - d_2 \prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \right) \right]^{-1} \int_{\underline{t}_*}^{1} \prod_{s<t_k<1} (1-\mathfrak{Q}_k)\, ds \\
&= -d_1 l \prod_{k=n}^{m} (1-\mathfrak{Q}_k) + d_1 \int_0^1 l g(v) f(v, x(v)) dv \\
&\quad \times \left[\prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \left(1 - d_2 \prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \right) \right]^{-1} \int_{\underline{t}_*}^{1} \prod_{s<t_k<1} (1-\mathfrak{Q}_k)\, ds.
\end{aligned} \tag{2.9}$$

It follows from (2.8) and (2.9), we see that

$$\begin{aligned}
& d_1 \prod_{k=n}^{m} (1-\mathfrak{Q}_k) \prod_{j=1}^{r} (1-\mathfrak{Q}_j) \\
& < \int_0^1 g(v) f(v, x(v)) dv \left[\prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \left(1 - d_2 \prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \right) \right]^{-1} \\
& \quad \times \left(d_1 \prod_{j=1}^{r} (1-\mathfrak{Q}_j) \int_{\underline{t}_*}^{1} \prod_{s<t_k<1} (1-\mathfrak{Q}_k)\, ds + \int_0^{\bar{t}^*} \prod_{s<t_k<\bar{t}^*} (1-\mathfrak{Q}_k)\, ds \right).
\end{aligned}$$

Multiply both sides of the above inequality by $\prod_{j=r+1}^{m} (1-\mathfrak{Q}_j)$, then we have

$$\begin{aligned}
& d_1 \prod_{k=n}^{m} (1-\mathfrak{Q}_k) \prod_{j=1}^{m} (1-\mathfrak{Q}_j) \\
& < \int_0^1 g(v) f(v, x(v)) dv \left[\prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \left(1 - d_2 \prod_{k=1}^{m} (1-\mathfrak{Q}_k^*) \right) \right]^{-1} \\
& \quad \times \left(d_1 \prod_{j=1}^{m} (1-\mathfrak{Q}_j) \int_{\underline{t}_*}^{1} \prod_{s<t_k<1} (1-\mathfrak{Q}_k)\, ds + \int_0^{\bar{t}^*} \prod_{s<t_k<\bar{t}^*} (1-\mathfrak{Q}_k)\, ds \prod_{j=r+1}^{m} (1-\mathfrak{Q}_j) \right)
\end{aligned}$$

$$\leq \int_0^1 g(v)f(v,x(v))dv \left[\prod_{k=1}^m (1-\mathfrak{Q}_k^*)\left(1-d_2\prod_{k=1}^m (1-\mathfrak{Q}_k^*)\right)\right]^{-1}$$
$$\times \left(\int_{\underline{t}_*}^1 \prod_{s<t_k<1}(1-\mathfrak{Q}_k)ds + \int_0^{\bar{t}^*} \prod_{s<t_k<1}(1-\mathfrak{Q}_k)ds\right)$$
$$\leq \int_0^1 g(v)f(v,x(v))dv \left[\prod_{k=1}^m (1-\mathfrak{Q}_k^*)\left(1-d_2\prod_{k=1}^m (1-\mathfrak{Q}_k^*)\right)\right]^{-1}$$
$$\times \int_0^1 \prod_{s<t_k<1}(1-\mathfrak{Q}_k)ds.$$

Therefore,

$$d_1\prod_{k=1}^m (1-\mathfrak{Q}_k)^2 < \int_0^1 g(v)f(v,x(v))dv \left[\prod_{k=1}^m (1-\mathfrak{Q}_k^*)\left(1-d_2\prod_{k=1}^m (1-\mathfrak{Q}_k^*)\right)\right]^{-1} \times \int_0^1 \prod_{s<t_k<1}(1-\mathfrak{Q}_k)\,ds,$$

which contradicts (2.4). Thus we get $x(t) \leq 0$ on J. □

Now we consider the following problem

$$\begin{cases} x''(t) = h(t) - g(t)\,f(t,x(t)),\ t \in J', \\ \triangle x(t_k) = \varphi_k - \mathfrak{Q}_k x(t_k),\quad \triangle x'(t_k) = \phi_k - \mathfrak{Q}_k^* x'(t_k),\ k=1,\cdots,m, \\ x(0) = d_1 x(1) + C_1,\ x'(0) = d_2 x'(1) - \sum_{i=1}^{m-2} a_i x(\eta_i) + C_2, \end{cases} \tag{2.10}$$

where $h \in PC(J,\mathbb{R})$, φ_k, ϕ_k, C_1, $C_2 \in \mathbb{R}$.

Lemma 2.5. *If $x(t) \in PC^1(J,\mathbb{R}) \cap C^2(J',\mathbb{R})$ is a solution of the impulsive differential system (2.10) if and only if $x(t) \in PC^1(J,\mathbb{R})$ is a solution of the impulsive integral equation*

$$\begin{aligned} x(t) = W_1 + tW_2 + \int_0^t (t-s)\,(h(s) - g(s)\,f(s,x(s)))\,ds \\ + \prod_{0<t_k<t}\left((\varphi_k - \mathfrak{Q}_k x(t_k)) + (t-t_k)(\phi_k - \mathfrak{Q}_k^* x'(t_k))\right), \end{aligned} \tag{2.11}$$

where

$$W_1 = \frac{d_1}{1-d_1}\Bigg(\int_0^1 (1-s)\left(h(s) - g\left(s\right) f\left(s, x\left(s\right)\right)\right) ds$$
$$+ \prod_{0<t_k<1} \big((\varphi_k - \mathfrak{Q}_k x(t_k)) + (1-t_k)(\phi_k - \mathfrak{Q}_k^* x'(t_k))\big) + W_2 + C_1\Bigg) + C_1,$$
$$W_2 = \frac{d_2}{1-d_2}\Bigg(\int_0^1 \left(h(s) - g\left(s\right) f\left(s, x\left(s\right)\right)\right) ds$$
$$+ \prod_{0<t_k<1} \left(\phi_k - \sum_{i=1}^{m-2} a_i x\left(\eta_i\right) - \mathfrak{Q}_k^* x'(t_k)\right) + C_2\Bigg) + C_2.$$

Proof. By a simple computation, we can get the results. So we omit the proof. □

Lemma 2.6. *For $h \in PC(J,\mathbb{R}), \varphi_k, \phi_k, C_1, C_2 \in \mathbb{R}, 0 \le \mathfrak{Q}_k < 1, 0 < d_1, d_2, \mathfrak{Q}_k^* < 1$. If*

$$\begin{cases} \dfrac{1}{1-d_1}\left(\displaystyle\int_0^1 (1-s)\, g(s) f(s,x(s)) ds + \sum_{k=1}^{m} (\mathfrak{Q}_k + (1-t_k)\mathfrak{Q}_k^*)\right) \\ \quad + \dfrac{d_2}{(1-d_1)(1-d_2)}\left(\displaystyle\int_0^1 g(s) f(s,x(s)) ds + \sum_{k=1}^{m} \mathfrak{Q}_k^*\right) < 1, \\ \dfrac{1}{1-d_2}\left(\displaystyle\int_0^1 g(s) f(s,x(s)) ds + \sum_{k=1}^{m} \mathfrak{Q}_k^*\right) < 1. \end{cases} \tag{2.12}$$

Then (2.10) *has a unique solution $x\left(t\right) \in PC^1(J,\mathbb{R}) \cap C^2(J',\mathbb{R})$.*

Lemma 2.7. *Let $x \in PC^1(J,\, \mathbb{R}) \cap C^2(J,\, \mathbb{R})$ and*

$$\begin{cases} x''(t) \le 0,\ t \in J', \\ x\left(t_k^+\right) = x\left(t_k^-\right) + \overline{I}_k\left(x\left(t_k\right)\right), \quad k = 1, \ldots, m, \\ x'\left(t_k^+\right) \le x'\left(t_k^-\right) + I_k^*\left(x'\left(t_k\right)\right),\ k = 1, \ldots, m, \\ x(0) - \eta_1 x'(0) \ge 0, \quad \eta_1 \in (0,1), \\ x(1) + \eta_2 x'(1) \ge 0, \quad \eta_2 \in (0,1). \end{cases} \tag{2.13}$$

Then $x(t) \ge 0$ for all $t \in J$.

Proof. By simple computation, we can easily obtain the result. Noticing that the graph of $x(t)$ on $[0,1]$ is concave. The proof is omitted. □

Lemma 2.8. *Suppose that condition* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Then there exists a constant* $\tau \in (0, \frac{1}{2})$ *satisfies*

$$0 < \int_{[\tau, 1-\tau]} g(s) ds < +\infty.$$

Proof. It follows from (H_2) that

$$0 < \int_{[\tau, 1-\tau]} g(s) ds < \int_{[0,1]} g(s) ds < +\infty.$$

The proof is completed. □

Lemma 2.9. *Assume that conditions* $(\boldsymbol{H}_1)$ *as well as* $(\boldsymbol{H}_2)$ *and* $(\boldsymbol{H}_3)$ *hold. Then* $\mathfrak{T} : \overline{K} \to \overline{K}$ *is a completely continuous operator.*

Proof. It is easily to prove that $\mathfrak{T} : \overline{K} \to \overline{K}$. Next, for any positive constants $0 < r < R < +\infty$, we will show

$$\sup_{y \in \partial \overline{K}_{r,R}} \int_{[0,1]} g(s) f(s,\, x(s)) ds < +\infty, \tag{2.14}$$

which implies that $\mathfrak{T} : \overline{K} \backslash \{0\} \to \overline{K}$ is well-defined.

By $(\mathbf{H}_2) - (\mathbf{H}_3)$, for any $0 < r < R < +\infty$, there exists a positive integer number j such that

$$\sup_{x \in \partial \overline{K}_{r,R}} \int_{\varrho(j)} g(s) f(s,\, x(s)) ds < 1. \tag{2.15}$$

For any $x \in \partial P_r$, let $x(t_0) = \max_{t \in [0,1]} |x(t)| = r, t_0 \in [0, 1]$. Denote

$$\chi_{\varrho[\underline{\alpha},\, \underline{\beta}]}(t) = \begin{cases} 1, & t \in [\underline{\alpha},\, \underline{\beta}], \\ 0, & t \notin [\underline{\alpha},\, \underline{\beta}], \end{cases}$$

is the eigenvalue function of the set $\varrho[\underline{\alpha},\, \underline{\beta}] = \{t \mid \underline{\alpha} \le t \le \underline{\beta}\}$.

Denote

$$\Theta^{*} = \max \left\{ f(t,\, x) \mid (t,\, x) \in ([0, 1] \setminus \varrho(j)) \times \left[\frac{r}{j},\, R \right],\, j \in \mathbb{Z}_+ \right\}. \tag{2.16}$$

It follows from $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ with (2.15)−(2.16) that

$$\begin{aligned}\sup_{x\in\partial\overline{K}_{r,R}}\int_{[0,1]} g(s)f(s,\,x(s))ds &\le \sup_{x\in\partial\overline{K}_{r,R}}\int_{\varrho(j)} g(s)f(s,\,x(s))ds\\ &\quad+\sup_{x\in\partial\overline{P}_{r,R}}\int_{[0,1]\setminus\varrho(j)} g(s)f(s,\,x(s))ds\\ &\le 1+\Theta^{*}\int_{[0,1]} g(s)ds\\ &< +\infty,\end{aligned} \tag{2.17}$$

that is, (2.14) holds, which also implies that $\mathfrak{T}:\overline{K}_{r,R}\to\overline{K}$ is well-defined and $\mathfrak{T}(\mathfrak{B})$ is uniformly bounded for any bounded set $\mathfrak{B}\subset\overline{K}_{r,R}$.

By simple computing and deducing, we know that $\mathfrak{T}(\overline{K}_{r,R})$ is *equicontinuous*. Thus, by the *Ascoli − Arzela* theorem, we see that $\mathfrak{T}:\overline{K}_{r,R}\to\overline{K}$ is a compact operator. Finally we show that $\mathfrak{T}:\overline{K}_{r,R}\to\overline{K}$ is continuous. In fact, for any $x_n, x_0\in\overline{K}_{r,R}$ and $\|x_n-x_0\|\to 0$ $(n\to\infty)$. Then $\|\mathfrak{T}x_n-\mathfrak{T}x_0\|\to 0$ $(n\to\infty)$. This completes the proof. □

3. Main results

In this section, we present and prove our main results.

Theorem 3.1. *Suppose that* $(H_1)-(H_3)$ *and* (2.12) *hold. Then the impulsive system* (1.1) *has the min-maximal solutions* $x^*,\,y^*\in[x_0,\,y_0]$*, respectively. Moreover, there exist monotone iterative sequences* $\{x_n\}_{n=1}^{\infty},\,\{y_n\}_{n=1}^{\infty}\subset[x_0,\,y_0]$ *such that* $x_n\to x^*$, $y_n\to y^*$ $(n\to\infty)$ *uniformly on* $t\in J$*, where* $\{x_n\}_{n=1}^{\infty},\,\{y_n\}_{n=1}^{\infty}$ *satisfy*

$$\begin{cases} x_n''(t)=g(t)\,f(t,x_n(t))-g^*(t)\,(x_n(t)-x_{n-1}(t))\,,\ t\in J',\\ \triangle\,x_n(t_k)=\bar{I}_k(x_{n-1}(t_k))-\mathfrak{Q}_k(t)\,(x_n(t)-x_{n-1}(t))\,,\ \ k=1,\cdots,m,\\ \triangle\,x_n'(t_k)=I_k^*(x_{n-1}(t_k))-\mathfrak{Q}_k^*(t)\,(x_n(t)-x_{n-1}(t))\,,\ \ k=1,\ldots m,\\ x_n(0)=x_{n-1}(0)+d_1\,(x_n(1)-x_{n-1}(1))\,,\ n=1,2,\cdots,\\ x_n'(0)=x_{n-1}'(0)+d_2\left(x_n'(1)-x_{n-1}'(1)\right),\ n=1,2,\cdots, \end{cases} \tag{3.1}$$

and

$$\begin{cases} y_n''(t)=g(t)\,f(t,y_n(t))-g^*(t)\,(y_n(t)-y_{n-1}(t))\,,\ t\in J',\\ \triangle\,y_n(t_k)=\bar{I}_k(y_{n-1}(t_k))-\mathfrak{Q}_k(t)\,(y_n(t)-y_{n-1}(t))\,,\ k=1,\cdots,m,\\ \triangle\,y_n'(t_k)=I_k^*(y_{n-1}(t_k))-\mathfrak{Q}_k^*(t)\,(y_n(t)-y_{n-1}(t))\,,\ k=1,\cdots,m,\\ y_n(0)=y_{n-1}(0)+d_1\,(y_n(1)-y_{n-1}(1))\,,\ n=1,2,\cdots,\\ y_n'(0)=y_{n-1}'(0)+d_2\left(y_n'(1)-y_{n-1}'(1)\right),\ n=1,2,\cdots \end{cases} \tag{3.2}$$

with

$$x_0 \le x_1 \le x_2 \le \cdots \le x_n \le \cdots \le x^* \le y^* \le \cdots \le y_n \le \cdots \le y_2 \le y_1 \le y_0.$$

Proof. For any $x_{n-1},\, y_{n-1} \in PC^1(J,\mathbb{R}) \cap C^2(J',\mathbb{R})$, it follows from Lemma 2.6 and the problems (3.1) and (3.2) have unique solutions x_n and y_n in $PC^1(J,\mathbb{R}) \cap C^2(J',\mathbb{R})$, respectively. Now, we verify that

$$x_{n-1} \le x_n \le y_n \le y_{n-1},\, n = 1,\, 2,\, \cdots. \tag{3.3}$$

Let $u(t) = x_0(t) - x_1(t), v(t) = y_1(t) - y_0(t), \varpi(t) = x_1(t) - y_1(t)$, and such that

$$d_1(y_0(1) - x_0(1)) \le x_0(0) - y_0(0)$$

with $x_0'(0) - y_0'(0) + d_1[y_0'(1) - x_0'(1)] \le 0$. From (3.1) and (3.2) with $(H_1)-(H_3)$, we have that

$$\begin{cases} u''(t) \le -g^*(t)\,u(t),\, t \in J', \\ \triangle u(t_k) \le -\mathfrak{Q}_k(t_k),\, \triangle u'(t_k) \le -\mathfrak{Q}_k^*(t_k),\, k = 1, \cdots, m, \\ u(0) \le d_1 u(1),\, u'(0) \le d_2 u'(1), \end{cases}$$

and

$$\begin{cases} v''(t) \le -g^*(t)\,v(t),\, t \in J', \\ \triangle v(t_k) \le -\mathfrak{Q}_k(t_k),\, \triangle v'(t_k) \le -\mathfrak{Q}_k^*(t_k),\, k = 1, 2, \cdots, m, \\ v(0) \le d_1 v(1),\, v'(0) \le d_2 v'(1), \end{cases}$$

with

$$\begin{cases} \varpi''(t) = -g^*(t)\,\varpi(t),\, t \in J', \\ \triangle\varpi(t_k) = \mathfrak{Q}_k\varpi(t_k),\ k = 1, \cdots, m, \\ \triangle\varpi'(t_k) = \mathfrak{Q}_k^*\varpi(t_k),\ k = 1, \cdots, m, \\ \varpi(0) = x_0(0) - y_0(0) + d_1[x_1(1) - x_0(1)] - d_1[y_1(1) - y_0(1)] \\ \qquad \le x_0(0) - y_0(0) + d_1\varpi(1) + d_1[y_0(1) - x_0(1)] \le d_1\varpi(1), \\ \varpi'(0) = x_0'(0) - y_0'(0) + d_1[x_1'(1) - x_0'(1)] - d_1[y_1'(1) - y_0'(1)] \\ \qquad \le x_0'(0) - y_0'(0) + d_1\varpi'(1) + d_1[y_0'(1) - x_0'(1)] \le d_1\varpi'(1). \end{cases}$$

Thus, by means of Lemma 2.4, we have $u(t) \le 0,\, v(t) \le 0,\, \varpi(t) \le 0,\, \forall t \in J$, i. e., $x_0 \le x_1 \le y_1 \le y_0$. Assume that $x_{k-1} \le x_k \le y_k \le y_{k-1}$ for some $k \ge 1$. Thus, employing the same technique once again, by Lemma 2.4, one can get $x_k \le x_{k+1} \le y_{k+1} \le y_k$, for $k \ge 1$. Then, one can easily show that

$$x_0 \le x_1 \le \cdots \le x_n \le \cdots \le x^* \le y^* \le \cdots \le y_n \le \cdots \le y_1 \le y_0,\, n = 1, 2, \cdots. \tag{3.4}$$

Employing the standard arguments, we have $\lim_{n\to\infty} x_n(t) = x^*(t),\ \lim_{n\to\infty} y_n(t) = y^*(t)$ uniformly on $t \in J$, and the limit functions x^* and y^* satisfy (1.1). Moreover, $x^*, y^* \in [x_0,\, y_0]$.

Next, we prove that $x^*(t)$ and $y^*(t)$ are the min-maximal solutions of impulsive differential equation (1.1) in $[x_0, y_0]$, respectively. If $\varpi \in [x_0, y_0]$ is any solution of the problem (1.1), and $x_{n-1}(t) \le \varpi(t) \le y_{n-1}(t)$ for some integer n. Let $\mu(t) = x_n(t) - \varpi(t)$ such that $\mu(t) \ge 1 + \varpi(t)$ and $x_{n-1}(0) \le d_1 x_{n-1}(1)$ with $y_0'(0) - x_0'(1) - d_1(y_0'(1) + \varpi'(1)) \le d_2\mu'(1)$. Then

$$\begin{cases} \mu''(t) = g(t) f(t, x_n(t)) - g^*(t)[x_n(t) - x_{n-1}(t)] + g^*(t)\varpi(t) \\ \qquad \le g^*(t)\mu(t), t \in J', \\ \triangle\mu(t_k) = \bar{I}_k(x_{n-1}(t_k)) - \mathfrak{Q}_k[x_n(t_k) - x_{n-1}(t_k)] - \mathfrak{Q}_k\varpi(t) \\ \qquad \le \mathfrak{Q}_k x_n(t_k) - \mathfrak{Q}_k\varpi(t_k) = \mathfrak{Q}_k\mu(t_k), \\ \triangle\mu'(t_k) = I_k^*(x_{n-1}(t_k)) - \mathfrak{Q}_k^*[x_n(t_k) - x_{n-1}(t_k)] - \mathfrak{Q}_k^*\varpi(t_k) \\ \qquad \le \mathfrak{Q}_k^* x_n(t_k) - \mathfrak{Q}_k^*\varpi(t_k) = \mathfrak{Q}_k\mu(t_k) = \mathfrak{Q}_k^*\mu(t_k), \\ \mu(0) = x_{n-1}(0) + d_1[x_n(1) - x_{n-1}(1)] - x_0(0) + y_0(0) \\ \qquad - d_1(x_1(1) - x_0(1)) + d_1(y_1(1) - y_0(1)) \\ \qquad \le x_{n-1}(0) - x_0(0) + y_0(0) + d_1\mu(1) - d_1 x_{n-1}(1) \\ \qquad = x_{n-1}(0) - d_1 x_{n-1}(1) - x_0(0) + y_0(0) + d_1\mu(1) \le d_1\mu(1), \\ \mu'(0) = x_{n-1}'(0) + d_2\left[x_n'(1) - x_{n-1}'(1)\right] - x_0'(0) + y_0'(0) \\ \qquad - d_1(x_1'(1) - x_0'(1)) + d_1(y_1'(1) - y_0'(1)) \\ \qquad \le y_0'(0) - x_0'(1) - d_1(y_0'(1) + \varpi'(1)) \le d_2\mu'(1). \end{cases} \tag{3.5}$$

By Lemma 2.4, we have $x_n(t) \le \varpi(t)$ for $t \in J$. By the same way as above, we can show $\varpi(t) \le y_n(t)$ for $t \in J$. Therefore, $x_n(t) \le \varpi(t) \le y_n(t)$ for $t \in J$. That is $x^*, y^* \in [x_0, y_0]$. Thus x^* and y^* are the min-maximal solutions of the impulsive differential equation boundary value problem (1.1). □

4. Discussions

As an application, we study an infinite system of scalar second-order impulsive differential equations

$$\begin{cases} x_n''(t) = \dfrac{1}{\sqrt{236}}\left(\dfrac{1}{n^3} - x_n\right) + \dfrac{1}{60n^2\sqrt{t}}\left(x_{n+1}^2 + x_{2n}^3\right), \quad 0 \le t \le 1,\, t \ne \dfrac{1}{2}, \\ \triangle\, x_n|_{t=t_i} = -\dfrac{1}{16}x_n(1) + \dfrac{1}{5}x_n'(1), \\ \triangle\, x_n'|_{t=t_i} = -\dfrac{1}{24}x_n(1), \\ x_n(0) = \dfrac{1}{n^3}, \quad x_n'(0) = 0, \ (n = 1, 2, 3, \cdots), \end{cases} \tag{4.1}$$

where $d_1 = d_2 = 0$, $a_i \equiv 0$, $i = 1, \cdots, m-2$. Evidently, $x_n(t) \equiv 0$ $(n = 1, 2, \cdots)$ is not a solution of (4.1).

Corollary 4.1. *System* (4.1) *has minimal and maximal solutions which are continuously differential on* $[0, \frac{1}{2}] \bigcup (\frac{1}{2}, 1]$, $(n = 1, 2, \cdots)$ *and satisfy*

$$0 \le x_n(t) \le \begin{cases} \dfrac{1}{n^3}, & 0 \le t \le \dfrac{1}{2}, \\ \dfrac{3}{n^2\sqrt{t}}, & \dfrac{1}{2} \le t \le 1, \ (n = 1, 2, 3, \cdots). \end{cases} \tag{4.2}$$

Remark 4.2. From above discussions, it is clear that our results improve and extend the results in [6] and [9].

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