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APPLICATIONS OF NEW INTEGRAL TRANSFORM OPERATOR IN QUANTUM CALCULUS

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Abstract. In this article, we have derived the basic analogue of a new integral transform operator using the methods of q-calculus. Further, we establish some properties of this integral transform operator which coincides with the corresponding classical ones when q tends to unity. Finally, the q-representation of this integral transform operator has been used to derive the analytical solution of analogous heat-diffusion equation.

1. Introduction

One of main applications of integral transforms is in finding the solution of processes governed by linear partial differential equations like heat equation, wave equation, etc. [16]. Integral transforms in the classical analysis, are the

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most widely used to solve differential equations and integral equations. A lot of work has been done on the theory and application of integral transforms. Integral transforms have been applied to the solution of ordinary and partial differential equations in control engineering problems.

In recent past the theory of q-calculus known as quantum calculus, have been applied in the many areas of mathematics and physics like ordinary fractional calculus, optimal control problems, q-transform analysis, geometric functional theory, in finding solutions of the q-differential and q-integral equations. Recently, a new Integral transform operator similar to Fourier transform has been proposed for finding the analytical solution for the heat-diffusion problem [17]. Bhat et al. [2] presented a novel integral transform known as the one-dimensional quaternion quadratic-phase Fourier transform. The natural transform was represented by two (p, q)-analogues, and their comparative characteristics were established by Altaf et al. [8]. The basic idea of the present paper is to develop basic analogue of new integral transform operator which will play a similar role in mathematical analysis as well as mathematical physics. The q-analogue of this integral transform operator is used to derive the analytical solution of analogous heat-diffusion problem [9]. We start with basic definitions and facts from the q-calculus which are necessary for understanding of this study. In this sequel, we assume that q satisfies the condition $0 < |q| < 1$, we have from [15]; for $a \in \mathbb{C}$, $[a]_q = \dfrac{1-q^n}{1-q}$.

The q-derivative $D_q f$, of a function f is given by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \qquad (D_q f)(0) = q^{-1} f'(0),$$

where $f'(0)$ exists.

The q-Jackson integrals are defined in [12] as

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} q^n f(aq^n),$$

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(aq^n).$$

Instead, of exponential function here we are concerned with the q-exponential function given in [11]

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}, \quad \text{further} \quad D_q\{e_q^x\} = e_q^x.$$

The q-Laplace transform of the real function, $f(t)$ is defined in [1] as

$$A_q(s) = L_q[f(t)](s) = \int_0^{\infty} [e_q^{-st}] f(t) d_q t \quad \text{for} \quad Re(s) > 0,$$

provided the integral exists for s, where,

$$e_q^t = [1 + (1-q)t]^{\frac{1}{1-q}}, \quad t \in \mathbb{R}, \quad s \in \mathbb{C}.$$

The q-Fourier transform of the real function, $f(t)$ is defined in [13] as

$$B_q(\omega) = F[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e_q^{-i\omega t} d_q t,$$

provided the integral exists for ω, where $t, \omega \in \mathbb{R}$. For further details in q-calculus go through [3, 4, 5, 6, 7, 10].

All these expressions reduce to their classical forms under the limiting case, when q tends to unity.

2. q-generalization of integral transform operators

In this section, we introduce the concepts of the integral transform operators, their q-generalizations are given and some results have been derived.

A new integral transform operator of the real function, $f(t)$ defined on the interval $(-\infty, \infty)$ is defined in [17]

$$C(\theta) = P[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-i\frac{t}{\theta}) dt,$$

provided the integral exists for θ, where $t \in \mathbb{R}$, $\theta \in \mathbb{R} - \{0\}$ and P is called a new integral transform operator.

The q-analogue of this integral transform is defined as

$$C_q(\theta) = P_q[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp_q(-i\frac{t}{\theta}) d_q t, \tag{2.1}$$

provided the integral exists for θ, where $t \in \mathbb{R}$, $\theta \in \mathbb{R} - \{0\}$. Note

$$e_q^{iz} = [1 + i(1-q)z]^{\frac{1}{1-q}}, \quad q < 1, \quad 0 < z < \frac{1}{1-q}.$$

The domains of q-Fourier transform and the transform defined in (2.1) are different so, they are different integral transforms.

Theorem 2.1. (Duality) *If $C_q(\theta) = P_q[f(t)]$ and $A_q(s) = L_q[f(t)](s)$, then we have*

$$C_q(\theta) = B_q(\frac{1}{\theta}) \quad and \quad B_q(\omega) = C_q(\frac{1}{\omega}).$$

Proof. Taking $\omega = \frac{1}{\theta}$, we have

$$C_q(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) exp_q(-i\frac{t}{\theta}) d_q t = B_q(\frac{1}{\theta}).$$

Similarly,

$$\begin{aligned} B_q(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e_q^{-i\omega t} d_q t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) exp_q(-i\frac{t}{\theta}) d_q t = C_q(\frac{1}{\omega}). \end{aligned}$$

□

Theorem 2.2. *If $C_q(\theta) = P_q[f(t)]$, $\lim_{|t|\to\infty} f(t) = 0$ and the derivative of $f(t)$ is $f'(t)$, then*

$$P_q[f'(t)] = \frac{i}{\theta} C_q(\theta).$$

Proof.

$$\begin{aligned} P_q[f'(t)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp_q(-i\frac{t}{\theta}) d_q f(t) \\ &= \frac{1}{\sqrt{2\pi}} \left\{ exp_q(\frac{-it}{\theta}) f(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(\frac{-i}{\theta}) exp_q(\frac{-it}{\theta}) d_q t \right\} \\ &= \frac{i}{\theta} C_q(\theta). \end{aligned}$$

□

Theorem 2.3. *If ${}_1C_q(\theta) = P_q[f_1(t)]$ and ${}_2C_q(\theta) = P_q[f_2(t)]$, then*

$$P_q[(f_1 * f_2)(t)] =_1 C_q *_2 C_q,$$

where the convolution of two integrable functions $f_1(t)$ and $f_2(t)$ denoted by $(f_1 * f_2)(t)$ and defined by

$$(f_1 * f_2)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau.$$

Proof.

$$\begin{aligned} P_q[(f_1 * f_2)(t)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp_q(-i\frac{\tau}{\theta}) f_1(\tau) d_q\tau \left(\int_{-\infty}^{\infty} f_2(t-\tau) \exp_q(-i\frac{(t-\tau)}{\theta}) d_q(t) \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\tau) \exp_q(-i\frac{t}{\theta}) d_q\tau \cdot \int_{-\infty}^{\infty} f_2(t) \exp_q(-i\frac{t}{\theta}) d_q(t) \\ &=_1 C_q *_2 C_q. \end{aligned}$$

□

Theorem 2.4. *If a is a positive constant, then*

$$P_q[e_q^{-at^2}] = \frac{1}{2a} \exp_q(\frac{-1}{4a\theta^2}).$$

Proof. We have

$$\int_{-\infty}^{\infty} e_q^{-at^2} d_q t = \sqrt{\frac{\pi}{a}},$$

then

$$\begin{aligned} P_q[e_q^{-at^2}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp_q(-at^2 - i\frac{t}{\theta}) d_q t \\ &= \exp_q(\frac{-1}{4a\theta^2}) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp_q \left(-a(t + \frac{i}{2a\theta})^2 \right) d_q t \\ &= \exp_q(\frac{-1}{4a\theta^2}) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e_q^{-a(t)^2} d_q t \\ &= \exp_q(\frac{-1}{4a\theta^2}) \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{a}} \\ &= \frac{1}{\sqrt{2a}} \exp_q(\frac{-1}{4a\theta^2}). \end{aligned}$$

□

Theorem 2.5. *If c is a constant, then*

$$P_q[\delta_q(t)] = \frac{1}{\sqrt{2\pi}}.$$

Proof. We have, from [14]

$$\int\limits_{-\infty}^{\infty} \delta_q(t)dt = 1,$$

so, $\delta_q(x)$ behaves like Dirac delta. For $q \neq 1$, we have

$$\begin{aligned} P_q[\delta_q(t)] &= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \delta_q(t) \exp_q(\frac{-it}{\theta}) d_q(t) \\ &= \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

□

3. An application to q-heat-diffusion problem

Let us consider the following q-diffusion equation [9]

$$\Phi_{q,t}(x,t) = k\Phi^{(2)}_{q,x}(x,t), \qquad x \in (-\infty, \infty), \quad t \in \mathbb{R}_+ \tag{3.1}$$

with the initial value condition $\Phi(x,0) = \delta(x)$, where $\Phi_q(x,t)$ is the generalized temperature function, and $k \in (0,\infty)$ is thermal diffusivity.

We have by taking q-integral transform on both sides of equation (3.1) with respect to x, then

$$P_q[\Phi_q(x,t)] = \frac{d_q}{d_q t}\Phi(\theta,t)$$

and

$$P_q[\Phi^{(2)}_{q,x}(x,t)] = \left(\frac{i}{\theta}\right)^2 = \frac{-1}{\theta^2}\Phi(\theta,t).$$

So, from equation (3.1)

$$\frac{d_q}{d_q t}\Phi(\theta,t) = \frac{-k}{\theta^2}\Phi(\theta,t).$$

This implies that

$$\frac{d_q\Phi(\theta,t)}{\Phi(\theta,t)} = \frac{-k}{\theta^2}d_q t,$$

where the initial value condition is $\Phi(\theta, 0) = P_q[\delta_q(x)] = \dfrac{1}{\sqrt{2\pi}}$. Then, we have

$$\Phi(\theta, t) = \frac{1}{\sqrt{2\pi}} \exp_q \left(\frac{-kt}{\theta^2} \frac{\log q}{q-1} \right).$$

Taking $a = \dfrac{1}{4k_1(t)}$ in Theorem 2.4, we get

$$\Phi(x, t) = \frac{1}{\sqrt{4\pi k_1 t}} \exp_q \left(\frac{-x^2}{4k_1 t} \right),$$

where $k_1 = \dfrac{k \log q}{q-1}$, $x \in (-\infty, \infty)$, $t \in (0, \infty)$ and $0 < q < 1$. When q tends to unity, the solution reduces to the solution of classical heat diffusion equation [17].

4. Conclusion

In this work, a new concept of q-integral transform operator has been introduced which is having wide range of applications in mathematical physics and engineering. The advantage of this transform has been illustrated by computing the solution of the generalized heat-diffusion equation. The method can be further applied to other generalized partial differential equations.

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