Nonlinear Functional Analysis and Applications
Vol. 29, No. 4 (2024), pp. 1057-1093
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2024.29.04.09
http://nfaa.kyungnam.ac.kr/journal-nfaa

APPROXIMATION OF SOLUTIONS OF THE ROBIN-DIRICHLET PROBLEMS FOR DAMPED WAVE EQUATIONS

Nguyen Vu Dzung[1], Nguyen Huu Nhan[2], Le Thi Phuong Ngoc[3] and Nguyen Thanh Long[4]

[1]Faculty of Mathematics and Computer Science, University of Science,
227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam;
Vietnam National University, Ho Chi Minh City, Vietnam;
Cao Thang Technical College, 65 Huynh Thuc Khang Street, Ben Nghe Ward,
District 1, Ho Chi Minh City, Vietnam
e-mail: `dzungngv@gmail.com`

[2]Ho Chi Minh City University of Foreign Language and Information Technology,
828 Su Van Hanh Str., Ward 13, Dist. 10, Ho Chi Minh City, Vietnam
e-mail: `nhannh1@huflit.edu.vn`

[3]University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam
e-mail: `ngoc1966@gmail.com`

[4]Faculty of Mathematics and Computer Science, University of Science,
227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam;
Vietnam National University, Ho Chi Minh City, Vietnam
e-mail: `longnt2@gmail.com`

Abstract. In this paper, we investigate the Robin-Dirichlet problems (P_n) for damped wave equations with arithmetic-mean terms $(S_n u)(t) = \dfrac{1}{n}\sum\limits_{i=1}^{n} u^2(\frac{i-1}{n}, t)$, and $(\hat{S}_n u)(t) = \dfrac{1}{n}\sum\limits_{i=1}^{n} u_x^2(\frac{i-1}{n}, t)$, where u is the unknown function. First, under suitable conditions, we prove that, for each $n \in \mathbb{N}$, (P_n) has a unique weak solution $\bar{u}_n$. Next, we prove that the sequence of solutions $\bar{u}_n$ converge strongly in appropriate spaces to the weak solution u_∞ of the corresponding problem (P_∞). Some remarks on open problems are also given in the end of paper.

[0]Received February 14, 2024. Revised September 3, 2024. Accepted September 7, 2024.
[0]2020 Mathematics Subject Classification: 35L20, 35L70, 35Q72.
[0]Keywords: Robin-Dirichlet problem, arithmetic-mean terms, Faedo-Galerkin method, linear recurrent sequence.
[0]Corresponding author: Nguyen Thanh Long(`longnt2@gmail.com`).

1. Introduction

In this paper, we consider the Robin-Dirichlet problems (P_n) for damped wave equations as follows

$$(P_n)\quad \begin{cases} u_{tt} - \lambda u_{txx} - \Phi\left(t, (S_n u)(t), (\hat{S}_n u)(t)\right) \dfrac{\partial}{\partial x}\left(\mu(x,t)u_x(x,t)\right) \\ \quad = f\left(x,t\right),\, 0 < x < 1,\, 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where Φ, μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, $\lambda > 0$, $\zeta \geq 0$ are given constants, and $(S_n u)(t)$, $(\hat{S}_n u)(t)$, $n \in \mathbb{N}$ are arithmetic-mean terms defined by

$$(S_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u^2(\tfrac{i-1}{n}, t)$$

and

$$(\hat{S}_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u_x^2(\tfrac{i-1}{n}, t).$$

The nonlinear wave equations with strong damping of this type have been investigated extensively and obtained many interesting results during the past decades. These equations arise naturally in various sciences such as classical mechanics, fluid dynamics, quantum field theory, see [2], [4]-[8], [10]-[14] and the references given therein. In those mentioned works, by using different methods together with various techniques in functional analysis, several results concerning the existence/global existence and the properties of solutions such as blow-up, decay, stability have been established.

In article [12], Pellicer and Morales considered a model for a damped spring-mass system, precisely a strongly damped wave equation with dynamic boundary conditions as follows

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} = 0,\, 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon\left[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)\right]. \end{cases} \tag{1.2}$$

It is well known that the motion of a mass in a spring-mass-damper system is usually modelled by the following second-order ordinary differential equation (ODE) of damped oscillations

$$m u''(t) = -k u(t) - d u'(t), \tag{1.3}$$

where $k > 0$ is recovery constant of spring and $d \geq 0$ stands for dissipation coefficient. The authors showed that, for some certain values of the parameters in (1.3), the large time behavior of the solutions is the same as for a classical spring-mass-damper ODE. For more details, they proved that for fixed constants α, $r > 0$ and ε small enough, the partial differential equation model (1.2) admitted two dominant eigenvalues. Therefore, this can be implied the existence of a second-order ODE of type (1.3) which can be considered as the limit of the model (1.2) when $t \to \infty$ and ε is sufficiently small.

In article [2], Gazzola and Squassina discussed the following viscoelastic equation with strong damping term Δu_t:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = u|u|^{p-2}, & \text{in } \Omega \times [0,T], \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,T], \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.4}$$

where Ω is an open bounded Lipschitz subset of $\mathbb{R}^n$, $T > 0$, $p > 2$, ω, $\mu > 0$. The authors established the global existence theorem and proved that the global solution is uniformly bounded. They also constructed the finite time blow up of solutions for low initial energy or arbitrarily high initial energy.

In article [5], Q. Li and L. He investigated the nonlinear viscoelastic wave equation with strong damping of the form

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \Delta u_t + u_t = u|u|^{p-2}, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is bounded domains with smooth boundary $\partial\Omega$. The authors proved results concerning local existence/global existence of solutions, established the general decay result for global solutions, and showed the finite time blow-up result for some solutions with negative initial energy and positive initial energy.

In [9], Nhan et al. considered the Robin problem for a nonlinear wave equation with source containing multi-point nonlocal terms as follows

$$\begin{cases} u_{tt} - u_{xx} = f\left(x, t, u(x,t), u_t(x,t), u(\eta_1, t), \cdots, u(\eta_q, t)\right), \\ \qquad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.6}$$

where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and h_0, $h_1 \geq 0$, η_1, $\eta_2, \cdots, \eta_q$ are given constants with $h_0 + h_1 > 0$, $0 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq 1$. Here, the authors proved the existence and uniqueness of a weak solution and established an asymptotic expansion of high order in a small parameter of a weak solution.

At the present time, to the best of our knowledge, less results are investigated for the damped wave equation containing multi-point nonlocal terms. Therefore, motivated by the above-mentioned inspiring works, we discuss here the existence and uniqueness of a weak solution $\bar{u}_n$ for the problem (P_n), $n \in \mathbb{N}$. Furthermore, the convergence of the sequence of solutions $\bar{u}_n$ in appropriate spaces is investigated.

Let us explain in some detail related to our main results. First, for each $n \in \mathbb{N}$ fixed, we prove the existence and uniqueness of a local weak solution $\bar{u}_n$ of Prob. (P_n). Then, we can consider the behavior of solutions $\bar{u}_n$, $n \in \mathbb{N}$. It is clear to see that, if $u \in L^\infty(0,T;H^2)$ then functions $y \mapsto u^2(y,t)$ and $y \mapsto u_x^2(y,t)$ are continuous on $[0,1]$, a.e. $t \in [0,T]$, it leads to

$$(S_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u^2(\tfrac{i-1}{n},t) \to \int_0^1 u^2(x,t)dx = \|u(t)\|^2 \quad \text{as } n \to \infty,$$
$$(\hat{S}_n u)(t) = \frac{1}{n}\sum_{i=1}^{n} u_x^2(\tfrac{i-1}{n},t) \to \int_0^1 u_x^2(x,t)dx = \|u_x(t)\|^2 \quad \text{as } n \to \infty.$$

Therefore, it is possible that Prob. (P_n) have a close relationship in a certain sense with Prob. (P_∞) defined as follows

$$(P_\infty) \quad \begin{cases} u_{tt} - \lambda u_{txx} - \Phi\left(t, \|u(t)\|^2, \|u_x(t)\|^2\right) \dfrac{\partial}{\partial x}\left(\mu(x,t)u_x(x,t)\right) \\ \quad = f(x,t),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x). \end{cases} \tag{1.7}$$

We shall prove this relationship to obtain a solution of Prob. (P_∞) via the convergence of the solution sequence $\{\bar{u}_n\}$ in appropriate spaces. To the best of our knowledge, there are relatively few results related to approximation problems (P_n), with nonlinear expressions containing arithmetic mean terms, to get the approximation of the solutions of Prob. (P_∞).

In one-dimensional case, the first equation of Eq. $(1.1)_1$ of Prob. (P_∞) is regarded as a model of nonlinear wave equations of the Kirchhoff-Carrier type with strong damping. It is well known that the mathematical model of Kirchhoff and Carrier comes from a description of small vibrations of an elastic stretched string. In [3], Kirchhoff first investigated the following nonlinear vibration of an elastic string

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \tag{1.8}$$

where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ is the mass density, h is the cross-section area, L is the length, E is the Young modulus, P_0 is the initial axial tension. And Carrier in [1] established a model of the type

$$u_{tt} - \left(P_0 + P_1 \int_0^L u^2(y,t)dy\right) u_{xx} = 0, \tag{1.9}$$

where P_0, P_1 are given constants, which models vibrations of an elastic string when changes in tension are not small.

This paper consists of four sections. In Section 2, we present some preliminaries. In Section 3, under suitable assumptions, we prove that (P_n) has a unique weak solution $\bar{u}_n$. In Section 4, we show that the solution sequence $\{\bar{u}_n\}$ of Probs. (P_n), $n \in \mathbb{N}$ strongly converges in the Banach space

$$H_T = \{v \in C^0\left([0,T]; H^2 \cap V\right) \cap C^1([0,T]; V) : v' \in L^2(0,T; H^2 \cap V)\}$$

to a weak solution u of the problem (P_∞) as $n \to \infty$, with the estimation $\|\bar{u}_n - u\|_{H_T} \leq C_T E_n$, for all $n \in \mathbb{N}$, where

$$\begin{aligned} E_n &= \|\Phi_n[\bar{u}_n] - \Phi[u]\|_{L^2(0,T)} \\ &\leq \tilde{K}_M \left[\left\|S_n\bar{u}_n - \|u(\cdot)\|^2\right\|_{L^2(0,T)} + \left\|\hat{S}_n\bar{u}_n - \|u_x(\cdot)\|^2\right\|_{L^2(0,T)}\right] \to 0 \end{aligned}$$

as $n \to \infty$, and C_T is the constant independent of n. In the proofs of results obtained here, the main tools of functional analysis such as the linear approximate method, the Galerkin method, the arguments of continuity with priori estimates, the compact method, the regularized technique are employed.

Moreover, in order to get a better priori evaluation, a suitable energy lemma (Lemma 3.4) is also built, where a piecewise linear function on $[0,T]$ and a regularized sequence in $C_c^\infty(\mathbb{R})$ are used to get an energy equality in the case the initial condition $\tilde{u}_0 = \tilde{u}_1 = 0$. Lemma 3.4 is a relative generalization of the inequality and equality of energy given in Lions's book [6, Lemma 1.6, p. 224], it is the key lemma to establish the convergence of linear approximate sequence associated with the problem (P_n). Finally, we remark that the methods used can be applied again for similar problems to obtain the same results (see Remarks 4.1, 4.2 below).

2. Preliminaries

In this paper, with $\Omega = (0,1)$, we will use the usual function spaces $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ denote either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. We denote by $\|\cdot\|$ the norm in L^2 and by $\|\cdot\|_X$ the norm in a Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \leq p \leq \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable such that

$\|u\|_{L^p(0,T;X)} < +\infty$ with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ ess \sup\limits_{0<t<T} \|u(t)\|_X, & \text{if} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

Let $T^* > 0$, with $\Phi \in C^k([0,T^*] \times \mathbb{R}_+^2)$, $\Phi = \Phi\,(t,y,z)$, we put $D_1\Phi = \frac{\partial \mu}{\partial t}$, $D_2\Phi = \frac{\partial \Phi}{\partial y}$, $D_3\Phi = \frac{\partial \Phi}{\partial z}$, and $D^\alpha \Phi = D_1^{\alpha_1} \cdots D_3^{\alpha_3}\Phi$, $\alpha = (\alpha_1, \cdots, \alpha_3) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_1 + \cdots + \alpha_3 \leq k$, $D^{(0,\cdots,0)}\Phi = \Phi$.

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

We put

$$V = \{v \in H^1 : v(1) = 0\}, \tag{2.2}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + \zeta u(0) v(0), \; u, v \in V. \tag{2.3}$$

Then, V is a closed subspace of H^1 and on V, three norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v_x\|$ and $v \mapsto \|v\|_a = \sqrt{a(v,v)}$ are equivalent norms.

We have the following lemmas, the proofs of which are straightforward hence we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1} \text{ for all } v \in H^1. \tag{2.4}$$

Lemma 2.2. *Let $\zeta \geq 0$. Then the imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \leq \|v\|_a, \\ \frac{1}{\sqrt{2}} \|v\|_{H^1} \leq \|v_x\| \leq \|v\|_a \leq \sqrt{1+\zeta}\, \|v_x\| \leq \sqrt{1+\zeta}\, \|v\|_{H^1}, \end{cases} \tag{2.5}$$

for all $v \in V$.

Lemma 2.3. *Let $\zeta \geq 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.3) *is continuous on $V \times V$ and coercive on V.*

Lemma 2.4. *Let $\zeta \geq 0$. Then there exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue λ_j such that*

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \ \lim_{j\to+\infty} \lambda_j = +\infty, \\ a(w_j, v) = \lambda_j \langle w_j, v\rangle \text{ for all } v \in V, \ j = 1, 2, \cdots . \end{cases} \tag{2.6}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also a Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$. On the other hand, we also have w_j satisfying the following boundary value problem

$$\begin{cases} -\Delta w_j = \lambda_j w_j, \text{ in } (0,1), \\ w_{jx}(0) - \zeta w_j(0) = w_j(1) = 0, \ w_j \in C^\infty(\overline{\Omega}). \end{cases} \tag{2.7}$$

Proof. The proof of Lemma 2.4 can be found in ([13, Theorem 7.7, p.87]) with $H = L^2$ and V, $a(\cdot,\cdot)$ as defined by (2.2), (2.3). □

Remark 2.5. The weak formulation of Prob. (P_n) can be given in the following manner: Find $u \in \tilde{V}_T = \{v \in L^\infty(0,T; H^2 \cap V) : v' \in L^\infty(0,T; H^2 \cap V),$ $v'' \in L^\infty(0,T; L^2) \cap L^2(0,T;V)\}$, such that u satisfies the following variational equation

$$\langle u''(t), w\rangle + \lambda a(u'(t), w) + \Phi_n[u](t) a_\mu(t; u(t), w) = \langle f(t), w\rangle \tag{2.8}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1, \tag{2.9}$$

where

$$\begin{aligned} \Phi_n[u](t) &= \Phi\left(t, (S_n u)(t), (\hat{S}_n u)(t)\right), \\ (S_n u)(t) &= \frac{1}{n}\sum_{i=1}^{n} u^2(\tfrac{i-1}{n}, t), \\ (\hat{S}_n u)(t) &= \frac{1}{n}\sum_{i=1}^{n} u_x^2(\tfrac{i-1}{n}, t), \\ a_\mu(t; u, w) &= \langle \mu(t) u_x, v_x\rangle + \zeta\mu(0,t) u(0) v(0), \ u, \ v \in V. \end{aligned} \tag{2.10}$$

3. Existence and uniqueness for (P_n)

We make the following assumptions:

(H_1) : $\tilde{u}_0, \ \tilde{u}_1 \in V \cap H^2, \ \ \tilde{u}_{0x}(0) - \zeta\tilde{u}_0(0) = 0$;
(H_2) : $\mu \in C^2([0,1] \times [0,T^*])$ such that
$\mu(x,t) \geq \mu_* > 0$, for all $(x,t) \in [0,1] \times [0,T^*]$;
(H_3) : $\Phi \in C^1([0,T^*] \times \mathbb{R}_+^2)$ such that $\Phi(t,y,z) \geq 1$,
for all $(t,y,z) \in [0,T^*] \times \mathbb{R}_+^2$;
(H_4) : $f \in L^2(0,T^*; L^2)$ such that $f' \in L^1(0,T^*; L^2)$.

For each $M>0$ given, we set the constants $\bar{K}_M(f)$, $\tilde{K}_M(\Phi)$, $\hat{K}_\mu$ as follows

$$\begin{aligned}
&\tilde{K}_M=\tilde{K}_M(\Phi)=\|\Phi\|_{C^1(\tilde{A}_M)}=\|\Phi\|_{C^0(\tilde{A}_M)}+\sum\nolimits_{i=1}^{3}\|D_i\Phi\|_{C^0(\tilde{A}_M)},\\
&\hat{K}_\mu=\|\mu\|_{C^2(\bar{Q}_{T^*})}=\sum\nolimits_{|\alpha|\le 2}\|D^\alpha\mu\|_{C^0(\bar{Q}_{T^*})},\\
&\|\Phi\|_{C^0(\tilde{A}_M)}=\sup_{(t,y,z)\in\tilde{A}_M}|\Phi(t,y,z)|,\\
&\|\mu\|_{C^0(\bar{Q}_{T^*})}=\sup_{(x,t)\in\bar{Q}_{T^*}}|\mu(x,t)|,
\end{aligned}\tag{3.1}$$

where

$$\tilde{A}_M=[0,T^*]\times[0,M^2]\times[0,2M^2],\ \bar{Q}_{T^*}=[0,1]\times[0,T^*].\tag{3.2}$$

For every $T\in(0,T^*]$, we put

$$V_T=\{v\in L^\infty(0,T;H^2\cap V):v'\in L^\infty(0,T;H^2\cap V),\ v''\in L^2(0,T;V)\},\tag{3.3}$$

then V_T is a Banach space with respect to the following norm (see Lions [6])

$$\|v\|_{V_T}=\max\left\{\|v\|_{L^\infty(0,T;H^2\cap V)},\ \|v'\|_{L^\infty(0,T;H^2\cap V)},\ \|v''\|_{L^2(0,T;V)}\right\}.\tag{3.4}$$

For every $M>0$, we put

$$\begin{aligned}
W(M,T)&=\{v\in V_T:\|v\|_{V_T}\le M\},\\
W_1(M,T)&=\{v\in W(M,T):v''\in L^\infty(0,T;L^2)\}.
\end{aligned}\tag{3.5}$$

We note that

$$H_T=\left\{v\in C^0\left([0,T];H^2\cap V\right)\cap C^1([0,T];V):v'\in L^2(0,T;H^2\cap V)\right\}\tag{3.6}$$

is a Banach space with respect to the norm

$$\|v\|_{H_T}=\|v\|_{C^0([0,T];H^2\cap V)}+\|v'\|_{C^0([0,T];V)}+\|v'\|_{L^2(0,T;H^2\cap V)}.\tag{3.7}$$

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0\equiv\tilde{u}_0$, suppose that

$$u_{m-1}\in W_1(M,T),\tag{3.8}$$

we associate Prob. (P_n) with the following problem.

Find $u_m\in W(M,T)$ $(m\ge 1)$ satisfying the linear variational problem

$$\begin{cases}\langle u_m''(t),w\rangle+\lambda a(u_m'(t),w)+A_m\left(t;u_m(t),w\right)=\langle f(t),w\rangle,\ \forall w\in V,\\ u_m(0)=\tilde{u}_0,\ u_m'(0)=\tilde{u}_1,\end{cases}\tag{3.9}$$

where

$$\begin{aligned}
A_m(t;u,w)&=\bar{\Phi}_m(t)a_\mu(t;u,w),\ \forall u,w\in V,\\
\bar{\Phi}_m(t)&=\Phi_n[u_{m-1}](t)=\Phi\left(t,(S_nu_{m-1})(t),(\hat{S}_nu_{m-1})(t)\right).
\end{aligned}\tag{3.10}$$

First, we need the following lemma, the proof of this lemma is not difficult so we omit the details.

Lemma 3.1. *The following inequalities are fulfilled.*

(i) $\mu_* \|v\|_a^2 \le a_\mu(t; v, v) \le \hat{K}_\mu \|v\|_a^2$, $\forall v \in V$, $\forall t \in [0, T^*]$,

(ii) $\left|a_{\mu'}(t; v, v)\right| \le \hat{K}_\mu \|v\|_a^2$, $\forall v \in V$, $\forall t \in [0, T^*]$,

(iii) $0 < \mu_* \le \mu(x, t) \le \hat{K}_\mu$,

(iv) $\left|\bar{\Phi}'_m(t)\right| \le \left(1 + 6M^2\right) \tilde{K}_M$,

(v) $\left\| \dfrac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t)\right) \right\| \le \sqrt{2} \hat{K}_\mu \sqrt{\bar{S}_m^{(k)}(t)}$,

(vi) $\left\| \dfrac{\partial^2}{\partial x \partial t} \left(\mu(t) u_{mx}^{(k)}(t)\right) \right\| \le 2 \hat{K}_\mu \sqrt{\bar{S}_m^{(k)}(t)}$,

(vii) $\left\| \dfrac{\partial}{\partial t} \left[\bar{\Phi}_m(t) \dfrac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t)\right)\right] \right\| \le (2 + \sqrt{2}) \tilde{K}_M \hat{K}_\mu \sqrt{\bar{S}_m^{(k)}(t)}$.

Now, we have the following theorem.

Theorem 3.2. *Let* $(H_1) - (H_4)$ *hold. Then, there exist positive constants* M, T *such that, for* $u_0 \equiv \tilde{u}_0$, *there exists a recurrent sequence* $\{u_m\} \subset W(M, T)$ *defined by* (3.8)-(3.10).

Proof. The proof consists of several steps.

Step 1. (The Faedo-Galerkin approximation: introduced by Lions [6]). Consider the basis $\{w_j\}$ for L^2 as in Lemma 2.4. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.11}$$

where the coefficients $c_{mj}^{(k)}$, $j = 1, \cdots, k$ satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j \rangle + \lambda a(\dot{u}_m^{(k)}(t), w_j) + A_m(t; u_m^{(k)}(t), w_j) = \langle f(t), w_j \rangle, \ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.12}$$

where

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H^2 \cap V, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap V. \end{cases} \tag{3.13}$$

The system (3.12) is written as follows

$$\begin{aligned} c_{mj}^{(k)}(t) = \quad & \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\lambda\lambda_j}\left(1 - e^{-\lambda\lambda_j t}\right) + \int_0^t e^{-\lambda\lambda_j r} dr \int_0^r e^{\lambda\lambda_j s} f_j(s) ds \\ & - \int_0^t e^{-\lambda\lambda_j r} dr \int_0^r e^{\lambda\lambda_j s} \sum_{i=1}^k a_{ij}^{(m)}(s) c_{mi}^{(k)}(s) ds, \ 1 \le j \le k, \end{aligned} \tag{3.14}$$

where

$$a_{ij}^{(m)}(t) = A_m(t; w_i, w_j), \ f_j(t) = \langle f(t), w_j \rangle, \ 1 \le i, j \le k.$$

Omitting the indices m and k, the system (3.14) is rewritten in the form of a fixed-point equation as follows

$$c(t) = U[c](t), \tag{3.15}$$

where

$$\begin{aligned} & c(t) = (c_1(t), \cdots, c_k(t)), \\ & U[c](t) = (U_1[c](t), \cdots, U_k[c](t)) = G(t) + L[c](t), \\ & G(t) = (G_1(t), \cdots, G_k(t)), \\ & L[c](t) = (L_1[c](t), \cdots, L_k[c](t)), \\ & G_j(t) = \alpha_j^{(k)} + \frac{\beta_j^{(k)}}{\lambda\lambda_j}\left(1 - e^{-\lambda\lambda_j t}\right) + \int_0^t e^{-\lambda\lambda_j r} dr \int_0^r e^{\lambda\lambda_j s} f_j(s) ds, \\ & L_j[c](t) = -\int_0^t e^{-\lambda\lambda_j r} dr \int_0^r e^{\lambda\lambda_j s} \sum_{i=1}^k a_{ij}^{(m)}(s) c_i(s) ds, \ 1 \le j \le k. \end{aligned}$$

Applying the contraction principle, system (3.15) has a unique solution $c(t)$ in $[0, T]$. The proof is given below.

Let $\gamma > \sqrt{A_{\max}}$, where we denote $A_{\max} \equiv \sup\limits_{0 \le t \le T} \left(\max\limits_{1 \le i \le k} \sum\limits_{j=1}^k \left| a_{ij}^{(m)}(t) \right| \right)$.

It is well known that $X = C^0\left([0, T]; \mathbb{R}^k\right)$ is a Banach space with respect to the norm

$$\|c\|_{\gamma, X} = \sup_{0 \le t \le T} e^{-\gamma t} |c(t)|_1, \ \ |c(t)|_1 = \sum\nolimits_{j=1}^k |c_j(t)|, \ c \in X.$$

Then, clearly, $U : X \to X$. Further, U is contractive. Indeed, first we note that, for all $c = (c_1, \cdots, c_k)$, $d = (d_1, \cdots, d_k) \in X$, $z = c - d$,

$$
\begin{aligned}
|U[c](t)-U[d](t)|_1 &= |L[z](t)|_1 \\
&= \sum\nolimits_{j=1}^{k} |L_j[z](t)| \\
&\leq \sum\nolimits_{j=1}^{k} \int_0^t e^{-\lambda\lambda_j r} dr \int_0^r e^{\lambda\lambda_j s} \sum_{i=1}^{k} \left| a_{ij}^{(m)}(s) z_i(s) \right| ds \\
&\leq \int_0^t dr \int_0^r \max_{1\leq i\leq k} \sum\nolimits_{j=1}^{k} \left| a_{ij}^{(m)}(s) \right| |z(s)|_1 \, ds \\
&\leq \sup_{0\leq s\leq T} \left(\max_{1\leq i\leq k} \sum\nolimits_{j=1}^{k} \left| a_{ij}^{(m)}(s) \right| \right) \int_0^t dr \int_0^r |z(s)|_1 \, ds \\
&\leq A_{\max} \|z\|_{\gamma,X} \int_0^t dr \int_0^r e^{\gamma s} ds \\
&\leq A_{\max} \frac{e^{\gamma t}}{\gamma^2} \|c-d\|_{\gamma,X} .
\end{aligned}
$$

It follows that

$$
e^{-\gamma t} |U[c](t)-U[d](t)|_1 \leq \frac{A_{\max}}{\gamma^2} \|c-d\|_{\gamma,X} ,
$$

it leads to

$$
\|U[c]-U[d]\|_{\gamma,X} \leq \frac{A_{\max}}{\gamma^2} \|c-d\|_{\gamma,X} , \ \forall c, \ d \in X.
$$

Since, $0 < \frac{A_{\max}}{\gamma^2} < 1$, $U : X \to X$ is contractive. Then, (3.15) has a unique solution $c \in X$. Thus, system (3.12) has a unique solution $u_m^{(k)}(t)$ in $[0,T]$.

Step 2. (A priori estimates). Put

$$
\begin{aligned}
S_m^{(k)}(t) = {} & \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \lambda \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 \\
& + \bar{\Phi}_m(t) \left[a_\mu(t; u_m^{(k)}(t), u_m^{(k)}(t)) + \left\| \sqrt{\mu(t)} \Delta u_m^{(k)}(t) \right\|^2 \right] \\
& + 2 \int_0^t \left[\lambda \left(\left\| \dot{u}_m^{(k)}(s) \right\|_a^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 \right) + \left\| \ddot{u}_m^{(k)}(s) \right\|_a^2 \right] ds,
\end{aligned} \tag{3.16}
$$

then we deduce from (3.12) that

$$
\begin{aligned}
\bar{\mu}_* \bar{S}_m^{(k)}(t) \leq S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\bar{\Phi}_m(0) \left\langle \frac{\partial}{\partial x} \left(\mu(0) \tilde{u}_{0kx}\right), \Delta \tilde{u}_{1k} \right\rangle \\
&+ 2 \left\langle f(0), \Delta \tilde{u}_{1k} \right\rangle \\
&+ \int_0^t \Big[\bar{\Phi}'_m(s) a_\mu(s; u_m^{(k)}(s), u_m^{(k)}(s)) + \bar{\Phi}_m(s) a_{\mu'}(s; u_m^{(k)}(s), u_m^{(k)}(s)) \Big] ds \\
&+ \int_0^t ds \int_0^1 \left[\frac{\partial}{\partial s} \left(\bar{\Phi}_m(s) \mu(x,s) \right) \right] \left| \Delta u_m^{(k)}(x,s) \right|^2 dx \\
&- 2 \int_0^t \bar{\Phi}_m(s) \langle \mu_x(s) u_{mx}^{(k)}(s), \Delta \dot{u}_m^{(k)}(s) \rangle ds \\
&+ 2 \int_0^t \bar{\Phi}'_m(s) \left\langle \frac{\partial}{\partial x} \left(\mu(s) u_{mx}^{(k)}(s) \right), \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\
&+ 2 \int_0^t \bar{\Phi}_m(s) \left\langle \frac{\partial^2}{\partial x \partial s} \left(\mu(s) u_{mx}^{(k)}(s) \right), \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\
&- 2 \bar{\Phi}_m(t) \left\langle \frac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t) \right), \Delta \dot{u}_m^{(k)}(t) \right\rangle \\
&+ 2 \int_0^t \left\langle f(s), \dot{u}_m^{(k)}(s) - \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\
&+ 2 \int_0^t \left\langle f'(s), \Delta \dot{u}_m^{(k)}(s) \right\rangle ds - 2 \left\langle f(t), \Delta \dot{u}_m^{(k)}(t) \right\rangle,
\end{aligned} \tag{3.17}
$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda, \}$ and

$$
\begin{aligned}
\bar{S}_m^{(k)}(t) &= \left\| u_m^{(k)}(t) \right\|_{V \cap H^2}^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_{V \cap H^2}^2 \\
&\quad + \int_0^t \left[\left\| \dot{u}_m^{(k)}(s) \right\|_{V \cap H^2}^2 + \left\| \ddot{u}_m^{(k)}(s) \right\|_a^2 \right] ds,
\end{aligned} \tag{3.18}
$$

$$
\|v\|_{V \cap H^2} = \sqrt{\|v\|_a^2 + \|\Delta v\|^2}, \ v \in V \cap H^2.
$$

so we omit the details. By using Lemma 3.1, we estimate the terms on the right-hand side of (3.17) as follows. We first have

$$
\begin{aligned}
I_1 &= \int_0^t \bar{\Phi}'_m(s) a_\mu(s; u_m^{(k)}(s), u_m^{(k)}(s)) ds + \int_0^t \bar{\Phi}_m(s) a_{\mu'}(s; u_m^{(k)}(s), u_m^{(k)}(s)) ds \qquad (3.19) \\
&\leq \int_0^t \left(\left| \bar{\Phi}'_m(s) \right| + \left| \bar{\Phi}_m(s) \right| \right) \hat{K}_\mu \left\| u_m^{(k)}(s) \right\|_a^2 ds \\
&\leq 2 \left(1 + 3M^2 \right) \tilde{K}_M \hat{K}_\mu \int_0^t \bar{S}_m^{(k)}(s) ds;
\end{aligned}
$$

$$\begin{aligned}
I_2 &= \int_0^t ds \int_0^1 \left[\frac{\partial}{\partial s} \left(\bar{\Phi}_m(s) \mu(x,s) \right) \right] \left| \Delta u_m^{(k)}(x,s) \right|^2 dx \\
&= \int_0^t ds \int_0^1 \left[\bar{\Phi}_m'(s) \mu(x,s) + \bar{\Phi}_m(s) \mu'(x,s) \right] \left| \Delta u_m^{(k)}(x,s) \right|^2 dx \\
&\leq 2 \left(1 + 3M^2 \right) \tilde{K}_M \hat{K}_\mu \int_0^t \bar{S}_m^{(k)}(s) ds;
\end{aligned}$$

$$\begin{aligned}
I_3 &= -2 \int_0^t \bar{\Phi}_m(s) \langle \mu_x(s) u_{mx}^{(k)}(s), \Delta \dot{u}_m^{(k)}(s) \rangle ds \\
&\leq 2 \tilde{K}_M \hat{K}_\mu \int_0^t \left\| u_{mx}^{(k)}(s) \right\| \left\| \Delta \dot{u}_m^{(k)}(s) \right\| ds \\
&\leq 2 \tilde{K}_M \hat{K}_\mu \int_0^t \bar{S}_m^{(k)}(s) ds;
\end{aligned}$$

$$\begin{aligned}
I_4 &= 2 \int_0^t \bar{\Phi}_m'(s) \left\langle \frac{\partial}{\partial x} \left(\mu(s) u_{mx}^{(k)}(s) \right), \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\
&\leq 2 \int_0^t \left| \bar{\Phi}_m'(s) \right| \left\| \frac{\partial}{\partial x} \left(\mu(s) u_{mx}^{(k)}(s) \right) \right\| \left\| \Delta \dot{u}_m^{(k)}(s) \right\| ds \\
&\leq 2\sqrt{2} \left(1 + 6M^2 \right) \tilde{K}_M \hat{K}_\mu \int_0^t \bar{S}_m^{(k)}(s) ds;
\end{aligned}$$

$$\begin{aligned}
I_5 &= 2 \int_0^t \bar{\Phi}_m(s) \left\langle \frac{\partial^2}{\partial x \partial s} \left(\mu(s) u_{mx}^{(k)}(s) \right), \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\
&\leq 2 \int_0^t \bar{\Phi}_m(s) \left\| \frac{\partial^2}{\partial x \partial s} \left(\mu(s) u_{mx}^{(k)}(s) \right) \right\| \left\| \Delta \dot{u}_m^{(k)}(s) \right\| ds \\
&\leq 4 \tilde{K}_M \hat{K}_\mu \int_0^t \bar{S}_m^{(k)}(s) ds.
\end{aligned}$$

Next, we get

$$\begin{aligned}
I_6 &= -2 \bar{\Phi}_m(t) \left\langle \frac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t) \right), \Delta \dot{u}_m^{(k)}(t) \right\rangle \\
&\leq \frac{\bar{\mu}_*}{4} \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 + \frac{4}{\bar{\mu}_*} \left\| \bar{\Phi}_m(t) \frac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t) \right) \right\|^2 \\
&\leq \frac{\bar{\mu}_*}{4} \bar{S}_m^{(k)}(t) + \frac{4}{\bar{\mu}_*} \left\| \bar{\Phi}_m(t) \frac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t) \right) \right\|^2.
\end{aligned}$$

Note that

$$\begin{aligned}\left\|\bar{\Phi}_m(t)\frac{\partial}{\partial x}\left(\mu(t)u_{mx}^{(k)}(t)\right)\right\|^2 &\leq \Bigg(\left\|\bar{\Phi}_m(0)\frac{\partial}{\partial x}\left(\mu(0)u_{mx}^{(k)}(0)\right)\right\| \\ &\quad + \int_0^t\left\|\frac{\partial}{\partial s}\left[\bar{\Phi}_m(s)\frac{\partial}{\partial x}\left(\mu(s)u_{mx}^{(k)}(s)\right)\right]\right\|ds\Bigg)^2 \\ &\leq 2\left\|\bar{\Phi}_m(0)\frac{\partial}{\partial x}\left(\mu(0)\tilde{u}_{0kx}\right)\right\|^2 \\ &\quad + \left(\int_0^t\left\|\frac{\partial}{\partial s}\left[\bar{\Phi}_m(s)\frac{\partial}{\partial x}\left(\mu(s)u_{mx}^{(k)}(s)\right)\right]\right\|ds\right)^2 \\ &\leq 2\left\|\bar{\Phi}_m(0)\frac{\partial}{\partial x}\left(\mu(0)\tilde{u}_{0kx}\right)\right\|^2 \\ &\quad + 2(2+\sqrt{2})^2T^*\tilde{K}_M^2\hat{K}_\mu^2\int_0^t\bar{S}_m^{(k)}(s)ds,\end{aligned}$$

we deduce that

$$\begin{aligned}I_6 &= -2\left\langle\bar{\Phi}_m(t)\frac{\partial}{\partial x}\left(\mu(t)u_{mx}^{(k)}(t)\right),\Delta\dot{u}_m^{(k)}(t)\right\rangle \qquad (3.20)\\ &\leq \frac{\bar{\mu}_*}{4}\bar{S}_m^{(k)}(t)+\frac{4}{\bar{\mu}_*}\left\|\bar{\Phi}_m(t)\frac{\partial}{\partial x}\left(\mu(t)u_{mx}^{(k)}(t)\right)\right\|^2 \\ &\leq \frac{\bar{\mu}_*}{4}\bar{S}_m^{(k)}(t)+\frac{8}{\bar{\mu}_*}\bar{\Phi}_m^2(0)\left\|\frac{\partial}{\partial x}\left(\mu(0)\tilde{u}_{0kx}\right)\right\|^2 \\ &\quad + \frac{8}{\bar{\mu}_*}(2+\sqrt{2})^2T^*\tilde{K}_M^2\hat{K}_\mu^2\int_0^t\bar{S}_m^{(k)}(s)ds.\end{aligned}$$

We also have

$$\begin{aligned}I_7 &= 2\int_0^t\left\langle f(s),\dot{u}_m^{(k)}(s)-\Delta\dot{u}_m^{(k)}(s)\right\rangle ds \qquad (3.21)\\ &\leq 2\int_0^t\|f(s)\|\left(\left\|\dot{u}_m^{(k)}(s)\right\|+\left\|\Delta\dot{u}_m^{(k)}(s)\right\|\right)ds \\ &\leq 2\sqrt{2}\,\|f\|_{L^\infty(0,T^*;L^2)}\int_0^t\sqrt{\bar{S}_m^{(k)}(s)}ds \\ &\leq 2T\,\|f\|_{L^\infty(0,T^*;L^2)}^2+\int_0^t\bar{S}_m^{(k)}(s)ds;\end{aligned}$$

$$
\begin{aligned}
I_8 &= 2\int_0^t \langle f'(s), \Delta \dot{u}_m^{(k)}(s)\rangle ds \\
&\leq \int_0^t \left\| f'(s) \right\| ds + \int_0^t \left\| f'(s) \right\| \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 ds \\
&\leq \left\| f' \right\|_{L^1(0,T^*;L^2)} + \int_0^t \left\| f'(s) \right\| \bar{S}_m^{(k)}(s) ds; \\
I_9 &= -2\left\langle f(t), \Delta \dot{u}_m^{(k)}(t) \right\rangle \leq \frac{\bar{\mu}_*}{4} \bar{S}_m^{(k)}(t) + \frac{4}{\bar{\mu}_*} \left\| f(t) \right\|^2 \\
&\leq \frac{\bar{\mu}_*}{4} \bar{S}_m^{(k)}(t) + \frac{4}{\bar{\mu}_*} \left\| f \right\|^2_{L^\infty(0,T^*;L^2)}.
\end{aligned}
$$

It follows from (3.17), (3.19)-(3.21) that

$$
\begin{aligned}
\bar{S}_m^{(k)}(t) \leq \bar{S}_{0m}^{(k)} &+ \frac{4T}{\bar{\mu}_*} \left\| f \right\|^2_{L^\infty(0,T^*;L^2)} \\
&+ \frac{2}{\bar{\mu}_*} \int_0^t \left(\bar{D}_1(M) + \left\| f'(s) \right\| \right) \bar{S}_m^{(k)}(s) ds
\end{aligned} \tag{3.22}
$$

with

$$
\begin{aligned}
\bar{S}_{0m}^{(k)} &= \frac{2}{\bar{\mu}_*} \left[S_m^{(k)}(0) + 2\left\langle f(0) + \bar{\Phi}_m(0) \frac{\partial}{\partial x} \left(\mu(0) \tilde{u}_{0kx} \right), \Delta \tilde{u}_{1k} \right\rangle \right] \\
&\quad + \frac{16}{\bar{\mu}_*^2} \bar{\Phi}_m^2(0) \left\| \frac{\partial}{\partial x} \left(\mu(0) \tilde{u}_{0kx} \right) \right\|^2 + \frac{2}{\bar{\mu}_*} \left\| f' \right\|_{L^1(0,T^*;L^2)} \\
&\quad + \frac{8}{\bar{\mu}_*^2} \left\| f \right\|^2_{L^\infty(0,T^*;L^2)}, \\
\bar{D}_1(M) &= 1 + 2\left(5 + \sqrt{2} + 6(1+\sqrt{2}) M^2 \right) \tilde{K}_M \hat{K}_\mu \\
&\quad + \frac{16}{\bar{\mu}_*} (1+\sqrt{2})^2 T^* \left(\tilde{K}_M \hat{K}_\mu \right)^2.
\end{aligned} \tag{3.23}
$$

By (3.13) and (3.16), the first formula in (3.23) leads to

$$
\bar{S}_{0m}^{(k)} \leq \frac{1}{2} M^2 \quad \text{for all} \quad m,\ k \in \mathbb{N}, \tag{3.24}
$$

where M is a constant depending only on μ, f, Φ, $\tilde{u}_0$, $\tilde{u}_1$, λ, ζ.

We choose $T \in (0, T^*]$ such that

$$
\left(\frac{1}{2} M^2 + \frac{4T}{\bar{\mu}_*} \left\| f \right\|^2_{L^\infty(0,T^*;L^2)} \right) \exp\left(\frac{2}{\bar{\mu}_*} \left[T\bar{D}_1(M) + \int_0^T \left\| f'(s) \right\| ds \right] \right) \leq M^2 \tag{3.25}
$$

and

$$k_T = 72\sqrt{2}\frac{M^2\tilde{K}_M\hat{K}_\mu}{\bar{\mu}_*}\sqrt{T}\exp\left(T\tilde{D}_*(M)\right) < 1, \tag{3.26}$$

in which

$$\tilde{D}_*(M) = \frac{4}{\bar{\mu}_*}\left(1 + 3M^2 + \frac{1}{\bar{\mu}_*}\tilde{K}_M\hat{K}_\mu\right)\tilde{K}_M\hat{K}_\mu. \tag{3.27}$$

By using Gronwall's Lemma, we deduce from (3.22), (3.24) and (3.25) that

$$\begin{aligned}\bar{S}_m^{(k)}(t) &\le M^2\exp\left(\frac{-2}{\bar{\mu}_*}\left[T\bar{D}_1(M) + \int_0^T \left\|f'(s)\right\| ds\right]\right)\\ &\qquad\times\exp\left[\frac{2}{\bar{\mu}_*}\int_0^t \left(\bar{D}_2(M) + \left\|f'(s)\right\|\right) ds\right]\\ &\le M^2\end{aligned} \tag{3.28}$$

for all $t \in [0,T]$, for all m and $k \in \mathbb{N}$. Therefore, we have

$$u_m^{(k)} \in W(M,T) \quad \text{for all} \quad m \text{ and } k \in \mathbb{N}. \tag{3.29}$$

Step 3. (Limit Procedure). From (3.29), there exists a subsequence of the sequence of $\{u_m^{(k)}\}$ with the same notation such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*},\\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*},\\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(0,T;V) \text{ weak},\\ u_m \in W(M,T). \end{cases} \tag{3.30}$$

Passing to limit in (3.12), we have u_m satisfying (3.9), (3.10) in $L^2(0,T)$ weak.

Furthermore, $(3.9)_1$ and $(3.30)_4$ imply that

$$u_m'' = \lambda\Delta u_m' + \bar{\Phi}_m(t)\frac{\partial}{\partial x}\left(\mu(t)u_{mx}\right) + f \in L^\infty(0,T;L^2),$$

so we obtain $u_m \in W_1(M,T)$, Theorem 3.2 is proved. □

Now, in order to be able to obtain a priori estimate for the sequence $\{w_m\}$ in the deeper function space and also to prove the uniqueness of Prob. (P_n), we need to establish the energy lemma as follows, which is a relative generalization of the inequality and equality of energy given in Lions's book [6].

Lemma 3.3. *Let $u \in \tilde{V}_T$ be the weak solution of the following problem*

$$\begin{cases} u'' - \lambda u'_{xx} - \Phi(t)\dfrac{\partial}{\partial x}\left(\mu(x,t)u_x(t)\right) = F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ \ u'(x,0) = \tilde{u}_1(x), \\ \tilde{u}_0,\ \tilde{u}_1 \in V \cap H^2,\ \ \tilde{u}_{0x}(0) - \zeta\tilde{u}_0(0) = 0, \\ F \in L^2(0,T;V),\ \mu \in C^1([0,1]\times[0,T]),\ \mu(x,t) \geq \mu_* > 0, \\ \Phi \in C([0,T]). \end{cases} \tag{3.31}$$

Then we have

$$\begin{aligned} &\frac{1}{2}\left\|u'(t)\right\|_a^2 + \frac{1}{2}\left\|\sqrt{\Phi(t)\mu(t)}\Delta u(t)\right\|^2 + \lambda\int_0^t \left\|\Delta u'(s)\right\|^2 ds \\ &\quad + \int_0^t \Phi(s)\langle\mu_x(s)u_x(s), \Delta u'(s)\rangle ds - \frac{1}{2}\int_0^t ds\int_0^1 \left(\Phi(s)\mu(x,s)\right)'\left|\Delta u(x,s)\right|^2 dx \\ &\geq \frac{1}{2}\left\|\tilde{u}_1\right\|_a^2 + \frac{1}{2}\left\|\sqrt{\Phi(0)\mu(0)}\Delta\tilde{u}_0\right\|^2 + \int_0^t \langle F(s), -\Delta u'(s)\rangle ds,\ \ a.e.,\ t \in (0,T). \end{aligned} \tag{3.32}$$

Furthermore, if $\tilde{u}_0 = \tilde{u}_1 = 0$ then there is an equality in (3.32).

Proof. This proof is similar to the argument of Lions to obtain the inequality and equality of energy in Lions's book [6, Lemma 1.6, p. 224]. The details are follows.

Fix t_1, t_2, $0 < t_1 < t_2 < T$ and let $w_{km}(x,t)$ be the function defined by

$$w_{km}(x,t) = \left[\left(\theta_m(t)\Delta u'(x,t)\right) * \rho_k(t) * \rho_k(t)\right]\theta_m(t), \tag{3.33}$$

where

(i) θ_m is a continuous, piecewise linear function on $[0,T]$ defined as follows

$$\theta_m(t) = \begin{cases} 0, & t \notin [t_1 + 1/m,\ t_2 - 1/m], \\ 1, & t \in [t_1 + 2/m,\ t_2 - 2/m], \\ m(t - t_1 - 1/m), & t \in [t_1 + 1/m,\ t_1 + 2/m], \\ -m(t - t_2 + 1/m), & t \in [t_2 - 2/m,\ t_2 - 1/m]; \end{cases} \tag{3.34}$$

(ii) $\{\rho_k\}$ is a regularizing sequence in $C_c^\infty(\mathbb{R})$, that is,

$$\rho_k \in C_c^\infty(\mathbb{R}), \mathrm{supp}\rho_k \subset [-1/k, 1/k],\ \ \rho_k(-t) = \rho_k(t), \int_{-\infty}^{\infty}\rho_k(t)dt = 1; \tag{3.35}$$

(iii) $(*)$ is the convolution product in the time variable, that is,

$$(u * \rho_k)(x,t) = \int_{-\infty}^{\infty} u(x,t-s)\rho_k(s)ds. \tag{3.36}$$

Taking the scalar product of the function $w_{km}(x,t)$ in (3.33) with the first equation in (3.31), then integrating with respect to the time variable from 0 to T, we have

$$A_{km} + B_{km} + C_{km} + D_{km} = E_{km}, \tag{3.37}$$

where

$$\begin{aligned} A_{km} &= \int_0^T \langle u''(t), w_{km}(t)\rangle dt, \quad B_{km} = \lambda \int_0^T a(u'(t), w_{km}(t))dt, \\ C_{km} &= \int_0^T \Phi(t) a_\mu(t; u(t), w_{km}(t))dt, \quad D_{km} = \int_0^T \langle F(t), w_{km}(t)\rangle dt. \end{aligned} \tag{3.38}$$

By using the properties of the functions $\theta_m(t)$ and $\rho_k(t)$, we can show after some lengthy calculations

$$\begin{aligned} \lim_{k\to\infty} A_{km} &= \int_0^T \theta_m(t)\theta_m'(t) \left\|u'(t)\right\|_a^2 dt, \\ \lim_{k\to\infty} B_{km} &= -\lambda \int_0^T \theta_m^2(t) \left\|\Delta u'(t)\right\|^2 dt, \\ \lim_{k\to\infty} C_{km} &= \int_0^T \theta_m(t)\theta_m'(t) \left\|\sqrt{\Phi(t)\mu(t)}\Delta u(t)\right\|^2 dt \\ &\quad - \int_0^T \theta_m^2(t)\Phi(t)\langle \mu_x(t) u_x(t), \Delta u'(t)\rangle dt \\ &\quad + \frac{1}{2}\int_0^T \theta_m^2(t) dt \int_0^1 (\Phi(t)\mu(x,t))' \left|\Delta u(x,t)\right|^2 dx, \\ \lim_{k\to\infty} D_{km} &= \int_0^T \theta_m^2(t)\langle F(t), \Delta u'(t)\rangle dt. \end{aligned} \tag{3.39}$$

Letting $m \to \infty$, from (3.37)-(3.39) we obtain

$$\begin{aligned} &\frac{1}{2}\left\|u'(t_1)\right\|_a^2 - \frac{1}{2}\left\|u'(t_2)\right\|_a^2 - \lambda\int_{t_1}^{t_2} \left\|\Delta u'(t)\right\|^2 dt \\ &\quad + \frac{1}{2}\left\|\sqrt{\Phi(t_1)\mu(t_1)}\Delta u(t_1)\right\|^2 - \frac{1}{2}\left\|\sqrt{\Phi(t_2)\mu(t_2)}\Delta u(t_2)\right\|^2 \\ &\quad - \int_{t_1}^{t_2} \Phi(t)\langle \mu_x(t) u_x(t), \Delta u'(t)\rangle dt + \frac{1}{2}\int_{t_1}^{t_2} dt \int_0^1 (\Phi(t)\mu(x,t))' \left|\Delta u(x,t)\right|^2 dx \\ &= \int_{t_1}^{t_2} \langle F(t), \Delta u'(t)\rangle dt, \ \text{a.e.}, \ t_1, \ t_2 \in (0,T), \ t_1 < t_2 < T, \end{aligned}$$

or

$$\frac{1}{2}\left\|u'(t_2)\right\|_a^2 + \frac{1}{2}\left\|\sqrt{\Phi(t_2)\mu(t_2)}\Delta u(t_2)\right\|^2 + \lambda\int_0^{t_2}\left\|\Delta u'(t)\right\|^2 dt \tag{3.40}$$
$$+ \int_0^{t_2}\Phi(t)\langle\mu_x(t)u_x(t), \Delta u'(t)\rangle dt$$
$$- \frac{1}{2}\int_0^{t_2} dt\int_0^1 (\Phi(t)\mu(x,t))'\left|\Delta u(x,t)\right|^2 dx + \int_0^{t_2}\langle F(t), \Delta u'(t)\rangle dt$$
$$= \frac{1}{2}\left\|u'(t_1)\right\|_a^2 + \frac{1}{2}\left\|\sqrt{\Phi(t_1)\mu(t_1)}\Delta u(t_1)\right\|^2 + \lambda\int_0^{t_1}\left\|\Delta u'(t)\right\|^2 dt$$
$$+ \int_0^{t_1}\Phi(t)\langle\mu_x(t)u_x(t), \Delta u'(t)\rangle dt$$
$$- \frac{1}{2}\int_0^{t_1} dt\int_0^1 (\Phi(t)\mu(x,t))'\left|\Delta u(x,t)\right|^2 dx + \int_0^{t_1}\langle F(t), \Delta u'(t)\rangle dt,$$

a.e., t_1, $t_2 \in (0,T)$, $t_1 < t_2 < T$.

From (3.40) we obtain (3.32), by taking $t_2 = t$ and passing to the limit as $t_1 \to 0_+$, and using the property of weak lower semicontinuity of the functional $v \longmapsto \|v\|^2$.

In the case of $\tilde{u}_0 = \tilde{u}_1 = 0$, we prolong u, F by 0 and $(\Phi(t),\ \mu(x,t))$ by $(\Phi(0),\ \mu(x,0))$, respectively as $t < 0$ and we deduce that equality (3.40) is true for almost $t_1 < t_2 < T$. Taking $t_1 < 0$ in (3.40), its right-hand side is 0, we take $t_1 \to 0_-$ and we have equality (3.32) when $\tilde{u}_0 = \tilde{u}_1 = 0$. The proof of Lemma 3.3 is completed. □

We will use the result of Theorem 3.2 and the compact imbedding theorems to prove the existence and uniqueness of a weak solution of Prob. (P_n). The following theorem is the main result in this section.

Theorem 3.4. *Let $(H_1) - (H_4)$ hold. Then, there exist positive constants M, T such that*

1. *Prob. (P_n) has a unique weak solution $u \in W_1(M,T)$.*
2. *The recurrent sequence $\{u_m\}$ defined by (3.8)-(3.10) converges to the solution u of Prob. (P_n) strongly in the Banach space H_T.*

Furthermore, we have the estimation

$$\|u_m - u\|_{H_T} \le C_T k_T^m \ \text{ for all } \ m \in \mathbb{N}, \tag{3.41}$$

where $k_T \in [0,1)$ and C_T are the constants depending only on T, μ, f, Φ, $\tilde{u}_0$, $\tilde{u}_1$, λ, ζ.

Proof. (a) **Existence of the solution:** We shall prove that $\{u_m\}$ is a Cauchy sequence in H_T. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \lambda a(w_m'(t), w) + \bar{\Phi}_{m+1}(t) a_\mu(t; w_m(t), w) \\ \qquad\qquad = \left[\bar{\Phi}_{m+1}(t) - \bar{\Phi}_m(t)\right] \langle \dfrac{\partial}{\partial x}\left(\mu(t) u_{mx}(t)\right), w\rangle, \ \forall w \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.42}$$

Taking $w = w_m'$ in $(3.42)_1$ and integrating in t, we get

$$\begin{aligned} X_m(t) = &\int_0^t \bar{\Phi}_{m+1}'(s) a_\mu(s; w_m(s), w_m(s)) ds \\ &+ \int_0^t \bar{\Phi}_{m+1}(s) a_{\mu'}(s; w_m(s), w_m(s)) ds \\ &+ 2\int_0^t \left[\bar{\Phi}_{m+1}(s) - \bar{\Phi}_m(s)\right] \langle \frac{\partial}{\partial x}\left(\mu(s) u_{mx}(s)\right), w_m'(s)\rangle ds, \end{aligned} \tag{3.43}$$

where

$$X_m(t) = \left\|w_m'(t)\right\|^2 + \bar{\Phi}_{m+1}(t) a_\mu(t; w_m(t), w_m(t)) + 2\lambda \int_0^t \left\|w_m'(s)\right\|_a^2 ds. \tag{3.44}$$

We note more that $w_m = u_{m+1} - u_m \in \tilde{V}_T$ is the weak solution of the problem (3.31) corresponding to $\tilde{u}_0 = \tilde{u}_1 = 0$, $\Phi(t) = \bar{\Phi}_{m+1}(t)$, $F(t) = \left[\bar{\Phi}_{m+1}(t) - \bar{\Phi}_m(t)\right] \dfrac{\partial}{\partial x}\left(\mu(t) u_{mx}(t)\right)$. By using Lemma 3.3 with $\tilde{u}_0 = \tilde{u}_1 = 0$, we get

$$\begin{aligned} Y_m(t) = &-2\int_0^t \bar{\Phi}_{m+1}(s) \langle \mu_x(s) w_{mx}(s), \Delta w_m'(s)\rangle ds \\ &+ \int_0^t ds \int_0^1 \left(\bar{\Phi}_{m+1}(s)\mu(x,s)\right)' \left|\Delta w_m(x,s)\right|^2 dx \\ &+ 2\int_0^t \langle \left[\bar{\Phi}_{m+1}(s) - \bar{\Phi}_m(s)\right] \frac{\partial}{\partial x}\left(\mu(s) u_{mx}(s)\right), -\Delta w_m'(s)\rangle ds, \end{aligned} \tag{3.45}$$

where

$$Y_m(t) = \left\|w_m'(t)\right\|_a^2 + \left\|\sqrt{\bar{\Phi}_{m+1}(t)\mu(t)}\Delta w_m(t)\right\|^2 + 2\lambda \int_0^t \left\|\Delta w_m'(s)\right\|^2 ds. \tag{3.46}$$

Put

$$\begin{aligned} S_m(t) = X_m(t) + Y_m(t) = &\left\|w_m'(t)\right\|^2 + \left\|w_m'(t)\right\|_a^2 \\ &+ \bar{\Phi}_{m+1}(t)\left(a_\mu(t; w_m(t), w_m(t)) + \left\|\sqrt{\mu(t)}\Delta w_m(t)\right\|^2\right) \\ &+ 2\lambda \int_0^t \left(\left\|w_m'(s)\right\|_a^2 + \left\|\Delta w_m'(s)\right\|^2\right) ds, \end{aligned} \tag{3.47}$$

we deduce from (3.43), (3.45) that

$$
\begin{aligned}
\bar{\mu}_* \bar{S}_m(t) \leq S_m(t) \quad & (3.48) \\
= \int_0^t & \left[\bar{\Phi}'_{m+1}(s) a_\mu(s; w_m(s), w_m(s)) + \bar{\Phi}_{m+1}(s) a_{\mu'}(s; w_m(s), w_m(s)) \right] ds \\
& - 2 \int_0^t \bar{\Phi}_{m+1}(s) \langle \mu_x(s) w_{mx}(s), \Delta w'_m(s) \rangle ds \\
& + \int_0^t ds \int_0^1 \frac{\partial}{\partial s} \left[\bar{\Phi}_{m+1}(s) \mu(x,s) \right] |\Delta w_m(x,s)|^2 dx \\
& + 2 \int_0^t \left[\bar{\Phi}_{m+1}(s) - \bar{\Phi}_m(s) \right] \left[\langle \frac{\partial}{\partial x} \left(\mu(s) u_{mx}(s) \right), w'_m(s) - \Delta w'_m(s) \rangle \right] ds \\
= z_1 & + \cdots + z_4,
\end{aligned}
$$

where $z_1, \cdots, z_4$ are defined as below, $\bar{\mu}_* = \min\{1, \mu_*, \lambda, \}$ and

$$
\begin{aligned}
\bar{S}_m(t) &= \left\| w'_m(t) \right\|^2 + \left\| w'_m(t) \right\|_a^2 + \| w_m(t) \|_a^2 \\
&\quad + \| \Delta w_m(t) \|^2 + \int_0^t \left(\left\| w'_m(s) \right\|_a^2 + \left\| \Delta w'_m(s) \right\|^2 \right) ds \\
&= \left\| w'_m(t) \right\|^2 + \left\| w'_m(t) \right\|_a^2 + \| w_m(t) \|_{H^2 \cap V}^2 \\
&\quad + \int_0^t \left\| w'_m(s) \right\|_{H^2 \cap V}^2 ds.
\end{aligned} \tag{3.49}
$$

Estimating z_1 : By Lemma 3.1, (i), (ii), (iv), we deduce that

$$
\begin{aligned}
z_1 &= \int_0^t \Big[\bar{\Phi}'_{m+1}(s) a_\mu(s; w_m(s), w_m(s)) \\
&\quad + \bar{\Phi}_{m+1}(s) a_{\mu'}(s; w_m(s), w_m(s)) \Big] ds \\
&\leq \int_0^t \left(\left| \bar{\Phi}'_{m+1}(s) \right| + \left| \bar{\Phi}_{m+1}(s) \right| \right) \hat{K}_\mu \| w_m(s) \|_a ds \\
&\leq 2 \left(1 + 3M^2 \right) \tilde{K}_M \hat{K}_\mu \int_0^t \| w_m(s) \|_a^2 ds \\
&\leq 2 \left(1 + 3M^2 \right) \tilde{K}_M \hat{K}_\mu \int_0^t \bar{S}_m(s) ds.
\end{aligned} \tag{3.50}
$$

Estimating z_2 : By $\bar{S}_m(t) \geq \|w_{mx}(t)\|^2 + \int_0^t \|\Delta w_m'(s)\|^2 ds$, we obtain

$$\begin{aligned} z_2 &= -2\int_0^t \bar{\Phi}_{m+1}(s)\langle \mu_x(s) w_{mx}(s), \Delta w_m'(s)\rangle ds \\ &\leq 2\tilde{K}_M \hat{K}_\mu \int_0^t \|w_{mx}(s)\| \left\|\Delta w_m'(s)\right\| ds \\ &\leq \frac{\bar{\mu}_*}{4}\int_0^t \left\|\Delta w_m'(s)\right\|^2 ds + \frac{4}{\bar{\mu}_*}\tilde{K}_M^2 \hat{K}_\mu^2 \int_0^t \|w_{mx}(s)\|^2 ds \\ &\leq \frac{\bar{\mu}_*}{4}\bar{S}_m(t) + \frac{4}{\bar{\mu}_*}\tilde{K}_M^2 \hat{K}_\mu^2 \int_0^t \bar{S}_m(s) ds. \end{aligned} \tag{3.51}$$

Estimating z_3 : We have

$$\begin{aligned} \left|\frac{\partial}{\partial s}\left[\bar{\Phi}_{m+1}(s)\mu(x,s)\right]\right| &= \left|\bar{\Phi}_{m+1}'(s)\mu(x,s) + \bar{\Phi}_{m+1}(s)\mu'(x,s)\right| \\ &\leq \left(\left|\bar{\Phi}_{m+1}'(s)\right| + \left|\bar{\Phi}_{m+1}(s)\right|\right)\hat{K}_\mu \\ &\leq 2\left(1+3M^2\right)\tilde{K}_M \hat{K}_\mu. \end{aligned}$$

Therefore,

$$\begin{aligned} z_3 &= \int_0^t ds \int_0^1 \frac{\partial}{\partial s}\left[\bar{\Phi}_{m+1}(s)\mu(x,s)\right] |\Delta w_m(x,s)|^2 dx \\ &\leq 2\left(1+3M^2\right)\tilde{K}_M \hat{K}_\mu \int_0^t \|\Delta w_m(s)\|^2 ds \\ &\leq 2\left(1+3M^2\right)\tilde{K}_M \hat{K}_\mu \int_0^t \bar{S}_m(s) ds. \end{aligned} \tag{3.52}$$

Estimating z_4 : First, we estimate, $\left|\bar{\Phi}_{m+1}(s) - \bar{\Phi}_m(s)\right|$.

$$\begin{aligned} &\left|\bar{\Phi}_{m+1}(t) - \bar{\Phi}_m(t)\right| \\ &= \left|\Phi\left(t, (S_n u_m)(t), (\hat{S}_n u_m)(t)\right) - \Phi\left(t, (S_n u_{m-1})(t), (\hat{S}_n u_{m-1})(t)\right)\right| \\ &\leq \tilde{K}_M\left[|(S_n u_m)(t) - (S_n u_{m-1})(t)| + \left|(\hat{S}_n u_m)(t) - (\hat{S}_n u_{m-1})(t)\right|\right]. \end{aligned} \tag{3.53}$$

We estimate the terms $|(S_n u_m)(t) - (S_n u_{m-1})(t)|$ and $\left|(\hat{S}_n u_m)(t) - (\hat{S}_n u_{m-1})(t)\right|$ respectively, as follows

$$
\begin{aligned}
|(S_n u_m)(t) - (S_n u_{m-1})(t)| &\leq \frac{1}{n}\sum_{i=1}^{n}\left|u_m^2(\tfrac{i-1}{n},t) - u_{m-1}^2(\tfrac{i-1}{n},t)\right| \\
&\leq \frac{1}{n}2M\sum_{i=1}^{n}\left|u_m(\tfrac{i-1}{n},t) - u_{m-1}(\tfrac{i-1}{n},t)\right| \\
&\leq \frac{1}{n}2M\sum_{i=1}^{n}\left|w_{m-1}(\tfrac{i-1}{n},t)\right| \\
&\leq \frac{1}{n}2M\sum_{i=1}^{n}\left\|\nabla w_{m-1}(t)\right\| \\
&= 2M\left\|\nabla w_{m-1}(t)\right\|.
\end{aligned} \tag{3.54}
$$

By the inequality $\left|\nabla u_m(\tfrac{i-1}{n},t)\right| \leq \sqrt{2}\left\|u_m(t)\right\|_{H^2\cap V}$, we obtain

$$
\begin{aligned}
&\left|(\hat{S}_n u_m)(t) - (\hat{S}_n u_{m-1})(t)\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left|\left|\nabla u_m(\tfrac{i-1}{n},t)\right|^2 - \left|\nabla u_{m-1}(\tfrac{i-1}{n},t)\right|^2\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left|\nabla u_m(\tfrac{i-1}{n},t)\right| + \left|\nabla u_{m-1}(\tfrac{i-1}{n},t)\right|\left|\nabla w_{m-1}(\tfrac{i-1}{n},t)\right| \\
&\leq \frac{1}{n}4M\sum_{i=1}^{n}\left\|w_{m-1}(t)\right\|_{H^2\cap V} \\
&= 4M\left\|w_{m-1}(t)\right\|_{H^2\cap V}.
\end{aligned} \tag{3.55}
$$

Then, it follows from (3.53)-(3.55), that

$$
\begin{aligned}
\left|\bar{\Phi}_{m+1}(t) - \bar{\Phi}_m(t)\right| &\leq \tilde{K}_M\left[2M\left\|\nabla w_{m-1}(t)\right\| + 4M\left\|w_{m-1}(t)\right\|_{H^2\cap V}\right] \\
&\leq 6M\tilde{K}_M\left\|w_{m-1}(t)\right\|_{H^2\cap V} \\
&\leq 6M\tilde{K}_M\left\|w_{m-1}\right\|_{H_T}.
\end{aligned} \tag{3.56}
$$

Next, we estimate $\left|\langle\frac{\partial}{\partial x}\left(\mu(s)u_{mx}(s)\right), w_m'(s) - \Delta w_m'(s)\rangle\right|$. By the following inequality

$$
\begin{aligned}
\left\|\frac{\partial}{\partial x}\left(\mu(s)u_{mx}(s)\right)\right\| &= \left\|\mu_x(s)u_{mx}(s) + \mu(s)\Delta u_m(s)\right\| \\
&\leq \hat{K}_\mu\sqrt{2}\left\|u_m(t)\right\|_{H^2\cap V} \leq \sqrt{2}M\hat{K}_\mu,
\end{aligned}
$$

we deduce that

$$\begin{aligned} &\left|\langle \frac{\partial}{\partial x}\left(\mu(s)u_{mx}(s)\right), w_m'(s) - \Delta w_m'(s)\rangle\right| \\ &\quad \leq \left\|\frac{\partial}{\partial x}\left(\mu(s)u_{mx}(s)\right)\right\| \left\|w_m'(s) - \Delta w_m'(s)\right\| \\ &\quad \leq 2M\hat{K}_\mu \left\|w_m'(s)\right\|_{H^2\cap V}. \end{aligned} \tag{3.57}$$

From (3.56), (3.57), the term z_4 is estimated as follows

$$\begin{aligned} z_4 &= 2\int_0^t \left[\bar{\Phi}_{m+1}(s) - \bar{\Phi}_m(s)\right] \left[\langle \frac{\partial}{\partial x}\left(\mu(s)u_{mx}(s)\right), w_m'(s) - \Delta w_m'(s)\rangle\right] ds \\ &\leq 2\int_0^t \left|\bar{\Phi}_{m+1}(s) - \bar{\Phi}_m(s)\right| \left|\langle \frac{\partial}{\partial x}\left(\mu(s)u_{mx}(s)\right), w_m'(s) - \Delta w_m'(s)\rangle\right| ds \\ &\leq 24M^2\tilde{K}_M\hat{K}_\mu \left\|w_{m-1}\right\|_{H_T} \int_0^t \left\|w_m'(s)\right\|_{H^2\cap V} ds \\ &\leq \frac{\bar{\mu}_*}{4}\int_0^t \left\|w_m'(s)\right\|_{H^2\cap V}^2 ds + \frac{4}{\bar{\mu}_*}144TM^4\left(\tilde{K}_M\hat{K}_\mu\right)^2 \left\|w_{m-1}\right\|_{H_T}^2 \\ &\leq \frac{\bar{\mu}_*}{4}\bar{S}_m(t) + \frac{4}{\bar{\mu}_*}144TM^4\left(\tilde{K}_M\hat{K}_\mu\right)^2 \left\|w_{m-1}\right\|_{H_T}^2. \end{aligned} \tag{3.58}$$

It follows from (3.48), (3.50)-(3.52) and (3.58) that

$$\bar{S}_m(t) \leq \left(24\sqrt{2}\right)^2 \frac{M^4\left(\tilde{K}_M\hat{K}_\mu\right)^2}{\bar{\mu}_*^2} T\left\|w_{m-1}\right\|_{H_T}^2 + 2\tilde{D}_*(M)\int_0^t \bar{S}_m(s)ds, \tag{3.59}$$

where $\tilde{D}_*(M)$ is defined as in (3.27).

Using Gronwall's Lemma, we deduce from (3.59) that

$$\left\|w_m\right\|_{H_T} \leq k_T\left\|w_{m-1}\right\|_{H_T}, \quad \forall\, m \in \mathbb{N}, \tag{3.60}$$

where $k_T \in (0,1)$ is defined as in (3.26), it leads to

$$\left\|u_m - u_{m+p}\right\|_{H_T} \leq \left\|u_0 - u_1\right\|_{H_T}(1-k_T)^{-1}k_T^m, \ \forall\, m,\ p \in \mathbb{N}. \tag{3.61}$$

It follows that $\{u_m\}$ is a Cauchy sequence in H_T. Then, there exists $u \in H_T$ such that

$$u_m \to u \text{ strongly in } H_T. \tag{3.62}$$

Because $u_m \in W(M,T)$, there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*}, \\ u_{m_j}' \to u' & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*}, \\ u_{m_j}'' \to u'' & \text{in} \quad L^2(0,T;V) \text{ weak}, \\ u \in W(M,T). \end{cases} \tag{3.63}$$

We note here that

$$\left|\bar{\Phi}_m(t) - \Phi\left(t, (S_n u)(t), (\hat{S}_n u)(t)\right)\right| \leq 6M\tilde{K}_M \left\|u_{m-1} - u\right\|_{H_T}. \tag{3.64}$$

Combining (3.62) and (3.64), we obtain

$$\bar{\Phi}_m \to \Phi\left(\cdot, (S_n u)(\cdot), (\hat{S}_n u)(\cdot)\right) \text{ strongly in } L^\infty(0,T). \tag{3.65}$$

Finally, passing to limit in (3.9), (3.10) as $m = m_j \to \infty$, it implies from (3.62), (3.63) and (3.64) that there exists $u \in W(M,T)$ satisfying (2.8)-(2.9). Furthermore, (2.8) and $(3.63)_4$ imply that

$$u'' = \lambda \Delta u' + \Phi\left(t, (S_n u)(t), (\hat{S}_n u)(t)\right) \frac{\partial}{\partial x}\left(\mu(x,t) u_x(t)\right) + f \in L^\infty(0,T;L^2),$$

so we obtain $u \in W_1(M,T)$. The existence is proved.

(b) **Uniqueness of the solution:** Let u_1, $u_2 \in W_1(M,T)$ be two weak solution of Prob. (P_n). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t), w\rangle + \lambda a(u'(t), w) + \tilde{\Phi}_1(t)\, a_\mu(t; u(t), w) \\ \qquad\qquad = -\left[\tilde{\Phi}_1(t) - \tilde{\Phi}_2(t)\right] a_\mu(t; u_2(t), w),\ \forall w \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.66}$$

where

$$\tilde{\Phi}_i(t) = \Phi\left(t, (S_n u_i)(t), (\hat{S}_n u_i)(t)\right),\ i = 1, 2. \tag{3.67}$$

Taking $w = u'$ in $(3.66)_1$ and integrating in t, we get

$$\begin{aligned} X(t) = &\int_0^t \tilde{\Phi}_1'(s) a_\mu(s; u(s), u(s)) ds + \int_0^t \tilde{\Phi}_1(s) a_{\mu'}(s; u(s), u(s)) ds \\ &- 2\int_0^t \left[\tilde{\Phi}_1(s) - \tilde{\Phi}_2(s)\right] a_\mu(s; u_2(s), u'(s)) ds, \end{aligned} \tag{3.68}$$

where

$$X(t) = \left\|u'(t)\right\|^2 + \tilde{\Phi}_1(t) a_\mu(t; u(t), u(t)) + 2\lambda \int_0^t \left\|u'(s)\right\|_a^2 ds. \tag{3.69}$$

Note that $u = u_1 - u_2$ is also the weak solution of the problem (3.31) corresponding to $\tilde{u}_0 = \tilde{u}_1 = 0$, $\Phi(t) = \tilde{\Phi}_1(t)$, $F(t) = \left[\tilde{\Phi}_1(t) - \tilde{\Phi}_2(t)\right] \frac{\partial}{\partial x}\left(\mu(t) u_{2x}(t)\right)$.

By using Lemma 3.3 with $\tilde{u}_0 = \tilde{u}_1 = 0$, we get

$$\begin{aligned} Y(t) = &-2\int_0^t \tilde{\Phi}_1(s)\langle \mu_x(s)u_x(s), \Delta u'(s)\rangle ds \\ &+ \int_0^t ds \int_0^1 \left(\tilde{\Phi}_1(s)\mu(x,s)\right)' |\Delta u(x,s)|^2 \, dx \\ &- 2\int_0^t \left[\tilde{\Phi}_1(s) - \tilde{\Phi}_2(s)\right] \langle \frac{\partial}{\partial x}\left(\mu(s)u_{2x}(s)\right), \Delta u'(s)\rangle ds, \end{aligned} \tag{3.70}$$

where

$$Y(t) = \left\|u'(t)\right\|_a^2 + \tilde{\Phi}_1(t)\left\|\sqrt{\mu(t)}\Delta u(t)\right\|^2 + 2\lambda\int_0^t \left\|\Delta u'(s)\right\|^2 ds. \tag{3.71}$$

Put $S(t) = X(t) + Y(t)$, we have

$$\begin{aligned} \bar{\mu}_*\bar{S}(t) &\le S(t) \\ &= \int_0^t \left[\tilde{\Phi}_1'(s)a_\mu(s;u(s),u(s)) + \tilde{\Phi}_1(s)a_{\mu'}(s;u(s),u(s))\right] ds \\ &\quad - 2\int_0^t \tilde{\Phi}_1(s)\langle \mu_x(s)u_x(s), \Delta u'(s)\rangle ds \\ &\quad + \int_0^t ds\int_0^1 \left(\tilde{\Phi}_1(s)\mu(x,s)\right)' |\Delta u(x,s)|^2 \, dx - 2\int_0^t \left[\tilde{\Phi}_1(s) - \tilde{\Phi}_2(s)\right] \\ &\quad \times \left[a_\mu(s;u_2(s),u'(s)) + \frac{\partial}{\partial x}\left(\mu(s)u_{2x}(s)\right), \Delta u'(s)\rangle\right] ds, \end{aligned} \tag{3.72}$$

where

$$\bar{S}(t) = \left\|u'(t)\right\|^2 + \left\|u'(t)\right\|_a^2 + \|u(t)\|_{H^2\cap V}^2 + \int_0^t \left\|u'(s)\right\|_{H^2\cap V}^2 ds. \tag{3.73}$$

With the similar estimations as in $\bar{S}_m(t)$, we obtain the following estimate

$$\bar{S}(t) \le \bar{D}_M \int_0^t \bar{S}(s) ds, \tag{3.74}$$

where

$$\bar{D}_M = \frac{8}{\bar{\mu}_*}\left[1 + 3M^2 + \left(1 + 144M^4\right)\frac{\tilde{K}_M\hat{K}_\mu}{\bar{\mu}_*}\right]\tilde{K}_M\hat{K}_\mu.$$

Using Gronwall's Lemma, it follows from (3.74) that $\bar{S}(t) \equiv 0$, that is, $u_1 \equiv u_2$. This completes the proof . □

4. Convergence of the sequence of solutions of (P_n) to a unique weak solution of (P_∞)

In this section, we shall consider the problems (P_n), (P_∞) as follows

$$(P_n)\qquad \begin{cases} u_{tt} - \lambda u_{txx} - \Phi_n[u](t)\dfrac{\partial}{\partial x}\left(\mu(x,t)u_x(x,t)\right) \\ \qquad\qquad\qquad = f\left(x,t\right),\, 0<x<1,\, 0<t<T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\; u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

$$(P_\infty)\qquad \begin{cases} u_{tt} - \lambda u_{txx} - \Phi[u](t)\dfrac{\partial}{\partial x}\left(\mu(x,t)u_x(x,t)\right) \\ \qquad\qquad\qquad = f\left(x,t\right),\, 0<x<1,\, 0<t<T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\; u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\zeta \geq 0$, $\lambda > 0$ are constants and Φ, μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, in which

$$\begin{aligned} \Phi_n[u](t) &= \Phi\left(t, (S_n u)(t), (\hat{S}_n u)(t)\right), \\ \Phi[u](t) &= \Phi\left(t, \|u(t)\|^2, \|u_x(t)\|^2\right), \\ (S_n u)(t) &= \frac{1}{n}\sum_{i=0}^{n-1} u^2(\tfrac{i}{n}, t),\; (\hat{S}_n u)(t) = \frac{1}{n}\sum_{i=0}^{n-1} u_x^2(\tfrac{i}{n}, t). \end{aligned} \tag{4.1}$$

Using the assumptions $(H_1) - (H_4)$ and the results of Theorem 3.4, there exist positive constants M, T independent of n such that the problem (P_n) have the unique weak solution $\bar{u}_n$ satisfying

$$\bar{u}_n \in W_1(M,T) \quad \text{for all} \;\; n \in \mathbb{N}. \tag{4.2}$$

From (4.2), we deduce that there exists a subsequence of $\{\bar{u}_n\}$, with the same notation, such that

$$\begin{cases} \bar{u}_n \to \bar{u} & \text{in} \quad L^\infty(0,T;H^2 \cap V) \text{ weak*}, \\ \bar{u}_n' \to \bar{u}' & \text{in} \quad L^\infty(0,T;H^2 \cap V) \text{ weak*}, \\ \bar{u}_n'' \to \bar{u}'' & \text{in} \quad L^2(0,T;V) \text{ weak}, \\ \bar{u} \in W(M,T). \end{cases} \tag{4.3}$$

Applying the lemma of Aubin-Lions, a classical compactness result in the space $C([0,T];V)$, there exists a subsequence $\{\bar{u}_n\}$, with the same symbol, such that

$$\begin{cases} \bar{u}_n \to \bar{u} & \text{in} \quad C([0,T];V) \text{ strongly}, \\ \bar{u}_n' \to \bar{u}' & \text{in} \quad C([0,T];V) \text{ strongly}. \end{cases} \tag{4.4}$$

Because $\bar{u}_n$ is the unique weak solution of (P_n), we have

$$\begin{aligned}
&\int_0^T \langle \bar{u}_n''(t), w\rangle \varphi(t)dt + \lambda \int_0^T a(\bar{u}_n'(t), w)\varphi(t)dt \\
&\quad + \int_0^T \Phi\left(t, (S_n\bar{u}_n)(t), (\hat{S}_n\bar{u}_n)(t)\right) a_\mu(t; \bar{u}_n(t), w)\varphi(t)dt \\
&= \int_0^T \langle f(t), w\rangle \varphi(t)dt, \ \forall w \in V, \ \forall \varphi \in C_c^\infty(0,T).
\end{aligned} \tag{4.5}$$

By (4.3) and (4.4), we get

$$\begin{aligned}
\int_0^T \langle \bar{u}_n''(t), w\rangle \varphi(t)dt &\to \int_0^T \langle \bar{u}''(t), w\rangle \varphi(t)dt, \\
\lambda \int_0^T a(\bar{u}_n'(t), w)\varphi(t)dt &\to \lambda \int_0^T a(\bar{u}'(t), w)\varphi(t)dt.
\end{aligned} \tag{4.6}$$

We need to check

$$\int_0^T \Phi_n[\bar{u}_n](t) a_\mu(t; \bar{u}_n(t), w)\varphi(t)dt \ \to \ \int_0^T \Phi[\bar{u}](t) a_\mu(t; \bar{u}(t), w)\varphi(t)dt, \tag{4.7}$$

so the following lemma is useful.

Lemma 4.1. *There exists a subsequence of $\{\bar{u}_n\}$, which is also denoted by $\{\bar{u}_n\}$, such that*

(i) $\left\| S_n\bar{u} - \|\bar{u}(\cdot)\|^2 \right\|_{L^2(0,T)}^2 = \int_0^T \left| (S_n\bar{u})(t) - \|\bar{u}(t)\|^2 \right|^2 dt \to 0,$

(ii) $\left\| \hat{S}_n\bar{u} - \|\bar{u}_x(\cdot)\|^2 \right\|_{L^2(0,T)}^2 = \int_0^T \left| (\hat{S}_n\bar{u})(t) - \|\bar{u}_x(t)\|^2 \right|^2 dt \to 0,$

(iii) $\|S_n\bar{u}_n - S_n\bar{u}\|_{C([0,T])} \to 0,$

(iv) $\left\| S_n\bar{u}_n - \|\bar{u}(\cdot)\|^2 \right\|_{L^2(0,T)}^2 = \int_0^T \left| (S_n\bar{u}_n)(t) - \|\bar{u}(t)\|^2 \right|^2 dt \to 0,$

(v) $\left\| \hat{S}_n\bar{u}_n - \hat{S}_n\bar{u} \right\|_{C([0,T])} \to 0,$

(vi) $\left\| \hat{S}_n\bar{u}_n - \|\bar{u}_x(\cdot)\|^2 \right\|_{L^2(0,T)}^2 = \int_0^T \left| (\hat{S}_n\bar{u}_n)(t) - \|\bar{u}_x(t)\|^2 \right|^2 dt \to 0,$

(vii) $E_n \equiv \|\Phi_n[\bar{u}_n] - \Phi[\bar{u}]\|_{L^2(0,T)} \to 0.$ (4.8)

Proof. We note that

$$\frac{1}{n}\sum_{i=0}^{n-1} h\left(\frac{i}{n}\right) \to \int_0^1 h(y)dy, \ \forall h \in C^0([0,1]). \tag{4.9}$$

Since $\bar{u} \in H_T \hookrightarrow C^0([0,T]; H^2 \cap V)$, we deduce that $\bar{u}$, $\bar{u}_x \in C^0([0,T]; H^1) \hookrightarrow C^0([0,T]; C^0(\bar{\Omega}))$, so the functions $y \mapsto \bar{u}^2(y,t)$, $y \mapsto \bar{u}_x^2(y,t)$ belongs to $C^0(\bar{\Omega})$, for all $t \in [0,T]$ then,

$$\begin{aligned}
(S_n\bar{u})(t) &= \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \bar{u}^2(\frac{i}{n},t) \to \int_0^1 \bar{u}^2(y,t)dy = \|\bar{u}(t)\|^2 \quad \text{as } n \to \infty, \\
(\hat{S}_n\bar{u})(t) &= \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \bar{u}_x^2(\frac{i}{n},t) \to \int_0^1 \bar{u}_x^2(y,t)dy = \|\bar{u}_x(t)\|^2 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.10}$$

Note that

$$\begin{aligned}
|(S_n\bar{u})(t)| &= \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \bar{u}^2(\frac{i}{n},t) \le \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \|\bar{u}\,(t)\|^2_{C^0(\bar{\Omega})} \\
&= \|\bar{u}\,(t)\|^2_{C^0(\bar{\Omega})} \le \|\bar{u}_x\,(t)\|^2 = \|\bar{u}\,(t)\|^2_V \le M^2,
\end{aligned} \tag{4.11}$$

so

$$\left|(S_n\bar{u})(t) - \|\bar{u}(t)\|^2\right| \le 2M^2,\ \forall n \in \mathbb{N} \text{ and } \forall t \in [0,T]. \tag{4.12}$$

Applying the dominated convergence theorem, we deduce that (4.8)(i) is true. Similarly,

$$\begin{aligned}
\left|(\hat{S}_n\bar{u})(t)\right| &= \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \bar{u}_x^2(\frac{i}{n},t) \le \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \|\bar{u}_x\,(t)\|^2_{C^0(\bar{\Omega})} \\
&= \|\bar{u}_x\,(t)\|^2_{C^0(\bar{\Omega})} \le 2\,\|\bar{u}\,(t)\|^2_{H^2\cap V} \le 2M^2
\end{aligned} \tag{4.13}$$

and

$$\left|(\hat{S}_n\bar{u})(t) - \|\bar{u}_x(t)\|^2\right| \le 3M^2 \quad \text{for all } n \in \mathbb{N} \text{ and } \forall t \in [0,T]. \tag{4.14}$$

By the dominated convergence theorem, (4.8)(ii) is also true.

Next, by $\bar{u}_n$, $\bar{u} \in W_1(M,T)$ and $V \hookrightarrow C^0\,([0,1]) \equiv E$, we deduce from the first limit in (4.4) that

$$\begin{aligned}
|(S_n\bar{u}_n)(t) - (S_n\bar{u})(t)| &\le \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \left|\bar{u}_n^2(\frac{i}{n},t) - \bar{u}^2(\frac{i}{n},t)\right| \\
&\le \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \left|\bar{u}_n(\frac{i}{n},t) + \bar{u}(\frac{i}{n},t)\right| \left|\bar{u}_n(\frac{i}{n},t) - \bar{u}(\frac{i}{n},t)\right| \\
&\le \frac{1}{n}\sum\nolimits_{i=0}^{n-1} \left(\|\bar{u}_n(t)\|_E + \|\bar{u}(t)\|_E\right) \|\bar{u}_n(t) - \bar{u}(t)\|_E \\
&= \left(\|\bar{u}_n(t)\|_E + \|\bar{u}(t)\|_E\right) \|\bar{u}_n(t) - \bar{u}(t)\|_E \\
&\le \left(\|\bar{u}_n(t)\|_V + \|\bar{u}(t)\|_V\right) \|\bar{u}_n(t) - \bar{u}(t)\|_V \\
&\le 2M\,\|\bar{u}_n - \bar{u}\|_{C([0,T];V)}\,,
\end{aligned}$$

hence

$$\|S_n\bar{u}_n - S_n\bar{u}\|_{C([0,T])} \le 2M\,\|\bar{u}_n - \bar{u}\|_{C([0,T];V)}\,. \tag{4.15}$$

By (4.15), $(4.4)_1$, it is clear to see that (4.8)(iii) holds.

By (i) and (iii) of (4.8), we obtain

$$\begin{aligned}\left\|S_n\bar{u}_n - \|\bar{u}(\cdot)\|^2\right\|_{L^2(0,T)} &\leq \|S_n\bar{u}_n - S_n\bar{u}\|_{L^2(0,T)} + \left\|S_n\bar{u} - \|\bar{u}(\cdot)\|^2\right\|_{L^2(0,T)} \\ &\leq \sqrt{T}\,\|S_n\bar{u}_n - S_n\bar{u}\|_{C([0,T])} + \left\|S_n\bar{u} - \|\bar{u}(\cdot)\|^2\right\|_{L^2(0,T)} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Thus, (4.8)(iv) holds.

Now, we prove (v) of (4.8). By $\bar{u}_n \in W_1(M,T)$, we deduce that

$$\begin{aligned}&\bar{u}_n \in C^0\left([0,T]; H^2 \cap V\right) \cap C^1\left([0,T]; V\right) \cap L^\infty(0,T; H^2 \cap V), \\ &\bar{u}_n' \in C^0\left([0,T]; V\right) \cap L^\infty(0,T; H^2 \cap V).\end{aligned} \tag{4.16}$$

Consider the sequence $\{w_n\}$ defined by $w_n = \bar{u}_{nx}$, by $H^1 \hookrightarrow C^0([0,1]) \equiv E$, we have $\{w_n\} \subset C^0\left([0,T]; H^1\right) \subset C^0\left([0,T]; E\right)$. We shall show that there exists a subsequence of $\{w_n\}$, it is also denoted by $\{w_n\}$, such that

$$w_n \to \bar{u}_x \text{ strongly in } C^0\left([0,T]; E\right). \tag{4.17}$$

Using Ascoli-Arzela theorem in $C^0([0,T]; E)$, we have the following

$$\begin{aligned}&\text{(j)} \quad \{w_n\} \text{ is equicontinuous in } C^0([0,T]; E), \\ &\text{(jj)} \quad \text{for every } t \in [0,T],\ \{w_n(t) : n \in \mathbb{N}\} \text{ is relatively compact in } E.\end{aligned} \tag{4.18}$$

Indeed, for all t_1, $t_2 \in [0,T]$, $t_1 \leq t_2$, for all $n \in \mathbb{N}$, by (4.16)(ii), we have

$$\begin{aligned}\|w_n(t_2) - w_n(t_1)\|_E &= \left\|\int_{t_1}^{t_2} w_n'(t)dt\right\|_E \\ &\leq \int_{t_1}^{t_2} \left\|w_n'(t)\right\|_E dt \\ &= \int_{t_1}^{t_2} \left\|\bar{u}_{nx}'(t)\right\|_E dt \\ &\leq \sqrt{2}\int_{t_1}^{t_2} \left\|\bar{u}_{nx}'(t)\right\|_{H^1} dt \\ &\leq \sqrt{2}\,|t_2 - t_1|\,\left\|\bar{u}_n'\right\|_{L^\infty(0,T;H^2\cap V)} \\ &\leq \sqrt{2}M\,|t_2 - t_1|.\end{aligned} \tag{4.19}$$

This implies that (4.18)(j) holds.

By $(4.16)_{(i)}$, we have

$$\|w_n(t)\|_{H^1} = \|\bar{u}_{nx}(t)\|_{H^1} = \|\bar{u}_n(t)\|_{H^2\cap V} \leq \|\bar{u}_n\|_{L^\infty(0,T;H^2\cap V)} \leq M. \tag{4.20}$$

Since the imbedding $H^1 \hookrightarrow C^0([0,1]) = E$ is compact, there exists a convergent subsequence of $\{w_n(t)\}$ (in E). This implies (4.18)(jj) holds.

From (4.18), we deduce that there exists a subsequence of $\{w_n\}$, also denoted by $\{w_n\}$ such that

$$w_n \to w \text{ strongly in } C^0([0,T];E). \tag{4.21}$$

By $C^0([0,T];E) \hookrightarrow C^0\left([0,T];L^2\right)$, we deduce that

$$w_n \to w \text{ strongly in } C^0\left([0,T];L^2\right). \tag{4.22}$$

On the other hand, from $(4.4)_{(\mathrm{i})}$, we obtain

$$w_n = \bar{u}_{nx} \to \bar{u}_x \text{ strongly in } C^0\left([0,T];L^2\right). \tag{4.23}$$

It follows from (4.22) and (4.23) that $w = \bar{u}_x$, thus (4.17) is proved.

On the other hand, from (4.2), we obtain the following estimation

$$\begin{aligned}\left|(\hat{S}_n\bar{u}_n)(t) - (\hat{S}_n\bar{u})(t)\right| &\le \frac{1}{n}\sum\nolimits_{i=0}^{n-1}\left|\bar{u}_{nx}^2(\frac{i}{n},t) - \bar{u}_x^2(\frac{i}{n},t)\right| \\ &\le (\|\bar{u}_{nx}(t)\|_E + \|\bar{u}_x(t)\|_E)\,\|\bar{u}_{nx}(t) - \bar{u}_x(t)\|_E \\ &\le \sqrt{2}\,(\|\bar{u}_{nx}(t)\|_{H^1} + \|\bar{u}_x(t)\|_{H^1})\,\|\bar{u}_{nx} - \bar{u}_x\|_{C^0([0,T];E)} \\ &\le 2\sqrt{2}M\,\|\bar{u}_{nx} - \bar{u}_x\|_{C^0([0,T];E)}.\end{aligned} \tag{4.24}$$

Hence

$$\left\|\hat{S}_n\bar{u}_n - \hat{S}_n\bar{u}\right\|_{C([0,T])} \le 2\sqrt{2}M\,\|\bar{u}_{nx} - \bar{u}_x\|_{C^0([0,T];E)}. \tag{4.25}$$

From (4.17) and (4.25), we obtain (4.8)(v) holds.

By (4.8)(ii) and (4.8)(v), we obtain

$$\begin{aligned}\left\|\hat{S}_n\bar{u}_n - \|\bar{u}_x(\cdot)\|^2\right\|_{L^2(0,T)} &\le \left\|\hat{S}_n\bar{u}_n - \hat{S}_n\bar{u}\right\|_{L^2(0,T)} + \left\|\hat{S}_n\bar{u} - \|\bar{u}_x(\cdot)\|^2\right\|_{L^2(0,T)} \\ &\le \sqrt{T}\left\|\hat{S}_n\bar{u}_n - \hat{S}_n\bar{u}\right\|_{C([0,T])} + \left\|\hat{S}_n\bar{u} - \|\bar{u}_x(\cdot)\|^2\right\|_{L^2(0,T)} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Thus, (4.8)(vi) holds.

By the following inequality

$$\begin{aligned}&|\Phi_n[\bar{u}_n](t) - \Phi[\bar{u}](t)| \\ &\quad = \left|\Phi\left(t, (S_n\bar{u}_n)(t), (\hat{S}_n\bar{u}_n)(t)\right) - \Phi\left(t, \|\bar{u}(t)\|^2, \|\bar{u}_x(t)\|^2\right)\right| \\ &\quad \le \tilde{K}_M\left[\left|(S_n\bar{u}_n)(t) - \|\bar{u}(t)\|^2\right| + \left|(\hat{S}_n\bar{u}_n)(t) - \|\bar{u}_x(t)\|^2\right|\right],\end{aligned} \tag{4.26}$$

we deduce from (4.8)(vi) that

$$\begin{aligned} E_n &\equiv \|\Phi_n[\bar{u}_n] - \Phi[\bar{u}]\|_{L^2(0,T)} \\ &\leq \tilde{K}_M \left[\left\| S_n\bar{u}_n - \|\bar{u}(\cdot)\|^2 \right\|_{L^2(0,T)} + \left\| \hat{S}_n\bar{u}_n - \|\bar{u}_x(\cdot)\|^2 \right\|_{L^2(0,T)} \right] \\ &\to 0 \end{aligned} \tag{4.27}$$

as $n \to \infty$. Thus, (4.8)(vii) holds. Lemma 4.1 is proved. □

Now, we continue with the proof of (4.7). Note more that

$$|\Phi_n[\bar{u}_n](t)| = \left|\Phi\left(t, (S_n\bar{u}_n)(t), (\hat{S}_n\bar{u}_n)(t)\right)\right| \leq \tilde{K}_M,$$

we obtain

$$\begin{aligned} &\left| \int_0^T \Phi_n[\bar{u}_n](t) a_\mu(t; \bar{u}_n(t), w)\varphi(t)dt - \int_0^T \Phi[\bar{u}](t) a_\mu(t; \bar{u}(t), w)\varphi(t)dt \right| \\ &\quad \leq \int_0^T \Phi_n[\bar{u}_n](t) \left| a_\mu(t; \bar{u}_n(t) - \bar{u}(t), w)\varphi(t) \right| dt \\ &\qquad + \int_0^T \left|\Phi_n[\bar{u}_n](t) - \Phi[\bar{u}](t)\right| \left| a_\mu(t; \bar{u}(t), w)\varphi(t) \right| dt \\ &\quad \leq \tilde{K}_M \hat{K}_\mu \|w\|_a \int_0^T \|\bar{u}_n(t) - \bar{u}(t)\|_a |\varphi(t)| \, dt \\ &\qquad + \hat{K}_\mu \|w\|_a \int_0^T \left|\Phi_n[\bar{u}_n](t) - \Phi[\bar{u}](t)\right| \|\bar{u}(t)\|_a |\varphi(t)| \, dt \\ &\quad \leq \tilde{K}_M \hat{K}_\mu \|w\|_a \|\varphi\|_{L^1(0,T)} \|\bar{u}_n - \bar{u}\|_{C([0,T];V)} \\ &\qquad + M\hat{K}_\mu \|w\|_a \|\varphi\|_{L^2(0,T)} E_n \to 0, \text{as } n \\ &\quad \to \infty. \end{aligned} \tag{4.28}$$

It follows from $(4.4)_1$, (4.27) and (4.28) that (4.7) holds.

Finally, letting $n \to \infty$ in (4.5), (4.6) and (4.7) lead to $\bar{u} \in W(M,T)$ satisfying the equation

$$\begin{aligned} &\int_0^T \langle \bar{u}''(t), w\rangle \varphi(t)dt + \lambda \int_0^T a(\bar{u}'(t), w)\varphi(t)dt \\ &\quad + \int_0^T \Phi[\bar{u}](t) a_\mu(t; \bar{u}(t), w)\varphi(t)dt = \int_0^T \langle f(t), w\rangle \varphi(t)dt \end{aligned} \tag{4.29}$$

for all $w \in V$, $\varphi \in C_c^\infty(0,T)$, together with the initial conditions

$$\bar{u}(0) = \tilde{u}_0, \ \bar{u}'(0) = \tilde{u}_1. \tag{4.30}$$

Furthermore, (4.29) and $(4.3)_4$ give

$$\bar{u}'' = \lambda\Delta\bar{u}' + \Phi[\bar{u}](t)\frac{\partial}{\partial x}\left(\mu(x,t)\bar{u}_x\right) + f \in L^\infty(0,T;L^2),$$

so we obtain $\bar{u} \in W_1(M,T)$. The existence proof is completed.

Next, it is not difficult to prove the uniqueness of a weak solution of (P) and so $\bar{u} = u_\infty$. Therefore, we obtain the main result in this section as follows.

Theorem 4.2. *Let $(H_1)-(H_4)$ hold. Then, there exist positive constants M, $T>0$ such that*

(i) *(P_∞) has a unique weak solution $u_\infty \in W_1(M,T)$.*

(ii) *the sequence $\{\bar{u}_n\}$ converges to the solution u_∞ of (P_∞) strongly in the Banach space H_T.*

Furthermore, we also have the estimation

$$\|\bar{u}_n - u_\infty\|_{H_T} \le C_T E_n \quad \text{for all } n\in\mathbb{N}, \tag{4.31}$$

where

$$E_n = \|\Phi_n[\bar{u}_n] - \Phi[u_\infty]\|_{L^2(0,T)} \to 0 \quad \text{as } n\to\infty,$$

and C_T is the constant depending only on T, Φ, μ, f, $\tilde{u}_0$, $\tilde{u}_1$, λ, ζ.

Proof. It remains to prove (ii). We put

$$\begin{aligned} v_n &= \bar{u}_n - u_\infty,\\ \bar{\Phi}_n(t) &= \Phi_n[\bar{u}_n](t) - \Phi[u_\infty](t)\\ &= \Phi\left(t,(S_n\bar{u}_n)(t),(\hat{S}_n\bar{u}_n)(t)\right) - \Phi\left(t,\|u_\infty(t)\|^2,\|u_{\infty x}(t)\|^2\right), \end{aligned} \tag{4.32}$$

then $v_n \in \tilde{V}_T$ satisfies the variational problem

$$\begin{cases} \langle v_n''(t),w\rangle + \lambda a(v_n'(t),w) + \Phi[u_\infty](t)a_\mu(t;v_m(t),w)\\ \qquad\qquad = \bar{\Phi}_n(t)\langle\frac{\partial}{\partial x}\left(\mu(t)\bar{u}_{nx}(t)\right),w\rangle,\ \forall w\in V,\\ v_n(0) = v_n'(0) = 0. \end{cases} \tag{4.33}$$

Taking $w = v_m'$ in $(4.33)_1$ and integrating in t, we get

$$\begin{aligned} Z_n(t) = &\int_0^t \left(\Phi[u_\infty]\right)'(s)a_\mu(s;v_n(s),v_n(s))ds\\ &+\int_0^t \Phi[u_\infty](s)a_{\mu'}(s;v_n(s),v_n(s))ds\\ &+2\int_0^t \bar{\Phi}_n(s)\langle\frac{\partial}{\partial x}\left(\mu(s)\bar{u}_{nx}(s)\right),v_m'(s)\rangle ds, \end{aligned} \tag{4.34}$$

where

$$Z_n(t) = \left\|v_n'(t)\right\|^2 + \Phi[u_\infty](t)a_\mu(t;v_n(t),v_n(t)) + 2\lambda\int_0^t \left\|v_n'(s)\right\|_a^2 ds. \tag{4.35}$$

On the other hand, $v_n \in \tilde{V}_T$ is also the weak solution of the problem (3.31) corresponding to $\tilde{u}_0 = \tilde{u}_1 = 0$, $\Phi(t) = \Phi[u_\infty](t)$, $F(t) = \bar{\Phi}_n(t)\frac{\partial}{\partial x}\left(\mu(t)\bar{u}_{nx}(t)\right).$

Similarly, by using Lemma 3.3 with $\tilde{u}_0 = \tilde{u}_1 = 0$, we have

$$\begin{aligned}\bar{Z}_n(t) = &-2\int_0^t \Phi[u_\infty](s)\langle \mu_x(s)v_{nx}(s), \Delta v_n'(s)\rangle ds \\ &+ \int_0^t ds\int_0^1 \left(\Phi[u_\infty](s)\mu(x,s)\right)' \left|\Delta v_n(x,s)\right|^2 dx \\ &+ 2\int_0^t \bar{\Phi}_n(s)\langle \frac{\partial}{\partial x}\left(\mu(s)\bar{u}_{nx}(s)\right), -\Delta v_n'(s)\rangle ds,\end{aligned} \tag{4.36}$$

where

$$\bar{Z}_n(t) = \left\|v_n'(t)\right\|_a^2 + \Phi[u_\infty](t)\left\|\sqrt{\mu(t)}\Delta v_n(t)\right\|^2 + 2\lambda\int_0^t \left\|\Delta v_n'(s)\right\|^2 ds. \tag{4.37}$$

Put

$$\begin{aligned}Y_n(t) &= Z_n(t) + \bar{Z}_n(t) \\ &= \left\|v_n'(t)\right\|^2 + \left\|v_n'(t)\right\|_a^2 \\ &\quad + \Phi[u_\infty](t)\left[a_\mu(t; v_n(t), v_n(t)) + \left\|\sqrt{\mu(t)}\Delta v_n(t)\right\|^2\right] \\ &\quad + 2\lambda\int_0^t \left(\left\|v_n'(s)\right\|_a^2 + \left\|\Delta v_n'(s)\right\|^2\right) ds, \\ \bar{Y}_n(t) &= \left\|v_n'(t)\right\|^2 + \left\|v_n'(t)\right\|_a^2 + \left\|v_n(t)\right\|_{H^2\cap V}^2 + \int_0^t \left\|v_n'(s)\right\|_{H^2\cap V}^2 ds,\end{aligned} \tag{4.38}$$

then, it follows from (4.34), (4.35), (4.37) and (4.38) that

$$\begin{aligned}\bar{\mu}_*\bar{Y}_n(t) &\le Y_n(t) \\ &= \int_0^t \left[\left(\Phi[u_\infty]\right)'(s)a_\mu(s; v_n(s), v_n(s)) + \Phi[u_\infty](s)a_{\mu'}(s; v_n(s), v_n(s))\right] ds \\ &\quad + \int_0^t ds\int_0^1 \left(\Phi[u_\infty](s)\mu(x,s)\right)' \left|\Delta v_n(x,s)\right|^2 dx \\ &\quad - 2\int_0^t \Phi[u_\infty](s)\langle \mu_x(s)v_{nx}(s), \Delta v_n'(s)\rangle ds \\ &\quad + 2\int_0^t \bar{\Phi}_n(s)\langle \frac{\partial}{\partial x}\left(\mu(s)\bar{u}_{nx}(s)\right), v_m'(s) - \Delta v_n'(s)\rangle ds,\end{aligned} \tag{4.39}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda, \}$. With the following estimations

$$\begin{aligned}
&|\Phi[u_\infty](t)| \leq \tilde{K}_M, \\
&\left|\left(\Phi[u_\infty]\right)'(t)\right| \leq (1+4M^2)\tilde{K}_M, \\
&\left|\left(\Phi[u_\infty](t)\mu(x,t)\right)'\right| \leq 2(1+2M^2)\tilde{K}_M\hat{K}_\mu, \\
&\left\|\frac{\partial}{\partial x}\left(\mu(t)\bar{u}_{nx}(t)\right)\right\| \leq \sqrt{2}M\hat{K}_\mu,
\end{aligned}$$

and putting $E_n \equiv \left\|\bar{\Phi}_n\right\|_{L^2(0,T)} = \left\|\Phi_n[\bar{u}_n] - \Phi[u_\infty]\right\|_{L^2(0,T)}$, we obtain the estimations for the terms in the right-hand side of (4.39) as follows

$$\begin{aligned}
J_1 &= \int_0^t \left[\left(\Phi[u_\infty]\right)'(s)a_\mu(s; v_n(s), v_n(s)) + \Phi[u_\infty](s)a_{\mu'}(s; v_n(s), v_n(s))\right] ds \\
&\leq 2(1+2M^2)\tilde{K}_M\hat{K}_\mu \int_0^t \bar{Y}_n(s)ds; \\
J_2 &= \int_0^t ds \int_0^1 \left(\Phi[u_\infty](s)\mu(x,s)\right)' |\Delta v_n(x,s)|^2 dx \\
&\leq 2(1+2M^2)\tilde{K}_M\hat{K}_\mu \int_0^t \bar{Y}_n(s)ds; \\
J_3 &= -2\int_0^t \Phi[u_\infty](s)\langle \mu_x(s)v_{nx}(s), \Delta v_n'(s)\rangle ds \\
&\leq 2\tilde{K}_M\hat{K}_\mu \int_0^t \|v_{nx}(s)\| \left\|\Delta v_n'(s)\right\| ds \\
&\leq \frac{\bar{\mu}_*}{4}\bar{Y}_n(t) + \frac{4}{\bar{\mu}_*}\left(\tilde{K}_M\hat{K}_\mu\right)^2 \int_0^t \bar{Y}_n(s)ds; \\
J_4 &= 2\int_0^t \bar{\Phi}_n(s)\langle \frac{\partial}{\partial x}\left(\mu(s)\bar{u}_{nx}(s)\right), v_m'(s) - \Delta v_n'(s)\rangle ds \\
&\leq 4M\hat{K}_\mu \int_0^t \left\|\bar{\Phi}_n(s)\right\| \left\|v_n'(s)\right\|_{H^2\cap V} ds \\
&\leq \frac{\bar{\mu}_*}{4}\bar{Y}_n(t) + \frac{16}{\bar{\mu}_*}\left(M\hat{K}_\mu E_n\right)^2.
\end{aligned} \tag{4.40}$$

It follows from (4.39) and (4.40) that

$$\bar{Y}_n(t) \leq \frac{32}{\bar{\mu}_*^2}\left(M\hat{K}_\mu E_n\right)^2 + 2\bar{d}_1(M)\int_0^t \bar{Y}_n(s)ds, \tag{4.41}$$

where

$$\bar{d}_1(M) = \frac{4}{\bar{\mu}_*}\left[1 + 2M^2 + \frac{1}{\bar{\mu}_*}\tilde{K}_M \hat{K}_\mu\right]\tilde{K}_M \hat{K}_\mu.$$

Using Gronwall's lemma, it follows from (4.41) that

$$\bar{Y}_n(t) \le \frac{32}{\bar{\mu}_*^2}\left(M\hat{K}_\mu\right)^2 \exp(2T\bar{d}_1(M))E_n^2. \tag{4.42}$$

Combining (3.7), (4.38) and (4.42), we get

$$\|v_n\|_{H_T} \le \frac{12}{\bar{\mu}_*}\sqrt{2}M\hat{K}_\mu \exp(T\bar{d}_1(M))E_n. \tag{4.43}$$

Theorem 4.2 is proved. □

Remark 4.3. The arguments and methods used for proving the unique existence of solutions of (P_n) can be applied to the problem $(\bar{P}_n)$ in which $(S_n u)(t)$, $(\hat{S}_n u)(t)$ are replaced by the following arithmetic-mean terms

$$(S_n u)(t) = \frac{1}{n}\sum_{i=0}^{n-1} u^2\left(\frac{i+\theta_i}{n}, t\right),\ (\hat{S}_n u)(t) = \frac{1}{n}\sum_{i=0}^{n-1} u_x^2\left(\frac{i+\bar{\theta}_i}{n}, t\right),$$

respectively, where θ_i, $\bar{\theta}_i \in [0,1)$, $i = \overline{0, n-1}$ are given constants.

Remark 4.4. The methods used in the above sections can be applied again to obtain the same results for the following problem

$$(\bar{P}_n) \quad \begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial}{\partial x}\left[\mu\left(x, t, (S_n u)(t), (\hat{S}_n u)(t)\right) u_x\right] \\ \quad = f\left(x, t, u, u_t, u_x, (S_n u)(t), (\hat{S}_n u)(t)\right),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\lambda > 0$, $\zeta \ge 0$ are given constants, Φ, μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and

$$(S_n u)(t) = \frac{1}{n}\sum_{i=0}^{n-1} u^2\left(\frac{i+\theta_i}{n}, t\right),\ (\hat{S}_n u)(t) = \frac{1}{n}\sum_{i=0}^{n-1} u_x^2\left(\frac{i+\bar{\theta}_i}{n}, t\right),$$

with θ_i, $\bar{\theta}_i \in [0,1)$, $i = \overline{0, n-1}$ are given constants. Furthermore, the weak solution of $(\bar{P}_n)$ converges strongly in appropriate spaces to the weak solution of the following problem

$$(\bar{P}_\infty) \quad \begin{cases} u_{tt} - \lambda u_{txx} - \dfrac{\partial}{\partial x}\left[\mu\left(x, t, \|u(t)\|^2, \|u_x(t)\|^2\right) u_x\right] \\ \quad = f\left(x, t, u, u_t, u_x, \|u(t)\|^2, \|u_x(t)\|^2\right),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\|u(t)\|^2 = \int_0^1 u^2(y,t)dy$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(y,t)dy$.

Acknowledgments: The authors wish to express their sincere thanks to the editor and the referees for the valuable comments and suggestions for the improvement of the paper.

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