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SOME NEW TOPOLOGICAL STRUCTURES IN LATTICE RANDOM NORMED SPACES

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Abstract. The paper is considered an expansion of the lattice random normed space concepts, as we presented some certain topological structures.

1. Introduction

Random lattices have garnered considerable attention in recent years from both a mathematical and algorithmic viewpoint. Probabilistic metrics were first introduced in 1942 by Menger [9]. He also made important contributions to the probabilistic metric space, and then the scientist Wald [12] followed. Schweizer and Sklar [10] discussed the development of probabilistic metric space which is presented in the first chapter of this book. And by them the theory of random normed space (RN-space) was then developed current version. which gave a new definition of random normed space.

Serstnev [11] defined random normed space as a generalization of a normed linear space. In real normalized linear space, the vector normal is represented by a non-negative real number, but in probabilistic normed space, the vector normal is represented by a probability distribution function instead of a positive number. The importance of random normed theory lies in modeling

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uncertainty resulting from various problems in science field such as computer programming, statistical convergence, nuclear physics, etc, such in 2009, Filipović et al. [5] worked on new independently random normed modules and first used them in financial applications. Which prompted us to work on this research which is primarily focused on examining the topological structure of lattice random normed space, also contains some expansion of lattice random normed space concepts. The study of lattice random normed spaces is important to an understanding of nonlinear analysis. For more information about RN space, check out [1, 2, 3, 6, 7].

2. Preliminaries

In order to make sense of the following section of this study, let us first describe some well-known concepts and findings.

Definition 2.1. ([8]) Suppose that $l = (\text{Ł}, \geq_{\text{Ł}})$ be a complete lattice, i.e., a partially ordered set in which every non-empty subset admits supremum and infimum, also $\inf \text{Ł} = 0_l, \sup \text{Ł} = 1_l$.

$\Delta^+_{\text{Ł}}$ is the set of distribution function

$$\Delta^+_{\text{Ł}} = \Big\{ g | g : \mathbb{R} \bigcup \{-\infty, \infty\} \to \text{Ł}, g(0) = 0_l, g(+\infty) = 1_l,$$
$$g \text{ is non-decreasing and left-continuous on } \mathbb{R} \Big\}.$$

The subset $D^+_{\text{Ł}}$ of $\Delta^+_{\text{Ł}}$ defined as $D^+_{\text{Ł}} = \left\{ g \in \Delta^+_{\text{Ł}} : \lim_{x \to +\infty} g(x) = 1_l \right\}$.

The function $\varepsilon_\circ(t)$ is given by

$$\varepsilon_\circ(t) = \begin{cases} 0_l, & if\ t \leq 0, \\ 1_l, & if\ t > 0, \end{cases}$$

which represent the maximal element for $D^+_{\text{Ł}}$.

Definition 2.2. ([8]) A function $\mathfrak{N} : \text{Ł} \to \text{Ł}$ is called a negation function, if

(1) $\mathfrak{N}(0_l) = 1_l$, $\mathfrak{N}(1_l) = 0_l$.
(2) $\mathfrak{N}(\beta) \leq \mathfrak{N}(\acute{\omega})$ if $\beta \geq \acute{\omega}$.

This function is involutive, iff $\mathfrak{N}(\mathfrak{N}(\hat{\alpha})) = \hat{\alpha}$, $\forall\ \hat{\alpha} \in \text{Ł}$.

Definition 2.3. ([8]) Let Ł be a complete lattice. A mapping $T : \text{Ł}^2 \to \text{Ł}$ is called a triangular norm (t-norm), if satisfying the following:

(1) $T(\varkappa, 1_l) = \varkappa$,$\forall \varkappa \in \text{Ł}$ (Boundary condition).
(2) $T(\varkappa, y) = T(y, \varkappa), \forall (\varkappa, y) \in \text{Ł}^2$ (Commutativity condition).
(3) $T(\varkappa, T(y, \cdot\mathcal{Z})) = T(T(\varkappa, y), \cdot\mathcal{Z}), \forall (\varkappa, y, \mathcal{Z}) \in \text{Ł}^3$ (Associativity condition).

(4) $y \geq_{Ł} y' \Longrightarrow T(\varkappa, y) \leq_{Ł} T(\varkappa, y'), \forall (x, y, y') \in Ł^3$ (Monotonicity condition).

Definition 2.4. ([4]) Let (W, Q, T) be a lattice random normed space (LRN-space) where W is vector space, T is a continuous triangular norm, $Q : W \times R \to D^+_{Ł}$ is a mapping satisfies the conditions:

(1) $Q(\varkappa, t) = \acute{\varepsilon}_{\circ}(t)$ if and only if $\varkappa = 0$ for all $t > 0$,

(2) $Q(\hat{\alpha}\varkappa, t) = Q\left(\varkappa, \frac{t}{|\hat{\alpha}|}\right)$ for all $\varkappa \in W,\ t > 0,\ \ \hat{\alpha} \neq 0$,

(3) $Q(\varkappa + y, t + \mathcal{Z}) \geq_{Ł} T(Q(\varkappa, t), Q(y, \mathcal{Z}))$ for all $\varkappa, y \in X,\ t, \mathcal{Z} \geq 0$.

Example 2.5. ([4]) Let $(Ł, \leq_{Ł})$ be defined by

$$Ł = \left\{(\varkappa_1, \varkappa_2) : (\varkappa_1, \varkappa_2) \in [0,1]^2,\ \varkappa_1 + \varkappa_2 \leq 1\right\},$$

$(\varkappa_1, \varkappa_2) \leq_{Ł} (y_1, y_2)$ if and only if $\varkappa_1 \leq y_1, y_2 \leq \varkappa_2$ for all $\varkappa = (\varkappa_1, \varkappa_2)$, $y = (y_1, y_2) \in Ł^2$. Then $(Ł, \leq_{Ł})$ is a complete lattice.

We denoted its units by $0_{Ł} = (0,1), 1_{Ł} = (1,0)$.

Assume that $(W, \|\cdot\|)$ is a normed space,

$$T(\varkappa, y) = (\min\{\varkappa_1, y_1\}, \max\{\varkappa_2, y_2\})$$

for all $\varkappa = (\varkappa_1, \varkappa_2), y = (y_1, y_2) \in [0,1]^2$ and

$$Q(\varkappa, t) = \left(\frac{t}{t + \|\varkappa\|}, \frac{\|\varkappa\|}{t + \|\varkappa\|}\right), \quad \forall t \in \mathbb{R}^+.$$

Then (W, Q, T) is an LRN-Space.

3. Lattice random topological structures

The section provides various topological structures in lattice random normed spaces.

Definition 3.1. The open ball $B(\varkappa, r, t)$ in LRN-Space (W, Q, T) is defined by

$$B(\varkappa, r, t) = \{y \in W : Q(\varkappa - y, t) >_{Ł} \mathfrak{N}(r)\}.$$

Also the closed ball $B[\varkappa, r, t]$ in LRN-space is defined by

$$B[\varkappa, r, t] = \{y \in W : Q(\varkappa - y, t) \geq_{Ł} \mathfrak{N}(r)\}$$

for all $\varkappa \in W, r \in Ł/\{0_l, 1_l\}, t > 0$.

Theorem 3.2. *Assume that $B(\varkappa, r, t)$ is an open ball in LRN-space (W, Q, T). Then it is an open set in LRN-space.*

Proof. Suppose that $B(\varkappa, r, t)$ is an open ball with center $\varkappa \in W$, and radius $r \in \text{Ł}/\{0_l, 1_l\}$, $t > 0$.

Suppose that $y \in B(\varkappa, r, t)$. Then $Q(\varkappa - y, t) >_{\text{Ł}} \mathfrak{N}(r)$. Since $Q(\varkappa - y, t)$ $>_{\text{Ł}} \mathfrak{N}(r)$, there exist $0 < t_\circ < t$ such that

$$Q(\varkappa - y, t_\circ) >_{\text{Ł}} \mathfrak{N}(r).$$

Put $\mathfrak{N}(r_\circ) = Q(\varkappa - y, t_\circ)$, since $\mathfrak{N}(r_\circ) >_{\text{Ł}} \mathfrak{N}(r)$, then there exists $\mathrm{s} \in \text{Ł}/\{0_l, 1_l\}$ such that $\mathfrak{N}(r) >_{\text{Ł}} \mathfrak{N}(\mathrm{s}) >_{\text{Ł}} \mathfrak{N}(r)$. If there exists $r_1 \in \text{Ł}/\{0_l, 1_l\}$, then $T(\mathfrak{N}(r_\circ), \mathfrak{N}(r_1)) >_{\text{Ł}} \mathfrak{N}(\mathrm{s})$. Consider the open ball $B(y, r_1, t - t_\circ)$. Then we will prove $B(y, r_1, t - t_\circ) \subset B(\varkappa, r, t)$. In fact, if $\mathcal{Z} \in B(y, r_1, t - t_\circ)$ then $Q(y - \mathcal{Z}, t - t_\circ) >_l \mathfrak{N}(r_1)$. Hence we have

$$\begin{aligned} Q(\varkappa - \mathcal{Z}, t) &\geq_{\text{Ł}} T(Q(\varkappa - y, t_\circ), Q(y - \mathcal{Z}, t - t_\circ)) \\ &\geq_L T(\mathfrak{N}(r_\circ), \mathfrak{N}(r_1)) \\ &>_{\text{Ł}} \mathfrak{N}(\mathrm{s}) \\ &>_{\text{Ł}} \mathfrak{N}(r). \end{aligned}$$

Therefore, $\mathcal{Z} \in B(\varkappa, r, t)$. □

Remark 3.3. There are different species of topologies in LRN-space. Each lattice random norm Q on W produce a topology on W, that has a base of the family of open sets (neighborhoods) which is denoted by

$$\{B(\varkappa, r, t)\}_{\varkappa \in W, t > 0, r \in \text{Ł}/\{0_l, 1_l\}}.$$

Theorem 3.4. *If (W, Q, T) is an LRN-Space, then W is a Hausdorff space.*

Proof. Let (W, Q, T) an LRN-space, and $\varkappa \neq y \in W$. Then

$$1_l >_{\text{Ł}} Q(\varkappa - y, t) >_{\text{Ł}} 0_l, \quad \forall t > 0.$$

Put $\mathfrak{N}(r) = Q(\varkappa - y, t)$, for each $r_\circ \in (0_l, \mathfrak{r})$, $\mathfrak{N}(r_\circ) >_{\text{Ł}} \mathfrak{N}(r)$. Then there exists r_1 such that $T(\mathfrak{N}(r_1), \mathfrak{N}(r_1)) \geq_{\text{Ł}} \mathfrak{N}(r_\circ)$.

Let $B\left(\varkappa, r_1, \frac{t}{2}\right), B\left(y, r_1, \frac{t}{2}\right)$ be open balls. Then, we can prove that

$$B\left(\varkappa, r_1, \frac{t}{2}\right) \cap B\left(y, r_1 \frac{t}{2}\right) = \emptyset.$$

If

$$\mathcal{Z} \in B\left(\varkappa, r_1, \frac{t}{2}\right) \cap B\left(y, r_1, \frac{t}{2}\right),$$

then, we have

$$Q\left(\varkappa - \mathcal{Z}, \frac{t}{2}\right) >_{\text{Ł}} \mathfrak{N}(r_1), \quad Q\left(y - \mathcal{Z}, \frac{t}{2}\right) >_{\text{Ł}} \mathfrak{N}(r_1).$$

Therefore,

$$\begin{aligned}\mathfrak{N}(r) &= Q(\varkappa - y, t) \\ &\geq_{Ł} T\left(Q\left(\varkappa - \mathcal{Z}, \frac{t}{2}\right), Q\left(y - \mathcal{Z}, \frac{t}{2}\right)\right) \\ &\geq_{Ł} T(\mathfrak{N}(r_1), \mathfrak{N}(r_1)) \\ &\geq_{Ł} \mathfrak{N}(r_\circ) \\ &>_{Ł} \mathfrak{N}(r),\end{aligned}$$

which is a contradiction, hence (W, Q, T) is a Hausdorff space. □

Definition 3.5. Let $\mathcal{A} \subset W$, and (W, Q, T) be an LRN-space. Then $\mathcal{A}$ is called LR-bounded if there exist $t > 0$, $r \in Ł/\{0_l, 1_l\}$ such that $Q(\varkappa - y, t) >_{Ł} \mathfrak{N}(r)$ for all $\varkappa, y \in \mathcal{A}$.

Theorem 3.6. *If (W, Q, T) is an LRN-space, and $\mathcal{A}$ is a compact subset of W, then $\mathcal{A}$ is LR-bounded.*

Proof. Assume that $\mathcal{A}$ is a compact subset of (W, Q, T). Consider an open cover $\{B(\varkappa, r, t) : \varkappa \in \mathcal{A}\}$ for all $t > 0$, $r \in Ł/\{0_l, 1_l\}$. Then, there exist $\varkappa_1, \varkappa_2, \ldots, \varkappa_n \in \mathcal{A}$ such that $\mathcal{A} \subseteq \bigcup_{\hbar=1}^{n} B(\varkappa_\hbar, r, t)$.

Suppose that $\varkappa, y \in \mathcal{A}$. Then $\varkappa \in B(\varkappa_\hbar, r, t), y \in B(\varkappa_{\mathcal{J}}, r, t),\ \hbar, \mathcal{J} \geq 1$. Thus, we have

$$Q(\varkappa - \varkappa_\hbar, t) >_{Ł} \mathfrak{N}(r),\ Q(y - \varkappa_{\mathcal{J}}, t) >_{Ł} \mathfrak{N}(r).$$

Let

$$\beta = \min\{Q(\varkappa_\hbar - \varkappa_{\mathcal{J}}, t) : 1 \leq \hbar, \mathcal{J} \leq n\}.$$

Then we get

$$\beta >_{Ł} 0_l.$$

Note that, there exist $\mathrm{s} \in Ł/\{0_l, 1_l\}$ such that

$$T^2(\mathfrak{N}(r), \beta, \mathfrak{N}(r)) >_{Ł} \mathfrak{N}(\mathrm{s}).$$

Hence, we have

$$\begin{aligned}Q(\varkappa - y, 3t) &\geq_{Ł} T^2(Q(\varkappa - \varkappa_\hbar, t), Q(\varkappa_\hbar - \varkappa_{\mathcal{J}}, t), Q(y - \varkappa_{\mathcal{J}}, t)) \\ &\geq_{Ł} T(\mathfrak{N}(r), \beta, \mathfrak{N}(r)) \\ &>_{Ł} \mathfrak{N}(\mathrm{s}).\end{aligned}$$

Taking $t' = 3t$, we get $Q(\varkappa - y, t') >_{Ł} \mathfrak{N}(\mathrm{s})$ for all $\varkappa, y \in W$. Hence, $\mathcal{A}$ is LR-bounded. □

Definition 3.7. Let (W, Q, T) be an LRN-space.

(1) Let $\{\varkappa_n\}$ be any sequence in W called as convergent to $\varkappa \in W$, if for any $\acute{\varepsilon} > 0, r \in Ł/\{0_l, 1_l\}$, there exists $\mathcal{N} \in \mathbb{Z}^+$ such that $Q(\varkappa_n - \varkappa, \acute{\varepsilon}) >_Ł \mathfrak{N}(r)$ for all $n \geq \mathcal{N}$.
(2) Let $\{\varkappa_n\}$ be any sequence in W called as Cauchy, if for any $\acute{\varepsilon} > 0$, $r \in Ł/\{0_l, 1_l\}$, there exist $\mathcal{N} \in \mathbb{Z}^+$ such that $Q(\varkappa_n - \varkappa_m, \acute{\varepsilon}) >_Ł \mathfrak{N}(r)$ for all $n \geq m \geq \mathcal{N}$.
(3) Let $\{\varkappa_n\}$ be a Cauchy sequence in LRN-space (W, Q, T). Then W is called complete, if $\varkappa_n \to \varkappa \in W$.

Theorem 3.8. *Assume that $\{\varkappa_n\}$ is a Cauchy sequence in LRN-space (W, Q, T), and it has a convergent subsequence. Then (W, Q, T) is complete.*

Proof. Suppose that $\{\varkappa_{\hbar_n}\} \subset \{\varkappa_n\}$ such that $\varkappa_{\hbar_n} \to \varkappa,\ \varkappa \in W$. We should prove that $\varkappa_n \to \varkappa$.

Let

$$\acute{\varepsilon} \in Ł/\{0_l, 1_l\}, \quad t > 0,$$

$$T(\mathfrak{N}(r), \mathfrak{N}(r)) \geq_Ł \mathfrak{N}(\acute{\varepsilon}),\ r \in Ł/\{0_l, 1_l\}.$$

Since $\{\varkappa_n\}$ is a Cauchy sequence, there exists $n_\circ \geq 1$ such that $Q(\varkappa_z - \varkappa_n, t) >_Ł \mathfrak{N}(r)$ for all $z, n \geq n_\circ$. Since $\varkappa_{\hbar_n} \to \varkappa$, there exists $\hbar_s \in \mathbb{Z}^+$ such that $\hbar_s > n_\circ$,

$$Q\left(\varkappa_{\hbar_s} - \varkappa, \frac{t}{2}\right) >_Ł \mathfrak{N}(r).$$

If $n \geq n_\circ$, then we have

$$\begin{aligned} Q(\varkappa_n - \varkappa, t) &\geq_Ł T\left(Q\left(\varkappa_n - \varkappa_{\hbar_s}, \frac{t}{2}\right), Q\left(\varkappa_{\hbar_s} - \varkappa, \frac{t}{2}\right)\right) \\ &>_Ł T(\mathfrak{N}(r), \mathfrak{N}(r)) \\ &\geq_Ł \mathfrak{N}(\acute{\varepsilon}). \end{aligned}$$

Therefore, $\varkappa_n \to \varkappa$, and (W, Q, T) is complete. □

Lemma 3.9. *Assume that (W, Q, T) is an LRN-space, for all $\varkappa, y \in W$, $t > 0$*

$$\mathcal{K}(\varkappa, y, t) = Q(\varkappa - y, t).$$

Then $\mathcal{K}$ is an LR-metric space on W, which is said to be LR-metric induced by LR-norm Q.

Proof. Let (W, Q, T) be an LRN -space and $\mathcal{K}(\varkappa, y, t) = Q(\varkappa - y, t)$.

Let $T(F, G)(\varkappa) = T(F(\varkappa), G(\varkappa))$ for all $\varkappa \in W$. Then

(1) $Q(\varkappa - y, t) = \varepsilon_o(t)$ iff $\varkappa = y, t > 0 \ \Rightarrow\ \mathcal{K}(\varkappa, y, t) = \varepsilon_o(t)$ iff $\varkappa = y$.
(2) $\mathcal{K}(\varkappa, y, t) = Q(\varkappa - y, t) = Q(y - \varkappa, t) = \mathcal{K}(y, \varkappa, t)$.

$$\begin{aligned}(3)\ \mathcal{K}(\varkappa,z,t+s)&=Q(\varkappa-z,t+s)\\&=Q(\varkappa-y+y-z,t+s)\\&\geq T(Q(\varkappa-y,t),Q(y-z,s))\\&=T(\mathcal{K}(\varkappa,y,t),\mathcal{K}(y,z,s)).\end{aligned}$$

□

Theorem 3.10. *A LR-metric $\mathcal{K}$ which is induced by a LR-norm Q has these properties:*

(1) $\mathcal{K}(\varkappa+\mathcal{Z},y+\mathcal{Z},t)=\mathcal{K}(\varkappa,y,t)$.

(2) $\mathcal{K}(\acute{\alpha}\varkappa,\acute{\alpha}y,t)=\mathcal{K}\left(\varkappa,y,\frac{t}{|\acute{\alpha}|}\right)$ *such that* $\acute{\alpha}\neq 0$, $\varkappa,y,\mathcal{Z}\in W$.

Proof. By Lemma 3.9

(1) $\mathcal{K}(\varkappa+\mathcal{Z},y+\mathcal{Z},t)=Q(\varkappa+\mathcal{Z}-y-\mathcal{Z},t)=Q(\varkappa-y,t)=\mathcal{K}(\varkappa,y,t)$.

(2) $\mathcal{K}(\acute{\alpha}\varkappa,\acute{\alpha}y,t)=Q(\acute{\alpha}\varkappa-\acute{\alpha}y,t)=Q\left(\varkappa-y,\frac{t}{|\acute{\alpha}|}\right)=\mathcal{K}\left(\varkappa,y,\frac{t}{|\acute{\alpha}|}\right)$.

□

Theorem 3.11. *Assume that (W,Q,T) is an LRN-space. Then we get:*

(1) *the mapping $(\varkappa,y)\to\varkappa+y$ is continuous.*

(2) *the mapping $(\acute{\alpha},\varkappa)\to\acute{\alpha}\varkappa$ is continuous.*

Proof. (1) Suppose that $\varkappa_n\to\varkappa, y_n\to y$, $n\to\infty$. Then, we get

$$\begin{aligned}Q\left((\varkappa_n+y_n)-(\varkappa+y),t\right)&\geq_{\text{Ł}} T\left(Q\left(\varkappa_n-\varkappa,\frac{t}{2}\right),Q\left(y_n-y,\frac{t}{2}\right)\right)\\&\to 1.\end{aligned}$$

(2) Suppose that $\varkappa_n\to\varkappa,\acute{\alpha}_n\to\acute{\alpha}$, $n\to\infty,\acute{\alpha}_n\neq 0$. Then we get

$$\begin{aligned}Q\left(\varkappa_n\acute{\alpha}_n-\acute{\alpha}\varkappa,t\right)&=Q\left(\acute{\alpha}_n\left(\varkappa_n-\varkappa\right)+\varkappa\left(\acute{\alpha}_n-\acute{\alpha}\right),t\right)\\&\geq_{\text{Ł}} T\left(Q\left(\acute{\alpha}_n\left(\varkappa_n-\varkappa\right),\frac{t}{2}\right),Q\left(\varkappa\left(\acute{\alpha}_n-\acute{\alpha}\right),\frac{t}{2}\right)\right)\\&=T\left(Q\left(\varkappa_n-\varkappa,\frac{t}{2\left|\acute{\alpha}_n\right|}\right),Q\left(\varkappa,\frac{t}{2\left|\acute{\alpha}_n-\acute{\alpha}\right|}\right)\right)\\&\to 1.\end{aligned}$$

□

Definition 3.12. The LR-Banach space (W,Q,T) is a complete LR-metric space induced by LR-norm.

Lemma 3.13. *Let (W, Q, T) be an LRN-space. And define*

$$E_{ß,Q} : W \to \mathbb{R}^{+} \cup \{0\}$$

is defined by:

$$E_{ß,Q}(\varkappa) = \inf\{t > 0 : Q(\varkappa, t) >_{Ł} \mathfrak{N}(ß)\}$$

for all $ß \in Ł/\{0_l, 1_l\}$, $\varkappa \in W$. Then, we have

(1) $E_{ß,Q}(\hat{\alpha}\varkappa) = |\hat{\alpha}| E_{ß,Q}$ *for all $\varkappa \in W$, $\hat{\alpha} \in \mathbb{R}$.*

(2) *For any $\gamma \in Ł/\{0_l, 1_l\}$, there exist $ß \in Ł/\{0_l, 1_l\}$ such that*

$$E_{\gamma,Q}(\varkappa_1 + \ldots + \varkappa_n) \leq_{Ł} E_{ß,Q}(\varkappa_1) + \ldots + E_{ß,Q}(\varkappa_n)$$

for all $\varkappa \in W$, $n \geq 1$.

(3) *Assume that $\{\varkappa_n\}$ is convergent sequence in LRN-space (W, Q, T), if and only if $E_{ß,Q}(\varkappa_n - \varkappa) \to 0$.*

Proof. (1) By definition of $E_{ß,Q}$,

$$\begin{aligned} E_{ß,Q}(\acute{\alpha}\varkappa) &= \inf\{t > 0 : Q(\acute{\alpha}\varkappa, t) >_{Ł} \mathfrak{N}(ß)\} \\ &= \inf\left\{t > 0 : Q\left(\varkappa, \frac{t}{|\acute{\alpha}|}\right) >_{Ł} \mathfrak{N}(ß)\right\} \\ &= |\acute{\alpha}| \inf\{t > 0 : Q(\varkappa, t) >_{Ł} \mathfrak{N}(ß)\} \\ &= |\acute{\alpha}| E_{ß,Q}(\varkappa). \end{aligned}$$

(2) For all $\gamma \in Ł/\{0_l, 1_l\}$, we can find $ß \in Ł/\{0_l, 1_l\}$ such that

$$T^{n-1}(\mathfrak{N}(ß), \ldots, \mathfrak{N}(ß)) \geq_{Ł} \mathfrak{N}(\gamma)$$

and

$$\begin{aligned} &Q(\varkappa_1 + \ldots + \varkappa_n, E_{ß,Q}(\varkappa_1) + \ldots + E_{ß,Q}(\varkappa_n) + n\xi) \\ &\quad \geq_{Ł} T^{n-1}(Q(\varkappa_1, E_{ß,Q}(\varkappa_1) + \xi), \ldots, Q(\varkappa_n, E_{ß,Q}(\varkappa_n) + \xi)) \\ &\quad \geq_{Ł} T^{n-1}(\mathfrak{N}(ß), \ldots, \mathfrak{N}(ß)) \\ &\quad \geq_{Ł} \mathfrak{N}(\gamma). \end{aligned}$$

This means that, for $\xi > 0$,

$$E_{\gamma,Q}(\varkappa_1 + \ldots + \varkappa_n) \leq_{Ł} E_{ß,Q}(\varkappa_1) + \ldots + E_{ß,Q}(\varkappa_n) + n\xi.$$

Since $\xi > 0$, we have for all $\varkappa \in W$, $n \geq 1$

$$E_{\gamma,Q}(\varkappa_1 + \ldots + \varkappa_n) \leq_{Ł} E_{ß,Q}(\varkappa_1) + \ldots + E_{ß,Q}(\varkappa_n).$$

(3) Since Q is continuous,

$$E_{ß,Q}(\varkappa) \in \{t > 0 : Q(\varkappa, \mathfrak{t}) >_{Ł} \mathfrak{N}(ß)\}, \ \forall \varkappa \in W, \varkappa \neq 0.$$

Therefore, we have $Q(\varkappa_n - \varkappa, \acute{\varepsilon}) \geq_{Ł} \mathfrak{N}(ß)$ if and only if $E_{ß,Q}(\varkappa_n - \varkappa) < \acute{\varepsilon}$. This completes the proof. □

Definition 3.14. Suppose that the mapping $f : W \to \Theta$ from LRN-space (W, Q, T) to LRN-space (Θ, V, T') is called uniformly continuous, if for all $r \in Ł/\{0_l, 1_l\}$, $t > 0, \varkappa, y \in W$, there exist $r_\circ \in Ł/\{0_l, 1_l\}, t_\circ > 0$ such that

$$Q(\varkappa - y, t_\circ) >_Ł \mathfrak{N}(r_\circ) \to \mathrm{V}(f(\varkappa) - f(y), t) >_Ł \mathfrak{N}(r).$$

Theorem 3.15. *If the mapping f from compact LRN-space (W, Q, T) to an LRN-space (Θ, V, T') is continuous, then f will be uniformly continuous.*

Proof. Suppose that $\mathrm{s} \in Ł/\{0_l, 1_l\}, t > 0$. Then, we can find $r \in Ł/\{0_l, 1_l\}$ such that

$$T'(\mathfrak{N}(r), \mathfrak{N}(r)) >_Ł \mathfrak{N}(\mathrm{s}).$$

Since f is continuous for any $\varkappa \in W$, there exist $r_\varkappa \in Ł/\{0_l, 1_l\}, t_\varkappa > 0$ such that

$$Q(\varkappa - y, t_\varkappa) >_Ł \mathfrak{N}(r_\varkappa) \to \mathrm{V}(f(\varkappa) - f(y), t) >_Ł \mathfrak{N}(r).$$

Since $r_\varkappa \in Ł/\{0_l, 1_l\}$, so we can find $\mathrm{s}_\varkappa < r_\varkappa$ such that

$$T(\mathfrak{N}(\mathrm{s}_\varkappa), \mathfrak{N}(\mathrm{s}_\varkappa)) >_Ł \mathfrak{N}(r_\varkappa).$$

Since W is compact, finite open covers $\left\{B\left(\varkappa, \mathrm{s}_\varkappa, \frac{t_\varkappa}{2}\right) : \varkappa \in W\right\}$ exist. That is, there exist $\varkappa_1, \varkappa_2, \ldots, \varkappa_m \in W$ such that

$$W = \cup_{\hbar=1}^{m} B\left(\varkappa_\hbar, \mathrm{s}_{\varkappa_\hbar}, \frac{t_{\varkappa_\hbar}}{2}\right), \quad \forall\, \hbar = 1, 2, \ldots, m.$$

Putting $\mathrm{s}_\circ = \min \mathrm{s}_{\varkappa_\hbar}, t_\circ = \min \frac{t_{\varkappa_\hbar}}{2}$, so if $Q(\varkappa - y, t_\circ) >_Ł \mathfrak{N}(\mathrm{s}_\circ)$ for all $\varkappa, y \in W$, then $Q\left(\varkappa - y, \frac{t_{\varkappa_\hbar}}{2}\right) >_Ł \mathfrak{N}(\mathrm{s}_{\varkappa_\hbar})$. For all $\varkappa \in W$, there exist $\varkappa_\hbar \in W$,

$$\begin{aligned} Q(y - \varkappa_\hbar, t_{\varkappa_\hbar}) &\geq_Ł T\left(Q\left(\varkappa - y, \frac{t_{\varkappa_\hbar}}{2}\right), Q\left(\varkappa - \varkappa_\hbar, \frac{t_{\varkappa_\hbar}}{2}\right)\right) \\ &\geq_Ł T(\mathfrak{N}(\mathrm{s}_{\varkappa_\hbar}), \mathfrak{N}(\mathrm{s}_{\varkappa_\hbar})) \\ &>_Ł \mathfrak{N}(r_{\varkappa_\hbar}). \end{aligned}$$

Therefore,

$$\mathrm{V}\left(f(y) - f(\varkappa_\hbar), \frac{t}{2}\right) >_Ł \mathfrak{N}(r)$$

and

$$\begin{aligned} \mathrm{V}(f(\varkappa) - f(y), t) &\geq_Ł T'\left(\mathrm{V}\left(f(\varkappa) - f(\varkappa_\hbar), \frac{t}{2}\right), \mathrm{V}\left(\mathcal{F}(y) - \mathcal{F}(\varkappa_\hbar), \frac{t}{2}\right)\right) \\ &\geq_Ł T'(\mathfrak{N}(r), \mathfrak{N}(r)) \\ &>_Ł \mathfrak{N}(\mathrm{s}). \end{aligned}$$

Thus $\mathcal{F}$ is uniformly continuous. □

Theorem 3.16. *Suppose that* (W, Q, T) *is a compact LRN-space. Then* W *is separable.*

Proof. Assume that (W, Q, T) is a compact LRN-space and $r \in Ł/\{0_l, 1_l\}$, $t > 0$. Since W is compact, there exist $\varkappa_1, \varkappa_2, \dots, \varkappa_n \in W$ such that

$$W = \cup_{\hbar=1}^{n} B(\varkappa_\hbar, r, t).$$

Let $\mathcal{A}_n \subset W$ for all $n \geq 1$, $W = \cup_{a \in \mathcal{A}_n} B\left(a, r_n, \frac{r}{n}\right)$ for all $r_n \in Ł/\{0_l, 1_l\}$.

Let $\mathcal{A} = \cup_{n \geq 1} \mathcal{A}_n$, therefore $\mathcal{A}$ is countable. Now, we should prove that $W \subset \bar{\mathcal{A}}$, if $\varkappa \in W$, then for all $n \geq 1$, there exist $a_n \in \mathcal{A}_n$ such that $\varkappa \in B\left(a_n, r_n, \frac{t}{n}\right)$. So $a_n \to \varkappa \in W$. But since $a_n \in \mathcal{A}$, $n \geq 1$, then $\varkappa \in \bar{\mathcal{A}}$, and so $\bar{\mathcal{A}} = W$. Hence W is separable. □

Definition 3.17. Suppose that $f : Z \to \Theta$ is function from $Z \neq \emptyset$ to LRN-space $(\Theta, \mathrm{V}, T\prime)$. Then $f_n \to f$ is called convergent uniformly to f, if for all $r \in Ł/\{0_l, 1_l\}$, $t > 0$, there exist $n_\circ \geq 1$ such that

$$\mathrm{V}(f_n(\varkappa) - f(\varkappa), t) >_{Ł} \mathfrak{N}(r), \ \ \forall\, n \geq n_\circ.$$

Definition 3.18. Family $\mathcal{F}$ of functions from an LRN-space (W, Q, T) to a complete LRN-space $(\Theta, \mathrm{V}, T\prime)$ is called equicontinuous for all $r \in Ł/\{0_\ell, 1_\ell\}$, $t > 0$, there exist $r_\circ \in Ł/\{0_l, 1_l\}, t_\circ > 0$ such that

$$Q(\varkappa - y, t_\circ) >_{Ł} \mathfrak{N}(r_\circ) \to \mathrm{V}(f(\varkappa) - f(y), t) >_{Ł} \mathfrak{N}(r)$$

for all $f \in \mathcal{F}$.

Theorem 3.19. *Suppose that* $\{f_n\}$ *is an equicontinuous sequence of functions from an LRN-space* (W, Q, T) *to a complete LRN-space* (Θ, V, T')*. If* $\{f_n\}$ *converges to each point of a dense subset* $\mathcal{A}$ *of* W*, then* $\{f_n\}$ *converges to each point of* W *and the limit function become continuous.*

Proof. Assume that we have $\mathrm{s} \in Ł/\{0_l, 1_l\}$, $t > 0$. Then we can find $r \in Ł/\{0_l, 1_l\}$ such that

$$T'^2(\mathfrak{N}(r), \mathfrak{N}(r), \mathfrak{N}(r)) >_{Ł} \mathfrak{N}(\mathrm{s}).$$

Since $\mathcal{F} = \{f_n\}$ is an equicontinuous family for all $r \in Ł/\{0_l, 1_l\}$, $t > 0$, there exist $r_1 \in Ł/\{0_l, 1_l\}$, $t_1 > 0$ such that for all $\varkappa, y \in W$,

$$Q(\varkappa - y, t_1) >_{Ł} \mathfrak{N}(r_1) \to \mathrm{V}\left(f_n(\varkappa) - f_n(y), \frac{t}{3}\right) >_{Ł} \mathfrak{N}(r)$$

for all $f_n \in \mathcal{F}$. Since $\bar{A} = W$, there exist $y \in B(\varkappa, r_1, t_1) \cap \mathcal{A}$, and to prove $f_n(y) \to f(y)$, we should prove $\{f_n(y)\}$ is a Cauchy sequence for all $r \in L/\{0_l, 1_l\}$, $t > 0$, there exist $n_\circ \geq 1$ such that

$$\mathrm{V}\left(f_n(y) - f_m(y), \frac{t}{3}\right) >_{Ł} \mathfrak{N}(r), \ \ \forall\, n, m \geq n_\circ$$

and

$$\begin{aligned}\mathrm{V}\left(f_n\left(\varkappa\right)-f_m\left(\varkappa\right),t\right) >_{\text{Ł}} T'^2\Bigg(&\mathrm{V}\left(f_n\left(\varkappa\right)-f\left(y\right),\frac{t}{3}\right),\\ &\mathrm{V}\left(f_n\left(y\right)-f_m\left(y\right),\frac{t}{3}\right),\mathrm{V}\left(f_n\left(\varkappa\right)-f_n\left(y\right),\frac{t}{3}\right)\Bigg)\\ &\geq_{\text{Ł}} T'^2\left(\mathfrak{N}\left(r\right),\mathfrak{N}\left(r\right),\mathfrak{N}\left(r\right)\right)\\ &>_{\text{Ł}} \mathfrak{N}\left(\mathrm{s}\right).\end{aligned}$$

Thus, $\{f_n\left(\varkappa\right)\}$ is a Cauchy sequence in Θ, then $\{f_n\left(\varkappa\right)\}$ is convergent sequence, since (Θ,V,T') is complete LRN- space.

We submit f is continuous. Suppose that $\mathrm{s}_\circ\in \text{Ł}/\left\{0_l,1_l\right\}$, $t_\circ>0$. Then we can find $r_\circ\in \text{Ł}/\left\{0_l,1_l\right\}$, $\mathrm{s}_\circ>r_\circ$ such that

$$T'^2\left(\mathfrak{N}\left(r_\circ\right),\mathfrak{N}\left(r_\circ\right),\mathfrak{N}\left(r_\circ\right)\right) >_{\text{Ł}} \mathfrak{N}\left(\mathrm{s}_\circ\right).$$

Since $\mathcal{F}$ is equicontinuous, for any $r_\circ\in \text{Ł}/\left\{0_l,1_l\right\}$, $t_\circ>0$, there exist $r_2\in \text{Ł}/\left\{0_l,1_l\right\}$, $t_2>0$ such that

$$Q\left(\varkappa-y,t_2\right) >_{\text{Ł}} \mathfrak{N}\left(r_2\right)\to \mathrm{V}\left(f_n\left(\varkappa\right)-f_n\left(y\right),\frac{t_\circ}{3}\right) >_{\text{Ł}} \mathfrak{N}\left(r_\circ\right)$$

for all $f_n\in\mathcal{F}$. Since $f_n\left(\varkappa\right)\to f\left(\varkappa\right)$ for all $r_\circ\in \text{Ł}/\left\{0_l,1_l\right\}$, $t_\circ>0$, there exist $n_1\geq 1$ such that

$$\mathrm{V}\left(f_n\left(\varkappa\right)-f\left(\varkappa\right),\frac{t_\circ}{3}\right) >_{\text{Ł}} \mathfrak{N}\left(r_\circ\right).$$

Also, since $f_n\left(y\right)\to f\left(y\right)$ for all $r_\circ\in \text{Ł}/\left\{0_l,1_l\right\}$, $t>0$, there exist $n_2\geq 1$ such that

$$\mathrm{V}\left(f_n\left(y\right)-f_n\left(y\right),\frac{t_\circ}{3}\right) >_{\text{Ł}} \mathfrak{N}\left(r_\circ\right)$$

for all $n\geq n_\circ$. Now, for all $n\geq \max\left\{n_1,n_2\right\}$, we get

$$\begin{aligned}\mathrm{V}\left(f\left(\varkappa\right)-f\left(y\right),t_\circ\right) \geq_{\text{Ł}} T'^2\Bigg(&\mathrm{V}\left(f\left(\varkappa\right)-f_n\left(\varkappa\right),\frac{t_\circ}{3}\right),\mathrm{V}\left(f_n\left(\varkappa\right)-f_n\left(y\right),\frac{t_\circ}{3}\right),\\ &\mathrm{V}\left(f_n\left(y\right)-f\left(y\right),\frac{t_\circ}{3}\right)\Bigg)\\ &\geq_{\text{Ł}} T'^2\left(\mathfrak{N}\left(r_\circ\right),\mathfrak{N}\left(r_\circ\right),N\left(r_\circ\right)\right)\\ &>_{\text{Ł}} \mathfrak{N}\left(\mathrm{s}_\circ\right).\end{aligned}$$

Hence, f is continuous. □

Theorem 3.20. *Assume that $\mathcal{A}\subset\mathbb{R}$ is LR-bounded in $(\mathbb{R},Q,T)$ if and only if it is bounded in $\mathbb{R}$.*

Proof. Suppose that $\mathcal{A} \subset \mathbb{R}$ is LR-bounded in $(\mathbb{R}, Q, T)$. Then there exist $r_\circ \in Ł/\{0_l, 1_l\}$, $t_\circ > 0$ such that

$$Q(\varkappa, t_\circ) >_Ł \mathfrak{N}(r_\circ), \quad \forall\, \varkappa \in \mathcal{A}.$$

Thus, we get

$$t_\circ \geq E_{r_\circ,Q}(\varkappa) = |\varkappa|\, E_{r_\circ,Q}(1).$$

So, $E_{r_\circ,Q}(1) \neq 0$. If we put $\delta = \frac{t_\circ}{E_{r_\circ,Q}(1)}$, then we have $|\varkappa| \leq \delta$ for all $\varkappa \in \mathcal{A}$. Hence, $\mathcal{A}$ is bounded in $\mathbb{R}$.

Conversely, suppose that $\mathcal{A} \subset \mathbb{R}$, which is bounded, we claim that $\mathcal{A}$ is LR-bounded in $(\mathbb{R}, Q, T)$. Then $|\varkappa| \leq \delta$ for all $\varkappa \in \mathcal{A}$. From

$$t_\circ \geq |\varkappa|\, E_{r_\circ,Q}(1) = E_{r_\circ,Q}(\varkappa),$$

we have $Q(\varkappa, t_\circ) >_Ł \mathfrak{N}(r_\circ)$. Hence, $\mathcal{A}$ is LR-bounded in $(\mathbb{R}, Q, T)$. □

Theorem 3.21. *Assume that* $\{\zeta_n\}$ *is sequence in an LRN-space* $(\mathbb{R}, Q, T)$. *Then* $\{\zeta_n\}$ *is convergent if and only if* $\{\zeta_n\}$ *is convergent in* $(\mathbb{R}, |\cdot|)$.

Proof. Suppose that $\zeta_n \to \zeta$ in $\mathbb{R}$. Then by Lemma 3.13 (1), we have

$$E_{ß,Q}(\zeta_n - \zeta) = |\zeta_n - \zeta|\, E_{ß,Q}(1) \xrightarrow{P} 0.$$

Thus by Lemma 3.13 (3), $\zeta_n \to \zeta$.

Conversely, suppose that $\zeta_n \to \zeta$, by Lemma 3.13,

$$\lim_{n\to+\infty} |\zeta_n - \zeta|\, E_{\zeta,Q}(1) = \lim_{n\to+\infty} E_{\zeta,Q}(\zeta_n - \zeta) = 0.$$

Since $E_{\zeta,Q}(1) \neq 0$, $\lim_{n\to+\infty}(\zeta_n - \zeta) = 0$. Hence, we have $\zeta_n \to \zeta$ in $\mathbb{R}$. □

4. Conclusion

We discussed the topological structure of an LRN-space, and we are trying to present the results related to the topological isomorphism, also we want to generalized the results to other spaces, such as lattice random Banach Algebra, lattice random para normed space, etc.

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