**Nonlinear Functional Analysis and Applications**
**Vol. 18, No. 4 (2013), pp. 453-472**

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm


# EXTENSION OF SOME WELL-KNOWN POLYNOMIAL INEQUALITIES

**Irshad Ahmad[1], A. Liman[2] and W. M. Shah[3]**

[1]Department of Mathematics, National Institute of Technology
Kashmir, India-190006
e-mail: irshadmaths84@rediffmail.com

[2]Department of Mathematics, National Institute of Technology
Kashmir, India-190006
e-mail: abliman22@rediffmail.com

[3]Jammu and Kashmir Institute of Mathematical Sciences
Srinagar, India-190008
e-mail: wmshah@rediffmail.com

**Abstract.** Let $P(z)$ be a polynomial of degree $n$ and $P^{'}(z)$ its derivative. In this paper we extend some well-known polynomial inequalities to operator $B$, which carries $P$ into

$$B[P(z)] = \lambda_0 P(z) + \lambda_1 \frac{nz}{2}\frac{P^{'}(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!},$$

where $\lambda_0$, $\lambda_1$ and $\lambda_2$ are real or complex numbers such that all the zeros of

$$\mathcal{U}(z) := \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2,\ C(n,r) = \frac{n!}{r!(n-r)!},$$

lie in the half plane

$$|z| \le \left|z - \frac{n}{2}\right|$$

and therefore obtain generalizations of these.

## 1. Introduction and Preliminaries

Let $P_n$ be the space of polynomials $\sum_{j=0}^{n} a_j z^j$ of degree at most $n$. If $P \in P_n$, then according to a famous result known as Bernstein's inequality ([9,11,12])

$$\max_{|z|=1} |P^{'}(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

[0]Received April 23, 2013. Revised October 7, 2013.
[0]2000 Mathematics Subject Classification: 30A06, 30A64, 30E10.
[0]Keywords: Polynomials, inequalities in the complex domain, B-operator, zeros.

Also concerning the estimate of the maximum of $|P(z)|$ on a large circle $|z| = R > 1$, we have

$$\max_{|z|=R>1} |P(z)| \le R^n \max_{|z|=1} |P(z)| \tag{1.2}$$

([for refrence see [9, p.158 Problem 269] or [10, p.346]). Inequalities (1.1) and (1.2) are sharp and equality holds for $P(z) = \lambda z^n, \lambda \neq 0$.

For the class of polynomials $P \in P_n$, which does not vanish inside unit disk, the inequality (1.1) and (1.2) can be replaced by

$$\max_{|z|=1} |P^{'}(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\max_{|z|=R>1} |P(z)| \le \frac{R^n+1}{2} \max_{|z|=1} |P(z)|, \tag{1.4}$$

respectively. Equality in (1.3) and (1.4) holds for $P(z) = \lambda z^n + \mu, |\lambda| = |\mu| = 1$. Inequality (1.3) was conjuctured by Erdös and later on verified by Lax [6]. Ankeny and Rivilin [1] used (1.3) to prove (1.4).

Inequalities (1.3) and (1.4) were improved by Aziz and Dawood [2], who with the same hypothesis proved

$$\max_{|z|=1} |P^{'}(z)| \le \frac{n}{2} \left[ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right] \tag{1.5}$$

and

$$\max_{|z|=R>1} |P(z)| \le \frac{R^n+1}{2} \max_{|z|=1} |P(z)| - \frac{R^n-1}{2} \min_{|z|=1} |P(z)|. \tag{1.6}$$

Inequalities (1.3) and (1.4) were generalized by Jain [4] by proving that, if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number $\beta$ with $|\beta| \le 1$, $|z| = 1$ and $R \ge 1$,

$$\left| zP^{'}(z) + \frac{n\beta}{2} P(z) \right| \le \frac{n}{2} \left\{ |1 + \frac{\beta}{2}| + |\frac{\beta}{2}| \right\} \max_{|z|=R>1} |P(z)| \tag{1.7}$$

and

$$\begin{aligned} &\left| P(Rz) + \beta \left( \frac{R+1}{2} \right)^n P(z) \right| \\ &\le \frac{1}{2} \left\{ \left| R^n + \beta \left( \frac{R+1}{2} \right)^n \right| + \left| 1 + \beta \left( \frac{R+1}{2} \right)^n \right| \right\} \max_{|z|=R>1} |P(z)|. \end{aligned} \tag{1.8}$$

Concerning to minimum modulus of polynomials Jain [5] proved that, if $P(z)$ has all the zeros in $|z| \le 1$, then for every real or complex number $\beta$ with $|\beta \le 1$,

$$\min_{|z|=1} \left| zP^{'}(z) + \frac{n\beta}{2} P(z) \right| \ge n \left| 1 + \frac{\beta}{2} \right| \min_{|z|=1} |P(z)|. \tag{1.9}$$

Jain [5] refines inequalities (1.7) and (1.8) and proved that, if $P \in P_n$ and has all the zeros in $|z| \geq 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1$,

$$\begin{aligned} &\left|zP^{'}(z) + \frac{n\beta}{2}P(z)\right| \\ &\leq \frac{n}{2}\left[\left\{|1+\frac{\beta}{2}| + |\frac{\beta}{2}|\right\}\max_{|z|=1}|P(z)| - \left\{|1+\frac{\beta}{2}| - |\frac{\beta}{2}|\right\}\min_{|z|=1}|P(z)|\right] \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &\max_{|z|=1}\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| \\ &\leq \frac{1}{2}\left[\left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|\right\}\max_{|z|=1}|P(z)| \right. \\ &\quad \left. - \left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| - \left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|\right\}\min_{|z|=1}|P(z)|\right]. \end{aligned} \tag{1.11}$$

Inequalities (1.6) and (1.7) were generalized recently by Mazerji, Baseri and Zireh [8], which also leads to a refinement of (1.8).

**Theorem 1.1.** *If $P \in P_n$ having all the zeros in $|z| \geq k \geq 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned} &\max_{|z|=1}\left|P(Rk^2z) + \beta\left(\frac{Rk+1}{k+1}\right)^n P(k^2z)\right| \\ &\leq \frac{1}{2}\left[\left\{k^n\left|R^n + \beta\left(\frac{Rk+1}{k+1}\right)^n\right| + \left|1 + \beta\left(\frac{Rk+1}{k+1}\right)^n\right|\right\}\max_{|z|=k}|P(z)| \right. \\ &\quad \left. - \left\{\left|R^n + \beta\left(\frac{Rk+1}{k+1}\right)^n\right| - \left|1 + \beta\left(\frac{Rk+1}{k+1}\right)^n\right|\right\}\min_{|z|=k}|P(z)|\right]. \end{aligned} \tag{1.12}$$

In this paper, we consider an operator $B$ which carries $P \in P_n$ into

$$B[P](z) = \lambda_0 P(z) + \lambda_1 \frac{nz}{2}\frac{P^{'}(z)}{1!} + \lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!}, \tag{1.13}$$

where $\lambda_0$, $\lambda_1$ and $\lambda_2$ are real or complex numbers such that all the zeros of

$$\mathcal{U}(z) := \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2,\ C(n,r) = \frac{n!}{r!(n-r)!}, \tag{1.14}$$

lie in the half plane

$$|z| \leq \left|z - \frac{n}{2}\right|, \tag{1.15}$$

and extend some polynomial inequalities have been recently developed.

## 2. Lemmas

For the proof of these theorems we require the following lemmas. First lemma is due to Aziz and Zargar [3].

**Lemma 2.1.** *If $P(z)$ is a polynomial of degree $n$ having all zeros in $|z| \le k \le 1$, then for $R \ge 1$ and $|z| = 1$*

$$|P(Rz)| \ge \left(\frac{R+k}{k+1}\right)^n |P(z)|.$$

Next Lemma follows from Corollary 18.3 of [7].

**Lemma 2.2.** *If $P(z)$ is a polynomial of degree $n$, having all zeros in the disk $|z| \le k, k \ge 0$, then all zeros of $B[P](z)$ lie in $|z| \le k$.*

**Lemma 2.3.** *Let $P(z)$ and $G(z)$ be two polynomials such that the degree of $P(z)$ does not exceed to that of $G(z)$. If $G(z)$ has all its zeros in $|z| \le k, k \ge 0$ and $|P(z)| \le |G(z)|$ for $|z| = k$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \le 1, |\beta| \le 1$ and $R > k$, we have for $|z| \ge 1$,*

$$\begin{aligned}&\left|B[P](Rz) - \alpha B[P](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[P](z)\right|\\ &\le \left|B[G](Rz) - \alpha B[G](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[G](z)\right|.\end{aligned} \tag{2.1}$$

*Proof.* Since $|P(z)| \le |G(z)|$ for $|z| = k$, and $G(z)$ has all the zeros in $|z| \le k$. Using Rouche's theorem, it follows that

$$F(z) = G(z) + \lambda P(z)$$

has all the zeros in $|z| \le k, |\lambda| < 1$. Applying Lemma 2.1 to $F(z)$, we have for $|z| = 1, R > k$,

$$|F(Rz)| \ge \left(\frac{R+k}{1+k}\right)^n |F(z)|.$$

Therefore, for any $\alpha$ with $|\alpha| \le 1$, we have for $|z| = 1$,

$$|F(Rz) - \alpha F(z)| \ge |F(Rz)| - |\alpha||F(z)| \ge \left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}|F(z)|.$$

Equivalently

$$|F(Rz) - \alpha F(z)| \ge \left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}|F(z)|. \tag{2.2}$$

Since $F(Rz)$ has all its zeros in $|z| < 1$. Again by Rouche's Theorem, it follows from (2.2) that for real or complex numbers $\alpha$, $\beta$ with $|\alpha| \le 1$, $|\beta| \le 1$ and

$R > k$, that all zeros of polynomial

$$F(Rz) - \alpha F(z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}F(z)$$

lie in $|z| < 1$. Applying Lemma 2.2, we have

$$\begin{aligned} S(z) &= B\left[F(Rz) - \alpha F(z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}F(z)\right] \\ &= B[F(Rz)] - \alpha B[F(z)] + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[F(z)] \end{aligned}$$

has all the zeros in $|z| < 1$, for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > k$. We conclude that all the zeros of the polynomial

$$\begin{aligned} S(z) = {} & B[G(Rz)] + \lambda B[P(Rz)] - \alpha(B[G(z)] + \lambda B[P(z)]) \\ & + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}(B[G(rz)] + \lambda B[P(z)])B[P](Rz) \\ & - \alpha B[G](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[G](z) \\ & + \lambda\left[B[P](Rz) - \alpha B[P](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[P](z)\right] \end{aligned} \tag{2.3}$$

lie in $|z| < 1$ for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > k$. This gives

$$\begin{aligned} & \Bigg|B[P](Rz) - \alpha B[P](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[P](z)| \\ & \leq \left|B[G](Rz) - \alpha B[G](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[G](z)\right|, \end{aligned} \tag{2.4}$$

for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > k$. If inequality (2.4) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\begin{aligned} & \left|B[P](Rz_0) - \alpha B[P](z_0) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[P](z_0)\right| \\ & > \left|B[G](Rz_0) - \alpha B[G](z_0) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[G](z_0)\right|. \end{aligned}$$

We take

$$\lambda = -\frac{B[G](Rz_0) - \alpha B[G](z_0) + \beta\{(\frac{R+k}{1+k})^n - |\alpha|\}B[G](z_0)}{B[P](Rz_0) - \alpha B[P](z_0) + \beta\{(\frac{R+k}{1+k})^n - |\alpha|\}B[P](z_0)},$$

so that $\lambda$ is a well defined real or complex number with $|\lambda| < 1$ and with this choice of $\lambda$, from (2.3), we get contradiction to the fact that all the zeros of

$S(z)$ lie in $|z| < 1$. Thus for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > k$,

$$\left|B[P](Rz) - \alpha B[P](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[P](z)\right|$$
$$\leq \left|B[G](Rz) - \alpha B[G](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[G](z)\right|.$$

Hence the result. □

**Lemma 2.4.** *If $P(z)$ is a polynomial of degree $n$, having no zeros in the disk $|z| < k$, $k \leq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > k$ and $|z| \geq 1$*

$$\left|B[P](Rz) - \alpha B[P](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[P](z)\right|$$
$$\leq \left|B[Q](Rz) - \alpha B[Q](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[Q](z)\right|,$$

*where* $Q(z) = \left(\frac{z}{k}\right)^n \overline{P\left(\frac{k^2}{\bar{z}}\right)}$.

*Proof.* Since $Q(z) = \left(\frac{z}{k}\right)^n \overline{P\left(\frac{k^2}{\bar{z}}\right)}$, obviously $Q(z)$ has all the zeros in $|z| \leq k$. Also $|P(z)| = |Q(z)|$ for $|z| = k$. Using Lemma 2.3, we get

$$\left|B[P](Rz) - \alpha B[P](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[P](z)\right|$$
$$\leq \left|B[Q](Rz) - \alpha B[Q](z) + \beta\left\{\left(\frac{R+k}{1+k}\right)^n - |\alpha|\right\}B[Q](z)\right|.$$

□

**Lemma 2.5.** *If $P \in P_n$ does not vanish in $|z| < k, k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$, and $|z| \geq 1$,*

$$\left|B[P](Rk^2z) - \alpha B[P](k^2z) + \beta\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\}B[P](k^2z)\right|$$
$$\leq k^n|B[Q](Rz) - \alpha B[Q](z) + \beta\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\}B[Q](z)|\Bigg|, \tag{2.5}$$

*where* $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.

*Proof.* Since $P(z)$ does not vanish in $|z| < k, k \geq 1$, therefore $Q(z)$ having all its zros in $|z| < \frac{1}{k} \leq 1$. As $k^n|Q(z)| = |P(k^2z)|$ for $|z| = \frac{1}{k}$. Applying Lemma

2.3 with $G(z)$ replaced by $k^nQ(z)$, we get for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > k \geq 1$ and $|z| \geq 1$,

$$\left|B[P](Rk^2z) - \alpha B[P](k^2z) + \beta\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\}B[P](k^2z)\right|$$
$$\leq k^n|B[Q](Rz) - \alpha B[Q](z) + \beta\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\}B[Q](z)\Big|.$$

Hence the result. $\square$

**Lemma 2.6.** *If $P \in P_n$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1, k \geq 1$ and $|z| \geq 1$,*

$$\left|B[P](Rk^2z) - \alpha B[P](k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}B[P](k^2z)\right|$$
$$+ k^n\left|B[Q](Rz) - \alpha B[Q](z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}B[Q](z)\right|$$
$$\leq \left[k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||B[z^n]|\right.$$
$$\left. + \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\left||\lambda_0|\right|\right] \max_{|z|=k}|P(z)|,$$

*where $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.*

*Proof.* Let $M = \max_{|z|=k}|P(z)|$, then $|P(z)| \leq M$ for $|z| = k$. Therefore by Rouche's Theorem, the polynomial

$$F(z) = P(z) - \zeta M$$

does not vanish in $|z| < k$, for real or complex numbers $\zeta$ such that $|\zeta| > 1$. Applying Lemma 2.5 to the polynomial $F(z)$, we get for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$, and $|z| \geq 1$,

$$\left|B[F(Rk^2z)] - \alpha B[F(k^2z)] + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}B[F(k^2z)]\right|$$
$$\leq k^n\left|B[G(Rz)] - \alpha B[G(z)] + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}B[G(z)]\right|,$$

where

$$G(z) = z^n\overline{F\left(\frac{1}{\bar{z}}\right)} = Q(z) - \bar{\zeta}Mz^n.$$

This implies,

$$\begin{aligned}&\Big|B[P(Rk^2z)-\zeta M]-\alpha B[P(k^2z)-\zeta M]\\&\quad+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[P(k^2z)-\zeta M]\Big|\\&\le k^n\Big|B[Q(Rz)-\zeta MR^nz^n]-\alpha B[Q(z)-\zeta Mz^n]\\&\quad+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[Q(z)-\zeta Mz^n]\Big|.\end{aligned}$$

Since $B$ is a linear operator, we have

$$\begin{aligned}&\Big|\Big\{B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[P(k^2z)]\Big\}\\&\quad-\zeta M\Big\{1-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big\}\lambda_0\Big|\\&\le k^n\Big|B[Q(Rz)]-\alpha B[Q(z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[Q(z)]\\&\quad-\bar{\zeta}M\Big\{R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big\}B[z^n]\Big|.\end{aligned}\tag{2.6}$$

Choosing arguement of $\zeta$ on the right hand side of inequality (2.6) such that

$$\begin{aligned}&k^n\Big|B[Q(Rz)]-\alpha B[Q(z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[Q(z)]\\&\quad-\bar{\zeta}M\Big\{R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big\}B[z^n]\Big|\\&=k^n|\bar{\zeta}|M\Big\{R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big\}||B[z^n]|\\&\quad-k^n\Big|B[Q(Rz)]|-\alpha B[Q(z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[Q(z)]\Big|,\end{aligned}$$

which is possible by Lemma 2.3., we get

$$\begin{aligned}&\bigg|\bigg\{B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}B[P(k^2z)]\bigg\}\bigg|\\&\quad-\bigg|\zeta M\bigg\{1-\alpha+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}\bigg\}\lambda_0\bigg|\\&\leq k^n|\bar{\zeta}|M\bigg|\{R^n-\alpha+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}\bigg\}\bigg||B[z^n]|\\&\quad-k^n\bigg|B[Q(Rz)]-\alpha B[Q(z)]+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}B[Q(z)]\bigg|.\end{aligned}$$

This gives,

$$\begin{aligned}&\bigg|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}B[P(k^2z)]\bigg|\\&\quad+k^n\bigg|B[Q(Rz)]-\alpha B[Q(z)]+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}B[Q(z)]\bigg|\\&\leq|\zeta|\left[k^n\bigg|R^n-\alpha+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}\bigg||B[z^n]|\right.\\&\quad\left.+\bigg|1-\alpha+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}\bigg||\lambda_0|\right]\max_{|z|=k}|P(z)|,\end{aligned}$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. Letting $|\zeta|\to 1$, we get the desired result. □

## 3. Statement and Proof of Main Results

**Theorem 3.1.** *If $P\in P_n$ has all its zeros in $|z|\leq k$, $k\geq 1$, then for real or complex numbers $\alpha,\beta$ with $|\alpha|\leq 1,|\beta|\leq 1$ and $R>1$,*

$$\begin{aligned}&\min_{|z|=1}\bigg|B[P](Rk^2z)-\alpha B[P](k^2z)+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}B[P](k^2z)\bigg|\\&\geq k^n\bigg|R^n-\alpha+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}\bigg|\bigg|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\bigg|\min_{|z|=k}|P(z)|.\end{aligned}\tag{3.1}$$

*Equivalently*

$$\begin{aligned}&\min_{|z|=1}\bigg|\lambda_0\bigg(P(Rk^2z)-\alpha P(k^2z)+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}P(k^2z)\bigg)\\&\quad+\lambda_1\frac{nz}{2}k^2\bigg(RP^{'}(Rk^2z)-\alpha P^{'}(k^2z)+\beta\bigg\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\bigg\}P^{'}(k^2z)\bigg)\end{aligned}$$

$$+\lambda_2\left(\frac{nz}{2}\right)^2k^4\frac{(R^2P''(Rk^2z)-\alpha P''(k^2z)+\beta\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\})}{2!}\Bigg|$$

$$\geq k^n\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\left|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\right|\min_{|z|=k}|P(z)|. \tag{3.2}$$

*Proof.* Let $m=\min_{|z|=k}|P(z)|$, the result is obvious if $P(z)$ has a zero on $|z|=k, k\geq 1$. We assume that $P(z)$ has all zeros in $|z|<k, k\geq 1$, therefore $P(k^2z)$ has all the zeros in $|z|<\frac{1}{k}\leq 1$ and $m=\min_{|z|=\frac{1}{k}}|P(k^2z)|>0$. This gives for $|z|=\frac{1}{k}$

$$m\leq |P(k^2z)|.$$

By Rouche's Theorem it follows that for every real or complex number $\lambda$ such that $|\lambda|<1$

$$F(z)=P(k^2z)-m\lambda k^nz^n$$

has all its zeros in $|z|<\frac{1}{k}\leq 1$. Applying Lemma 2.1 to the polynomial $F(z)$, we get for $|z|=1$

$$|F(Rz)|\geq\left(\frac{R+k}{k+1}\right)^n|F(z)|.$$

This implies, for $R>1$ and $|z|=1$

$$|F(Rz)|>|F(z)|.$$

Thus by Rouche's Theorem for every real or complex number $\alpha$ with $|\alpha|\leq 1$, all the zeros of

$$G(z):=F(Rz)-\alpha F(z)$$

lie in $|z|<1$. Therefore for $|z|=1$ and $R>1$

$$|F(Rz)-\alpha F(z)|\geq |F(Rz)|-|\alpha F(z)|>\left\{\left(\frac{R+k}{1+k}\right)^n-|\alpha|\right\}|F(z)|.$$

Once again by Rouche's Theorem, for every real or complex number $\beta$ with $|\beta|\leq 1$, the zeros of the polynomial

$$\begin{aligned}H(z):&=F(Rz)-\alpha F(z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}F(z)\\&=\left\{P(Rk^2z)-\alpha P(k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}P(k^2z)\right\}\\&\quad-\lambda k^n\left\{R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right\}z^n\min_{|z|=k}|P(z)|\end{aligned}$$

lie in $|z|<1$. Therefore by Lemma 2.2, all the zeros of $B[H(z)]$ lie in $|z|<1$, that is, all the zeros of

$$\Big\{B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[P(k^2z)]\Big\}$$
$$-\lambda k^n\Big\{R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big\}B[z^n]\min_{|z|=k}|P(z)|,$$

lie in $|z|<1$. This gives for $|z|\geq 1$

$$\min_{|z|=1}\Big|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[P(k^2z)]\Big|$$
$$\geq k^n\Big|R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big||\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}|\min_{|z|=k}|P(z)|. \tag{3.3}$$

If inequality (3.3) is not true, then there is a point $z=z_0$ with $|z_0|\geq 1$ such that

$$\min_{|z|=1}\Big|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[P(k^2z)]\Big|$$
$$<k^n\Big|R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big||\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}|\min_{|z|=k}|P(z)|.$$

We take

$$\lambda=\frac{\{B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\}B[P(k^2z)]\}_{z=z_0}}{mk^n(R^n-\alpha+\beta\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\})(\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8})_{z=z_0}},$$

then $|\lambda|<1$ and with this choice of $\lambda$, we have $B[H(z_0)]=0$, with $|z_0|\geq 1$, which is a contradiction, since all the zeros of $B[H(z)]$ lie in $|z|<1$. Thus for $|z|\geq 1, R>1$

$$\min_{|z|=1}\Big|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[P(k^2z)]\Big|$$
$$\geq k^n\Big|R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\Big||\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}|\min_{|z|=k}|P(z)|.$$

This completes the proof the theorem. □

Now taking $\lambda_1=\lambda_2=0$ in (3.2) and noting that all zeros of $\mathcal{U}(z)$ defined by (1.14) lie in the half plane (1.15) we get the following intresting result:

**Corollary 3.2.** *If $P \in P_n$ has all its zeros in $|z| \leq k,\ k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$,*

$$\begin{aligned} &\min_{|z|=1}\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \\ &\geq k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||z^n|\min_{|z|=k}|P(z)|. \end{aligned} \tag{3.4}$$

The result is best possible as shown by polynomial $P(z) = az^n, a \neq 0$.

If we divide both sides of the inequality (3.4) by $R-1$ with $\alpha = 1$ and make $R \to 1$, we get

**Corollary 3.3.** *If $P \in P_n$ has all its zeros in $|z| \leq k, k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$,*

$$\min_{|z|=1}\left|kzP^{'}(k^2z) + \frac{n\beta}{k+1}P(k^2z)\right| \geq nk^{n-1}\left|1 + \frac{n\beta k}{k+1}\right|\min_{|z|=k}|P(z)|. \tag{3.5}$$

**Remark 3.4.** For $k = 1$, inequality (3.5) reduces to inequality (1.9).

Setting $\beta = 0$ in inequality (3.4), we get:

**Corollary 3.5.** *If $P \in P_n$ has all its zeros in $|z| \leq k, k \geq 1$, then for every real or complex number $\alpha$ with $|\alpha| \leq 1$ and $R > 1$,*

$$\min_{|z|=1}|P^{'}(Rk^2z) - \alpha P(k^2z)| \geq k^n|R^n - \alpha|\min_{|z|=k}|P(z)|.$$

**Theorem 3.6.** *If $P \in P_n$ does not vanish in $|z| < k, k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,*

$$\begin{aligned} &\left|B[P](Rk^2z) - \alpha B[P](k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}B[P](k^2z)\right| \\ &\leq \frac{1}{2}\left[|\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right. \\ &\quad \left. + k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||B[z^n]|\right]\max_{|z|=k}|P(z)|. \end{aligned}$$

*Equivalently*

$$\begin{aligned}
&\min_{|z|=1}\Bigg|\lambda_0\Bigg(P(Rk^2z)-\alpha P(k^2z)+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}P(k^2z)\Bigg)\\
&\quad+\lambda_1\frac{nz}{2}k^2\Bigg(RP^{'}(Rk^2z)-\alpha P^{'}(k^2z)+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}P^{'}(k^2z)\Bigg)\\
&\quad+\lambda_2\Bigg(\frac{nz}{2}\Bigg)^2k^4\frac{(R^2P^{''}(Rk^2z)-\alpha P^{''}(k^2z)+\beta\{\big(\frac{Rk+1}{k+1}\big)^n-|\alpha|\})}{2!}\Bigg|\\
&\leq\frac{1}{2}\Bigg[|\lambda_0|\Bigg|1-\alpha+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}\Bigg|\\
&\quad+k^n\Bigg|R^n-\alpha+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}\Bigg|\\
&\quad\times\Bigg|\lambda_0z^n+\lambda_1\frac{nz}{2}\frac{nz^{n-1}}{1!}+\lambda_2\left(\frac{nz}{2}\right)^2\frac{n(n-1)z^{n-2}}{2!}\Bigg|\Bigg]\max_{|z|=k}|P(z)|.
\end{aligned}\tag{3.6}$$

*Proof.* Since $P(z)$ does not vanish in $|z|<k, k\geq 1$, therefore by Lemma 2.5, we have

$$\begin{aligned}
&\Bigg|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}B[P(k^2z)]\Bigg|\\
&\leq k^n\Bigg|B[Q(Rz)]-\alpha[Q(z)]+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}B[Q(z)]\Bigg|
\end{aligned}\tag{3.7}$$

for real or complex numbers $\alpha, \beta$ with $|\alpha|\leq 1, |\beta|\leq 1$, $R>1$ and $|z|\geq 1$, where $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. Above inequality (3.7) in conjuction with Lemma 2.6 gives for real or complex numbers $\alpha, \beta$ with $|\alpha|\leq 1, |\beta|\leq 1$, $R>1$ and $|z|\geq 1$

$$\begin{aligned}
&2\Bigg|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}B[P(k^2z)]\Bigg|\\
&\leq\Bigg|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}B[P(k^2z)]\Bigg|\\
&\quad+k^n\Bigg|B[Q(Rz)]-\alpha B[Q(z)]+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}B[Q(z)]\Bigg|\\
&\leq\Bigg[|\lambda_0|\Bigg|1-\alpha+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}\Bigg|\\
&\quad+k^n\Bigg|R^n-\alpha+\beta\Bigg\{\Bigg(\frac{Rk+1}{k+1}\Bigg)^n-|\alpha|\Bigg\}\Bigg|\Big|B[z^n]\Big|\Bigg]\max_{|z|=k}|P(z)|.
\end{aligned}$$

Hence the result. □

Putting $\lambda_1=\lambda_2=0$, in inequality (3.6), we get:

**Corollary 3.7.** *If $P \in P_n$ does not vanish in $|z| < k, k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,*

$$\begin{aligned} &\left|P(Rk^2z) - \alpha P(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P(k^2z)\right| \\ &\leq \frac{1}{2}\left[\left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\right. \\ &\quad \left. + k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||z^n|\right]\max_{|z|=k}|P(z)|. \end{aligned} \tag{3.8}$$

Choosing $\alpha = k = 1$ in Corollory 3.7 and divide two sides of inequality (3.8) by $R - 1$ and then making $R \to 1$, we obtain inequality (1.7), whereas inequality (1.8) follows from Corollary 3.7, when $\alpha = 0$ and $k = 1$.

Taking $\lambda_0 = \lambda_2 = 0$, in inequality (3.6), we get the following result:

**Corollary 3.8.** *If $P \in P_n$ does not vanish in $|z| < k, k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$, and $|z| \geq 1$,*

$$\begin{aligned} &\left|RP'(Rk^2z) - \alpha P'(k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}P'(k^2z)\right| \\ &\leq nk^{n-2}\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||z^{n-1}|\max_{|z|=k}|P(z)|. \end{aligned}$$

**Theorem 3.9.** *If $P \in P_n$ does not vanish in $|z| < k, k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$ and $|z| = 1$,*

$$\begin{aligned} &\max_{|z|=1}\left|B[P](Rk^2z) - \alpha B[P](k^2z) + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}B[P](k^2z)\right| \\ &\leq \frac{1}{2}\left[\left\{k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\left|\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}\right|\right.\right. \\ &\quad \left. + \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||\lambda_0|\right\}\max_{|z|=k}|P(z)| \\ &\quad - \left\{k^n\left|R^n - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right|\left|\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}\right|\right. \\ &\quad \left.\left. - \left|1 - \alpha + \beta\left\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\right\}\right||\lambda_0|\right\}\min_{|z|=k}|P(z)|\right]. \end{aligned}$$

*Equivalently*

$$\begin{aligned}
&\max_{|z|=1}\bigg|\lambda_0\Big(P(Rk^2z)-\alpha P(k^2z)+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}P(k^2z)\Big)\\
&\quad+\lambda_1\frac{nz}{2}k^2\Big(RP^{'}(Rk^2z)-\alpha P^{'}(k^2z)+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}P^{'}(k^2z)\Big)\\
&\quad+\lambda_2(\frac{nz}{2})^2k^4\frac{(R^2P^{''}(Rk^2z)-\alpha P^{''}(k^2z)+\beta\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\})}{2!}\bigg|\\
&\le\frac{1}{2}\bigg[\bigg\{k^n\bigg|R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\bigg|\\
&\quad\times\bigg|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\bigg|\\
&\quad+\bigg|1-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\bigg||\lambda_0|\bigg\}\max_{|z|=k}|P(z)|\\
&\quad-\bigg\{k^n\bigg|R^n-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\bigg|\bigg|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\bigg|\\
&\quad-\bigg|1-\alpha+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}\bigg||\lambda_0|\bigg\}\min_{|z|=k}|P(z)|\bigg].
\end{aligned}\tag{3.9}$$

*Proof.* The result is obvious if $P(z)$ has a zero on $|z|=k, k\ge 1$. Therefore we assume that $P(z)$ has all zeros in $|z|>k$. Then $m=\min_{|z|=k}|P(z)|>0$ and for $|z|=k$

$$m\le|P(z)|.$$

This gives for every $\lambda$ with $|\lambda|<1$,

$$|\lambda|m<|P(z)|,$$

for $|z|=k$ . By Rouche's Theorem, it follows that all the zeros of polynomial

$$S(z)=P(z)-\lambda m$$

lie in $|z|>k$ for every real or complex number $\lambda$ with $|\lambda|<1$. Applying Lemma 2.5 to the polynomial $S(z)$, we get for real or complex numbers $\alpha,\beta$ with $|\alpha|\le 1, |\beta|\le 1$, $R>1$ and $|z|\ge 1$

$$\begin{aligned}
&\bigg|B[S(Rk^2z)]-\alpha B[S(k^2z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[S(k^2z)]\bigg|\\
&\le k^n\bigg|B[T(Rk^2z)]-\alpha B[T(k^2z)]+\beta\Big\{\Big(\frac{Rk+1}{k+1}\Big)^n-|\alpha|\Big\}B[T(k^2z)]\bigg|,
\end{aligned}$$

where $T(z) = z^n\overline{S\left(\frac{1}{\bar{z}}\right)}$. That is, for $|z| = 1$

$$\begin{aligned}&\Bigg|B[P(Rk^2z)] - \alpha B[P(k^2z)] + \beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}B[P(k^2z)]\\ &\quad - \lambda\lambda_0\Bigg\{1-\alpha+\beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}\Bigg\}m\Bigg|\\ &\le k^n\Bigg|B[Q(Rz)] - \alpha B[Q(z)] + \beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}B[Q(z)]\\ &\quad - \bar{\lambda}\Bigg\{R^n-\alpha+\beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}\Bigg\}[\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}]m\Bigg|.\end{aligned} \tag{3.10}$$

Since all the zeros of $Q(\frac{z}{k^2})$ lie in $|z| \le k, k \ge 1$, then applying Theorem 3.1 to $Q(\frac{z}{k^2})$, for $R > 1$, we have

$$\begin{aligned}&\min_{|z|=1}\Bigg|B[Q(Rz)] - \alpha B[Q(z)] + \beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}B[Q(z)]\Bigg|\\ &\ge k^n\Bigg|R^n-\alpha+\beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}\Bigg||\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}|\min_{|z|=k}|Q(\frac{z}{k^2})|\\ &= \Bigg|R^n-\alpha+\beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}\Bigg||\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}|\min_{|z|=k}|P(z)|.\end{aligned} \tag{3.11}$$

Choosing the arguement of $\lambda$ on the right hand side of inequality (3.10) such that

$$\begin{aligned}&k^n\Bigg|B[Q(Rz)] - \alpha B[Q(z)] + \beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}B[Q(z)]\\ &\quad - \bar{\lambda}m\Bigg\{R^n-\alpha+\beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}\Bigg\}[\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}]\Bigg|\\ &= k^n\Bigg[\Bigg|B[Q(Rz)] - \alpha B[Q(z)] + \beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}B[Q(z)]\Bigg|\\ &\quad - |\bar{\lambda}|\Bigg|m\Bigg\{R^n-\alpha+\beta\Bigg\{\left(\frac{Rk+1}{k+1}\right)^n - |\alpha|\Bigg\}\Bigg\}[\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}]\Bigg|\Bigg],\end{aligned}$$

for $|z|=1$, which is possible by inequality (3.11). We get for $|z|=1$,

$$\begin{aligned}&\left|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}B[P(k^2z)]\right|\\&\quad-|\lambda||\lambda_0|\left|\left\{1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right\}m\right|\\&\leq k^n\left|B[Q(Rz)]-\alpha B[Q(z)]+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}B[Q(z)]\right|\\&\quad-|\bar{\lambda}|mk^n\left|\left\{R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right\}[\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}]\right|.\end{aligned}$$

Equivalently for $|z|=1$, we have

$$\begin{aligned}&\left|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}B[P(k^2z)]\right|\\&\quad-k^n\left|B[Q(Rz)]-\alpha B[Q(z)]+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}B[Q(z)]\right|\\&\leq|\lambda||\lambda_0|m\left[\left|\left\{1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right\}\right|\right.\\&\quad\left.-\left|\left\{R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right\}[\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}]\right|\right].\end{aligned}\tag{3.12}$$

Letting $|\lambda|\to 1$ in (3.12), we obtain for every real or complex number $\alpha,\beta$ with $|\alpha|\leq 1,|\beta|\leq 1,R>1$ and $|z|=1$,

$$\begin{aligned}&\left|B[P(Rk^2z)]-\alpha B[P(k^2z)]+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}B[P(k^2z)]\right|\\&\quad-k^n\left|B[Q(Rz)]-\alpha B[Q(z)]+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}B[Q(z)]\right|\\&\leq m|\lambda_0|\left[\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right.\\&\quad\left.-\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}[\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}]\right|\right].\end{aligned}\tag{3.13}$$

Inequality (3.13) in conjuction with lemma 2.6 yields

$$\begin{aligned}
&\max_{|z|=1}\left|B[P](Rk^2z)-\alpha B[P](k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}B[P](k^2z)\right|\\
&\leq\frac{1}{2}\left[\left\{k^n\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right||\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}|\right.\right.\\
&\quad\left.+\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right||\lambda_0|\right\}\max_{|z|=k}|P(z)|\\
&\quad-\left\{k^n\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right||\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}|\right.\\
&\quad\left.\left.-\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right||\lambda_0|\right\}\min_{|z|=k}|P(z)|\right].
\end{aligned}$$

□

The following result is immediate by taking $\lambda_1=\lambda_2=0$, in inequality (3.9).

**Corollary 3.10.** *If $P\in P_n$ does not vanish in $|z|<k, k\geq 1$, then for real or complex numbers $\alpha,\beta$ with $|\alpha|\leq 1, |\beta|\leq 1$, $R>1$ and $|z|=1$,*

$$\begin{aligned}
&\max_{|z|=1}\left|P(Rk^2z)-\alpha P(k^2z)+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}P(k^2z)\right|\\
&\leq\frac{1}{2}\left[\left\{k^n\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right.\right.\\
&\quad\left.+\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right\}\max_{|z|=k}|P(z)| \qquad (3.14)\\
&\quad-\left\{k^n\left|R^n-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right.\\
&\quad\left.\left.-\left|1-\alpha+\beta\left\{\left(\frac{Rk+1}{k+1}\right)^n-|\alpha|\right\}\right|\right\}\min_{|z|=k}|P(z)|\right].
\end{aligned}$$

Now taking $\alpha=0$, Corollary 3.10 reduces to Theorem 1.1.

Dividing the two sides of inequality (3.14) by $R-1$, and taking $\alpha=1$ and also letting $R\to 1$, we get:

**Corollary 3.11.** *If $P \in P_n$ does not vanish in $|z| < k, k \geq 1$, then for real or complex numbers $\alpha, \beta$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > 1$ and $|z| = 1$,*

$$\begin{aligned}\max_{|z|=1}\left|kzP^{'}(k^2z)+\frac{n\beta}{k+1}P(k^2z)\right| \\ \leq \frac{n}{2}\Bigg[\left\{k^{n-1}\left|1+\frac{\beta k}{k+1}\right|+\left|\frac{\beta}{k+1}\right|\right\}\max_{|z|=k}|P(z)| \\ -\left\{k^{n-1}\left|1+\frac{\beta k}{k+1}\right|-\left|\frac{\beta}{k+1}\right|\right\}\min_{|z|=k}|P(z)|\Bigg].\end{aligned} \tag{3.15}$$

**Remark 3.12.** For $k = 1$, inequality (3.15) reduces to inequality (1.11).

The following result is consequence of of Corollary 3.10 by taking $\beta = 0$ and $k = 1$.

**Corollary 3.13.** *If $P \in P_n$ does not vanish in $|z| < k, k \geq 1$, then for real or complex number $\alpha$ with $|\alpha| \leq 1$, $R > 1$,*

$$\begin{aligned}\max_{|z|=1}|P(Rz)-\alpha P(z)| \leq \left(\frac{|R^n-\alpha|+|1-\alpha|}{2}\right)\max_{|z|=1}|P(z)| \\ -\left(\frac{|R^n-\alpha|-|1-\alpha|}{2}\right)\min_{|z|=1}|P(z)|.\end{aligned} \tag{3.16}$$

**Remark 3.14.** For $\alpha = 0$, inequality (3.16) reduces to inequality (1.6). Also if we divide the two sides of inequality (3.16) by $R - 1$, and taking $\alpha = 1$ and also letting $R \to 1$, we get inequality (1.5).

## References


[1] N.C. Ankeny and T.J. Rivilin, *On a Theorem of S. Bernstein*, Pacific J. Math., **5** (1955), 849–852.

[2] A. Aziz and Q.M. Dawood, *Inequalities for polynomials and its derivative*, Approx. Theory, **54** (1988), 306–313.

[3] A. Aziz and B.A. Zargar, *Inequalities for a polynomial and its derivative*, Math. Ineq. and Appl., **I(4)** (1998), 543–550.

[4] V.K. Jain, *Generalization of certain well known inequalities for polynomials*, Glas. Mat. Ser., **111(32)** (1997), 45–52.

[5] V.K. Jain, *Inequalities for a polynomial and its derivative*, Proc. Indian Acad. Sci. Math. Sci., **110(2)** May (2000), 137–146.

[6] P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. (N.S), **50** (1944), 509–513.

[7] M. Marden, *Geometry of polynomials*, 2nd ed., Mathematical Surveys No.3, Amer. Math. Soc., Providence, RI, 1966.

[8] S. Mazerji, A. Baseri and A. Zireh, *Generalization of certain inequalities fot a polynomial and its derivative*, Lobachevskii J. Math., **33** (2012), 68–74.
[9] G.V. Milovanovic, D.S. Mitrinovic and Th.M. Rassias, Topics in Polynomials, Extremal problems, Inequalities, Zeros (Singapore: World Scientific), (1994).
[10] G. Polya and G. Szegö, Problems and Theorems in Analysis (New York: Springer verlag) (1972) vol.1.
[11] Q.I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, New York (2002).
[12] A.C. Schaffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc., **47** (1941), 565–579.