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ON THE ERROR OF THE TWO-SIDES COMONOTONIC APPROXIMATION

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Abstract. In this research, after finding some important definitions in the work, the approximation error preserving monotonicity on both sides was found, which represents the lower error and the upper error. Some of the properties of the weighted average modulus of smoothness. In this work, reference is made to the relationship between the modulus of smoothness at interval lengths $h^*_{r,s}$ and $\delta, r = 2, \ldots, n$. The best approximation between the polynomial and the weighting function f in the weighted space $L_{w,q}(I), I = [-d, d]\,,\ 0 < q < 1$.

1. Introduction

In mathematics, approximation theory is concerned with how functions can best be approximated with simpler functions, and with quantitatively characterizing the errors introduced thereby. There are also important applications in many areas of mathematics, including functional analysis, differential equations, dynamical systems theory, mathematical physics, control theory, probability theory, mathematical statistics, and other.

The reader can learn more about approximation and its applications through (see [1, 9, 12]).

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The monotonic-preserving approximation was found using a polynomial $p_n \in P_n$ whose degree is less than n, was interpolated with the weighting function in an interval $I = [-d, d]$ in the weighted $L_{w,q}(I)$ space $0 < q < 1$. And $w(x)$ it represents the weighted and bounded function.

Where $f(x)\,w(x) > 0$ for all $f(x) \geq 0, w(x) > 0$ for all $x \in I$, there exists $k \in N$ such that $|w(x)| \leq k$.

Let $A_s = \{a_2, \ldots, a_s, -d = a_1 < a_2 < \ldots < a_s < a_{s+1} = d\}$ and $\Delta^1(A_s)$ be a class of all functions that changes monotonict at all points $a_i \in A_s,\ i = 2, \ldots, s$.

In particular the function $f \in \Delta^1(A_s)$ if and only if $f'(x) \prod(x) \geq 0$, where

$$P_n = \prod_{j=2}^{s} (x - a_j), \quad x \in I.$$

Let $Y_i = d \cos \frac{i\pi}{n},\ 0 \leq i \leq n, d \in N$ be Chebyshev partition.

For $|I_i| = h_i, I_i = [Y_i, Y_{i+1}]$, let

$$\begin{aligned} \ell_{r,s} &= \left[c_{r,s}, c_{r-1,s}\right],\ s = 1, \ldots, n,\ \ell_{r,s} \subset I_i, \\ C_s &= \{c_2, \ldots, c_{n-1},\ Y_{i+1} = c_{1,s} < c_{2,s} < \ldots < c_{n-1,s} < c_{n,s} = Y_i\}. \end{aligned}$$

And that $\Delta^1(C_s)$ contains a class of all functions that changes monotonic at all points $c_{r,s} \in C_s,\ r = 2, \ldots, n,\ s = 1, \ldots, n$. And that

$$\cup_{s=1}^{n} \ell_{r,s} = I, \quad |\ell_{r,s}| = h^*_{r,s}, \quad s = 1, \ldots, n.$$

Let $\varphi(x) = \sqrt{1 - \left(\frac{x}{d}\right)^2}$, then

$$\Delta_n(x) = \frac{\varphi(x)}{n} + \frac{1}{n^2}, \quad x \in I.$$

And

$$\delta = \min_{2 \leq j \leq s} |a_{j+1} - a_j|,$$

if it is $n \to \infty$ it can get confusing

$$|\ell_{r,s}| \approx h_i \approx \Delta_n(x),$$

also

$$n\,|\ell_{r,s}| \approx n\Delta_n(x) \approx \varphi(x),$$

and that

$$\varphi(x) \approx 1 \text{ as } n \to \infty \text{ for all } x \in I.$$

2. Definitions

In this section some important definitions were found that are the basis of the work.

Definition 2.1. (Weighted Normed Space) Let $L_{w,q}(I)$ be the set of all measurable functions on an interval $I = [-d, d]$. Then $L_{w,q}(I)$ is called the weighted normed space where $0 < q < 1$, $I \subset R$ and $w(x)$, $x \in I$, is the weighted function $f(x)\,w(x) > 0$ for all $f(x) \geq 0$, $w(x) > 0$ such that

$$L_{w,q}(I) = \left\{ f : I \to R : \left(\int_I |f(x)\,w(x)|^q\,dx \right)^{\frac{1}{q}} \leq \infty, \;\; 0 < q < 1 \right\}.$$

And for all $f \in L_{w,q}(I)$ then

$$\| f \|_{L_{w,q}(I)} = \left(\int_I |f(x)\,w(x)|^q\,dx \right)^{\frac{1}{q}}.$$

Recall that for $f \in L_{w,q}(I) \cap \Delta^1(A_s)$ then

$$\| f \|_{L_{w,q}(I)} \leq 2^{\frac{1}{q}-1} \| f \|_{L_{w,1}(I)}, \;\; L_{w,1} \subset L_{w,q}.$$

Definition 2.2. (Modulus of Smoothness) Let $f \in L_{w,q}(I) \cap \Delta^1(A_s)$. Then the weighted Ditzien and toteke modulus of smoothness is defined in the form:

$$\begin{aligned} w_m^{\varphi}(f, \delta, I)_{w,q} &= \sup_{0<h\leq\delta} \| \Delta^m_{\frac{\varphi h}{m}}(f, .) \|_{L_{w,q}(I)} \\ &= \sup_{0<h\leq\delta} \left(\int_I \left| \Delta^m_{\frac{\varphi(x)h}{m}}(f, x)_w \right|^q dx \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} w_m^{\varphi}(f, x, \delta)_{w,q} = \sup_{0<h\leq\delta} \Bigg\{ \left| \Delta^m_{\frac{\varphi(x)h}{m}}(f, y)_w \right| &: y + \frac{\varphi(x)\,h}{m}, y + \frac{t\varphi(x)\,h}{m} \\ &\in \left[x - \frac{\varphi(x)\,h}{m}, x + \frac{\varphi(x)\,h}{m} \right] \cap I \Bigg\} \end{aligned}$$

is the m-th local modulus of smoothness of the function f where

$$\Delta^m_{\frac{\varphi(x)h}{m}}(f, x)_w = \begin{cases} \sum_{t=0}^{m} (-1)^{m-t} \binom{m}{t} f\left(x + \frac{t\varphi(x)h}{m}\right) w\left(x + \frac{\varphi(x)h}{m}\right), \\ \qquad\qquad \text{if } \; x + \frac{\varphi(x)h}{m}, \;\; x \in I, \\ 0, \qquad\qquad \text{otherwise}, \end{cases}$$

represent symmetric m-th difference.

Definition 2.3. (Average Modulus of Smoothness) Let $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, $0 < q < 1$. Then the weighted Sendov-Popov modulus or τ-modulus an averaged modulus of Smoothness is defined by

$$\tau_m(f, \delta, [-d, d])_{w,q} = \| w_m^{\varphi}(f, ., \delta) \|_{L_{w,q}(I)},$$

where $w_m^{\varphi}(f, x, \delta)_{w,q}$ is m-th weighted local modulus of smoothness of f which is define in Definition 2.2.

Definition 2.4. (Lower and Upper Error of Best Approximation) Let $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, $0 < q < 1$ of order $m \in N$. Then

$$E_n^-(f, A_s)_{w,q} = \inf_{p_1 \in P_n^- \cap \Delta^1(A_s)} \| f - p_1 \|_{L_{w,q}(I)}$$

is called the weighted error of best lower approximation and

$$E_n^+(f, A_s)_{w,q} = \inf_{p_2 \in P_n^+ \cap \Delta^1(A_s)} \| f - p_2 \|_{L_{w,q}(I)},$$

where

$$P_n^-(f) = \{p_1 \in P_n : p_1(x) \le f(x),\ x \in I\},$$

and

$$P_n^+(f) = \{p_2 \in P_n : f(x) \le p_2(x),\ x \in I\}$$

is the set of all algebraic polynomials.

Definition 2.5. (The Error Approximation of Two Sides) Let $\Delta^1(A_s)$ be the set of all measurable functions, that change monotonic at the points $a_j \in A_s, j = 2, \ldots, s$. For A_s suppose that $P_n(x) = P_n(x, A_s) = \prod_{j=2}^{s}(x - a_j)$ consist of all algebraic polynomials of degree is less than n and $p_1, p_2 \in P_n \cap \Delta^1(A_s)$ those satisfy $p_1(x) \le f(x) \le p_2(x)$, $x \in I$. Then

$$E_n^*(f, A_s)_{w,q} = \inf\{\| p_1 - p_2 \|_{L_{w,q}(I)}, p_1, p_2 \in P_n \cap \Delta^1(A_s), x \in I,\ 0 < q < 1\}$$

or written by

$$E_n^*(f, A_s)_{w,q} = \inf_{p_1, p_2 \in P_n \cap \Delta^1(A_s)} \| p_1 - p_2 \|_{L_{w,q}(I)}$$

is called the weighted error approximation of two sides of the function $f \in L_{w,q}(I) \cap \Delta^1(A_s)$ by polynomials from P_n.

Definition 2.6. Let $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, $0 < q < 1$. Then the function $w_m^{\varphi}(f, x, \delta)_{w,q}$ is defined for every $x \in I$, and satisfies:

$$w_m^{\varphi}(f, \delta)_{w,q} = \| w_m^{\varphi}(f, ., \delta) \|_{L_{w,q}(I)},$$

where $w_m^{\varphi}(f, x, \delta)_{w,q}$ is a local modulus of smoothness of the function f of order m at a point $x \in I$ which is defined in Definition 2.2.

3. Auxiliary results

In this section, some results were used, which are helpful factors in finding the main results of the work. For all $x \in I$, inequality is satisfied.

Lemma 3.1. ([3, 14]) *For all $x \in I$, the inequality is satisfied:*

$$\left| X_j(x) - T_j(x) \right| \leq \frac{c}{2m-1} h_j \Psi_j^{2m-1}(x),$$

where

$$T_j(x) = \frac{\int_{-b}^{x} t_j^m(y) \sum_{(y)} dy}{\int_{-b}^{b} t_j^m(y) \sum_{(y)} dy}$$

is an algebraic polynomial and

$$T_j(x) = \begin{cases} 0, & x = -b, \\ 1, & x = b, \end{cases}$$

$$T_j'(x) \sum_{(x)} \sum_{(x_{j+1})} \geq 0, \quad x \in I.$$

And

$$X_j(x) = \begin{cases} 0, & x < x_j \\ 1, & x \geq x_j \end{cases}.$$

Lemma 3.2. ([2, 4, 8]) *For an arbitrary algebraic polynomial $p_{n-1} \in P_{n-1}$, and for $r = 0$, we have*

$$\left| p_{n-1}^{(r)}(x) w(x) \right| \leq C \left| I_i \right|^{-(n-1)-\frac{1}{q}} \parallel p_{n-1} \parallel_{L_{w,q}(I)}, |I_i| = h_i, \ 0 \leq i \leq n, \ 0 < q < 1.$$

Lemma 3.3. ([11]) *Let a function g be defined for all real x. Then for every natural number k and every choice of the real numbers h and u, we have the identity:*

$$\Delta_h^k g(x) = \sum_{i=1}^{k} (-1)^i \binom{k}{i} \left[\Delta_{\frac{(u-h)}{k}}^k g(x + ih) - \Delta_{h + \frac{i(u-h)}{k}}^k g(x) \right].$$

For a function $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, $0 < q < 1$ and h, u are choice of the real numbers such that $0 < u \leq h$, $m \in N$, we have

$$\Delta_{\frac{\varphi(x)h}{m}}^m (f, x)_w = \sum_{J=1}^{m} (-1)^{m-J} \binom{m}{J} \left[\Delta_{\frac{J(\varphi(x)u - \varphi(x)h)}{m}}^m (f, x + Jh\varphi(x))_w - \Delta_{\frac{\varphi(x)h}{m} + \frac{J(\varphi(x)u - \varphi(x)h)}{m}}^m (f, x)_w \right]. \tag{3.1}$$

Lemma 3.4. *Let* $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, $0 < q < 1$, $0 < h \leq \delta_1 \leq \delta_2$. *Then*

$$w_m^{\varphi}(f, \delta_1)_{w,q} \leq w_m^{\varphi}(f, \delta_2)_{w,q}.$$

Proof. Let $\delta_1, \delta_2 \in [0, 2d]$ and $\delta_1 \leq \delta_2$, that is,

$$\sup_{0<h\leq\delta_1} \| \Delta^m_{\frac{\varphi h}{m}}(f,.) \|_{L_{w,q}(I)} \leq \sup_{0<h\leq\delta_2} \| \Delta^m_{\frac{\varphi h}{m}}(f,.) \|_{L_{w,q}(I)} .$$

Then from Definition 2.2 we get

$$w_m^{\varphi}(f, \delta_1)_{w,q} \leq w_m^{\varphi}(f, \delta_2)_{w,q} \text{ for all } I \subset R.$$

□

4. Main results

In this section, we will discuss how to find the approximate error on both sides:

Theorem 4.1. *Let* $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, $0 < p < 1$, $m \in N$. *Then*

$$\left(\sum_{L=2}^{n} w_m^{\varphi}(f, h_L^*, \mathfrak{T}_L^*)_{w,q}^q \right)^{\frac{1}{q}} \leq c(m,q) w_m^{\varphi}\left(f, n^{-1}\right),$$

where

$$\mathfrak{T}_L^* = [a_{L-1}, a_L] \subset [a_{L-1}, a_{L+1}] = \mathfrak{T}_L.$$

Proof. Let $|\mathfrak{T}_L^*| = h_L^*$ and $|\mathfrak{T}_L| = h_L$. Then it's clear that $h_L^* \leq h_L$. Suppose that $0 < h_L \leq h_L^* \leq \delta$. By Lemma 3.3 for the weighted m-th symmetric difference (2.2), and by the identity (3.1), we have

$$\Delta^m_{\frac{\varphi(x)h}{m}}(f,x)_w = \sum_{J=1}^{m} (-1)^{m-J} \binom{m}{J} \Big[\Delta^m_{\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f, x + Jh\varphi(x))_w - \Delta^m_{\frac{\varphi(x)h}{m} + \frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x)_w \Big].$$

Then

$$\left| \Delta^m_{\frac{\varphi(x)h}{m}}(f,x)_w \right|^q \leq \sum_{J=1}^{m} c(m,q) \left| \Delta^m_{\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f, x + Jh\varphi(x))_w \right|^q + \sum_{J=1}^{m} c(m,q) \left| \Delta^m_{\frac{\varphi(x)h}{m} + \frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x)_w] \right|^q. \tag{4.1}$$

By taking the integration of both sides to the identity (4.1) to $x \in [a_{L-1}, a_L] = \mathfrak{T}_L, L = 2, \ldots, n-1$, then we get

$$\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{\varphi(x)h}{m}}(f,x)_w\right|^q dx$$
$$\leq \sum_{L=1}^{m} c\,(m,q)\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x+Jh\varphi(x))_{w,q}\right|^q dx$$
$$+\sum_{L=1}^{m} c\,(m,q)\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{\varphi(x)h}{m}+\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x)_w\right|^q dx.$$

Now since

$$\delta\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{\varphi(x)h}{m}}(f,x)_w\right|^q dx=\int_{t=0}^{n=\delta}\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{\varphi(x)h}{m}}(f,x)_w\right|^q dxdt,$$

we have

$$\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{\varphi(x)h}{m}}(f,x)_w\right|^q dx$$
$$\leq \sum_{J=1}^{m} c\,(m,q)\int_{t=0}^{\delta}\int_{\mathfrak{T}_L^*}^{\frac{1}{\delta}}\left|\Delta^m_{\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x+Jh\varphi(x))_w\right|^q dxdt$$
$$+\sum_{J=1}^{m} c\,(m,q)\int_{t=0}^{\delta}\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{\varphi(x)h}{mk}+\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x)_w\right|^q dxdt$$
$$=\sum_{J=1}^{m} c(m,q)T_1+\sum_{j=1}^{m} c(m,q)T_2,$$

where

$$T_1=\int_{t=0}^{\delta}\int_{\mathfrak{T}_L^*}\frac{1}{\delta}\left|\Delta^m_{\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x+Jh\varphi(x))_{w,q}\right|^q dxdt,$$

and

$$T_2=\int_{t=0}^{\delta}\int_{\mathfrak{T}_L^*}\frac{1}{\delta}\left|\Delta^m_{\frac{\varphi(x)h}{m}+\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x)_w\right|^q dxdt.$$

Now,

$$T_1 = \int_0^{\delta} \int_{a_{L-1}}^{a_L} \frac{1}{\delta} \Big| \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} f\left(x + Jh\varphi(x) + \frac{rJ\left(\varphi(x)u - \varphi(x)h\right)}{m}\right)$$

$$\times\, w(x + \frac{J(\varphi(x)u - \varphi(x)h)}{m})\Big|^q dxdt$$

$$= \int_0^{\delta} \int_{a_{L-1}}^{a_L} \frac{1}{\delta} \Big| \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} f\left(x + Jh\varphi(x) + \frac{rJ\varphi(x)u}{m} - \frac{rJ\varphi(x)h}{m}\right)$$

$$\times\, w(x + \frac{J\left(\varphi(x)u - \varphi(x)h\right)}{m})\Big|^q dxdt$$

$$= \int_{t=-\frac{rJ\varphi(x)u}{m}}^{\delta-\frac{rJ\varphi(x)u}{m}} \int_{x=a_{L-1}+J\varphi(x)h}^{a_L+J\varphi(x)h} \frac{1}{\delta}\Big|\Delta^m_{\frac{J\varphi(x)h}{m}}(f,x)_w\Big|^q dxdt.$$

When $\varphi(x)r \geq m$ for some r and $\varphi(x) \approx 1$ as $n \to \infty$. Then we conclude from this $\delta \leq \frac{\varphi(x)r}{m}\delta$ and for all values of $J = 1, \ldots, m$, we get

$$\delta \leq \frac{J\varphi(x)r}{m}\delta.$$

Hence

$$\delta - \frac{Jr\varphi(x)h}{m} \leq \frac{J\varphi(x)r}{m}\delta - \frac{J\varphi(x)rh}{m},$$

and since $h > 0$ then $-\frac{J\varphi(x)r}{m}h < 0$. Therefore,

$$T_1 \ \leq \int_0^{\frac{J\varphi(x)r(\delta-h)}{m}} \int_{a_{L-1}+J\varphi(x)h}^{a_L+J\varphi(x)h} \frac{1}{\delta}\left|\Delta^m_{\frac{J\varphi(x)h}{m}}(f,x)_w\right|^q dxdt.$$

Since $\delta - h < \delta$, hence $\frac{\varphi(x)r}{m}(\delta - h) \leq \frac{\varphi(x)r}{m}\delta$, then we get for all values of $\frac{J\varphi(x)r}{m}(\delta - h) \leq \frac{J\varphi(x)r}{m}\delta \in N$, hence

$$T_1 \leq \int_0^{\frac{J\varphi(x)r}{m}\delta} \int_{a_{L-1}+J\varphi(x)h}^{a_L+J\varphi(x)h} \frac{1}{\delta}\left|\Delta^m_{\frac{J\varphi(x)h}{m}}(f,x)_w\right|^q dxdt.$$

Since

$$\mathfrak{T}^*_L \ = [a_{L-1} + J\varphi(x)h, a_L + J\varphi(x)h] \ \subset \mathfrak{T}_L,$$

we have

$$\begin{aligned}
&\sum_{J=1}^{m} c(m,q)T_1 \\
&\le \sum_{J=1}^{m} c(m,q)\int_0^{\frac{J\varphi(x)r}{m}} \int_{a_{L-1}}^{a_{L+1}} \frac{1}{\delta}\left|\Delta^m_{\frac{J\varphi(x)h}{m}}(f,x)_w\right|^q dxdt \\
&\le \sum_{J=1}^{m} \frac{c(m,q)}{\delta}\int_0^{\frac{J\varphi(x)r}{m}\delta} \left(\int_{x=a_{L-1}}^{a_{L+1}} \frac{1}{\delta}\left|\Delta^m_{\frac{J\varphi(x)h}{m}}(f,x)_w\right|^q dx\right) dt \\
&= \sum_{J=1}^{m} \frac{c\,(m,q)}{\delta}\int_0^{\frac{J\varphi(x)r}{m}\delta} \parallel \Delta^m_{\frac{J\varphi(x)h}{m}}(f,.)\parallel^q_{L_{w,q}(\mathfrak{T}_L)}\, dt \\
&= \sum_{J=1}^{m} \frac{c\,(m,q)}{\delta}\int_0^{\frac{J\varphi(x)r}{m}\delta} w^{\varphi}_m(f,Ju,\mathfrak{T}_L)^q_{w,q}dt.
\end{aligned}$$

Since $0 < u < h$ and $h \le h_L$ and by Lemma 3.4, we get

$$\begin{aligned}
&\sum_{J=1}^{m} c(m,q)T_1 \\
&\le \sum_{J=1}^{m} \int_0^{\frac{J\varphi(x)r}{m}\delta} \frac{c\,(m,q)}{\delta}\, w^{\varphi}_m(f,Jh_L,\mathfrak{T}_L)^q_{w,q}dt \\
&= \sum_{J=1}^{m} \frac{c\,(m,q)}{\delta}\cdot\frac{J\varphi(x)r\delta}{m}\, w^{\varphi}_m(f,Jh_L,\mathfrak{T}_L)^q_{w,q}dt.
\end{aligned}$$

Now for all $J = 1,\ldots,m$, we get

$$\sum_{J=1}^{m} c\,(m,q)\,T_1 \le \frac{c\,(m,q)}{m}\varphi(x)r\; w^{\varphi}_m(f,Jh_L,\mathfrak{T}_L)^q_{w,q},$$

we have $\varphi(x) \approx 1$ and $\varphi(x)r \ge m$, hence

$$\sum_{I=1}^{m} c\,(m,q)\,T_1 \le c\,(m,q)\,w^{\varphi}_m(f,hL,\mathfrak{T}_L)^q_{w,q}. \tag{4.2}$$

Now again

$$
\begin{aligned}
T_2 &= \int_0^{\delta}\int_{\mathfrak{T}_L^*}\frac{1}{\delta}\left|\Delta^m_{\frac{\varphi(x)h}{m}+\frac{J(\varphi(x)u-\varphi(x)h)}{m}}(f,x)_w\right|^q dxdt\\
&= \int_0^{\delta}\int_{\mathfrak{T}_L^*}\frac{1}{\delta}\left|\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}f\left(x+\frac{r\varphi(x)h}{m}+\frac{rJ\left(\varphi(x)u-\varphi(x)h\right)}{m}\right)\right.\\
&\qquad\qquad \left.\times\, w(x+\frac{\varphi(x)h+J\left(\varphi(x)u-\varphi(x)h\right)}{m})\right|^q dxdt\\
&= \int_0^{\delta}\int_{\mathfrak{T}_L^*}\frac{1}{\delta}\left|\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}f\left(x+\frac{r\varphi(x)h}{m}+\frac{rJ\varphi(x)u}{m}-\frac{rJ\varphi(x)h}{m}\right)\right.\\
&\qquad\qquad \left.\times\, w\left(x+\frac{\varphi h+J\left(\varphi(x)u-\varphi(x)h\right)}{m}\right)\right|^q dxdt\\
&= \frac{1}{\delta}\int_{-\frac{rJ\varphi(x)h}{m}}^{\delta-\frac{rJ\varphi(x)h}{m}}\int_{a_i+\frac{r\varphi(x)h}{m}}^{a_i+\frac{r\varphi(x)h}{m}}\left|\Delta^m_{\frac{J\varphi(x)u}{m}}(f,x)_w\right|^q dxdt.
\end{aligned}
$$

By the same way in the work of T_1 and since

$$
\left[a_{L-1}+\frac{r\varphi(x)h}{m},a_L+\frac{r\varphi(x)h}{m}\right]\subset\ [a_{L-1},a_{L+1}]=\mathfrak{T}_L,
$$

we have

$$
T_2\le\int_0^{\frac{J\varphi(x)r\delta}{m}}\int_{\mathfrak{T}_L}\frac{1}{\delta}\left|\Delta^m_{\frac{J\varphi(x)u}{m}}(f,x)_w\right|^q dxdt.
$$

Now

$$
\begin{aligned}
\sum_{J=1}^{m}c\,(m,q)\,T_2 &\le \sum_{J=1}^{m}\frac{C\,(m,q)}{\delta}\int_0^{\frac{J\varphi(x)r\delta}{m}}\int_{\mathfrak{T}_L}\left|\Delta^m_{\frac{J\varphi(x)u}{m}}(f,x)_w\right|^q dxdt\\
&\le \sum_{J=1}^{m}\frac{C\,(m,q)}{\delta}\int_0^{\frac{J\varphi(x)r\delta}{m}}\left(\left(\int_{\mathfrak{T}_L}\left|\Delta^m_{\frac{J\varphi(x)u}{m}}(f,x)_w\right|^q dx\right)^{\frac{1}{q}}\right)^q dt\\
&= \sum_{J=1}^{m}\frac{c\,(m,q)}{\delta}\int_0^{\frac{J\varphi(x)r\delta}{m}}\|\ \Delta^m_{\frac{J\varphi(x)u}{m}}(f,.)_w\ \|^q_{L_{w,q}(\mathfrak{T}_L)}\,dt\\
&= \sum_{J=1}^{m}\frac{c\,(m,q)}{\delta}\int_0^{\frac{J\varphi(x)r\delta}{m}}w_m^{\varphi}\left(f,Ju,\mathfrak{T}_L\right)^q_{L_{w,q}}dt\\
&= \sum_{J=1}^{m}\frac{c\,(m,q)}{\delta}\cdot\frac{J\varphi(x)r\delta}{m}w_m^{\varphi}\left(f,Ju,\mathfrak{T}_L\right)^q_{w,q}.
\end{aligned}
$$

Hence,

$$\sum_{J=1}^{m} c\,(m,q)\,T_2 \le c(m,q)w_m^{\varphi}(f,Ju,\mathfrak{T}_L)_{w,q}^{q}. \tag{4.3}$$

Therefore, from (4.2) and (4.3), we get

$$\int_{\mathfrak{T}_L^*}\left|\Delta^m_{\frac{\varphi(x)h}{m}}(f,x)_w\right|^q dx \le c(m,q)w_m^{\varphi}(f,Ju,\mathfrak{T}_L)_{w,q}^{q},$$

$$\parallel \Delta^m_{\frac{\varphi(x)h}{m}}(f,.)_w \parallel^q_{L_{w,q}\left(\mathfrak{T}_L^*\right)} \le c(m,q)w_m^{\varphi}(f,Ju,\mathfrak{T}_L)_{w,q}^{q}. \tag{4.4}$$

Since the inequality (4.4) also holds for $i=n$, finally

$$\sum_{L=2}^{n}\sup_{0<h\le\delta}\parallel \Delta^m_{\frac{\varphi(x)h}{m}}(f,.)_w \parallel^q_{L_{w,q}\left(\mathfrak{T}_L^*\right)} \le \sum_{L=2}^{n} c(m,q)w_m^{\varphi}(f,Ju,\mathfrak{T}_L)_{w,q}^{q}.$$

Since

$$\bigcup_{L=2}^{n}\mathfrak{T}_L=\left[Y_j,Y_{j+1}\right],\quad j=0,\ldots,n.$$

We conclude that for these $j=0,...,n,\ \bigcup_{i=2}^{n}\mathfrak{T}_L \subset I=[-d,d],\ h_L\le\delta$. Therefore, we have

$$\left(\sum_{L=2}^{n} w_m^{\varphi}(f,h_L^*,\mathfrak{T}_L^*)_{w,q}^{q}\right)^{\frac{1}{q}} \le c(m,q)w_m^{\varphi}(f,n^{-1}),\quad m\in N,\quad 0<q<1.$$

□

Theorem 4.2. *Let* $f\in L_{w,q}\left(I\right)\cap\Delta^1\left(A_s\right),\ 0<q<1$. *Then there is a constant* $c>0$ *such that* $c=c\left(m,v,d,q\right),\ m\in N,\ r\ge 2,\ d$ *is positive integer, which is satisfies*

$$E_n^*(f,A_s)_{w,q}\le ct^{-c_4}\tau_n(f,\delta)_{w,q}, \tag{4.5}$$

where $c_4=c_4\left(m,v,d,q\right),\ m\in N,\ 0<q<1,\ v\ge 2,\ d$ *is positive integer.*

Proof. In order to prove the approximation of polynomial in two sides on an interval I, let there exists a pair of polynomial $\{P_1,P_2\}$ of degree is less than n, interpolate f at every point of inside $I=[-d,d]$.

Let $P_n(-d)$ be a constant polynomial, and suppose the algebraic polynomials $\{P_1,P_2\}$ satisfy the monotonic condition $P_2\left(x\right)\le f\left(x\right)\le P_1\left(x\right),x\in I$, and $P_1\in P_n^+\cap\Delta^1\left(A_s\right),P_2\in P_n^-\cap\Delta^1\left(A_s\right)$ for $P_n^+,P_n^-\in\prod_n(x,A_s)\cap\Delta^1\left(A_s\right)$, where $\prod_n(x,A_s)$ consist of all algebraic polynomials of degree is at most n,by the Beatsons Lemma (see [5, 10]).

There are splines $S_{i,1},\ S_{i,2}$ with knotes $\{Y_i\}_{i=1}^n$ monotonic with f in I, so that the polynomials p_1 and p_2 result from a linear combination of splines $S_{i,1}$

and $S_{i,2}$, respectively. Hence, $S_{i,1}, S_{i,2} \in L_{w,q}(I) \cap \Delta^1(A_s)$. Let P_1, P_2 be polynomials in Chebyshev form:

$$P_1(x) = P_n(-d) + \sum_{r=1}^{2} \sum_{i=0}^{n-1} S_{i,1}(x)\, T_{i,r}(x)$$

and

$$P_2(x) = P_n(-d) + \sum_{r=1}^{2} \sum_{i=0}^{n-1} S_{i,2}(x)\, T_{i,r}(x), \quad x \in I,$$

where

$$T_{i,r}(x) = \begin{cases} T_{i,r}(x), & i = 1, \\ \chi_{i,r}(x), & i = 2, \end{cases}$$

is polynomial with degree is less than $\leq n-1$, and

$$\chi_{i,r}(x) = \begin{cases} 0, & x < Y_i, \\ 1, & x \geq Y_i, \end{cases}$$

(see [7]), Y_i denotes the Chebyshev partition $i = 0, \ldots, n$. Now, for all $x \in \ell_{r,s} \subset I_i$, we estimate

$$\begin{aligned}
&|(P_1(x) - P_2(x))\, w(x)| \\
&= \Big| \Big(P_n(-d) + \sum_{r=1}^{2} \sum_{r=0}^{n-1} S_{i,1}(x)\, T_{i,r}(x) - P_n(-d) \\
&\qquad - \sum_{r=1}^{2} \sum_{r=0}^{n-1} S_{i,2}(x)\, T_{i,r}(x) \Big) w(x) \Big| \\
&= \Big| \sum_{r=1}^{2} \sum_{i=0}^{n-1} (S_{i,1}(x) - S_{i,2}(x))\, T_{i,r}(x) w(x) \Big| \\
&= \Big| \sum_{i=0}^{n-1} (S_{i,1}(x) - S_{i,2}(x))\, w(x)\, T_{i,1}(x) + \sum_{i=0}^{n-1} (S_{i,1} - S_{i,2}) w(x)\, T_{i,2}(x) \Big| \\
&= \left| \sum_{i=0}^{n-1} (S_{i,1}(x) S_{i,2}(x)) w(x) (T_{i,1}(x) - T_{i,2}(x)) \right| \\
&= \Big| \sum_{i=0}^{n-1} (S_{i,1}(x) - S_{i,2}(x)) w(x) \Big|\, \big| T_{i,1}(x) - T_{i,2}(x) \big|.
\end{aligned}$$

Through Chebyshev's partition $Y_i = d\ cos \frac{i\pi}{n}$, $i = 0, \ldots, n$, and the relationship of the interval with this partition, we notice that $\ell_{r,s} = [c_{r,s}, c_{r-1,s}]$, $s = 1, \ldots, n$.

$$Y_1 = a_1 < a_2 < \ldots < a_{n-1} < a_n = Y_0.$$

It is a partial relationship of this partition, that is $\ell_{r,s} \subset I_i = [Y_i, Y_{i+1}]$, $i = 0, \ldots, n$, and $x < Y_0$, $x < Y_1$, $\ldots$, $x < Y_{n-1}$. We conclude from this statement that $x < Y_i$, $i = 0, \ldots, n-1$, hence, when $\mid \mathcal{T}_{i,1} - \mathcal{T}_{i,2} \mid \leq \frac{c}{2v-1} \Psi_i^{2v-1}(x)\, h_i$, $c = c(v,d)$, $v > 0$, d is positive integer, $x \in \ell_{r,s} \subset I_i$. Then

$$|(P_1(x) - P_2(x))\, w(x)| \leq \sum_{i=0}^{n-1} \frac{c}{2v-1} h_i \Psi_i^{2v-1}(x)\, |(S_{i,1}(x) - S_{i,2}(x))\, w(x)|\,.$$

After interpolating the polynomial with the function, we will now work to find the best approximation between the splines $S_{i,1}$ and $S_{i,2}$ which represent the linear combinations of the polynomials P_1 and P_2 and the weighted function f on an interval $\ell_{r,s}$, $s = 1, \ldots, n$. Since the polynomials P_1 and P_2 are comonotonic with the function f, $S_{i,1}$ and $S_{i,2}$ are comonotonic with the function f.

By using the fact $\left| P_n^{(r)}(Y_i) w(Y_i) \right| \leq c_1 |I_i|^{-n-\frac{1}{q}} \parallel P_n \parallel$, on an interval $\ell_{r,s}$, $s = 1, \ldots, n$ ([13]), we have for some r that

$$|(S_{i,1}(x) - S_{i,2}(x) w(x)| \leq c_2 |I_i|^{-n-\frac{1}{q}} \parallel S_{i,1}(x) - S_{i,2}(x) \parallel_{L_{w,q}(\ell_{r,s})}\,.$$

Hence,

$$|(P_1(x) - P_2(x))\, w(x)| \leq \sum_{i=0}^{n-1} \frac{c_3}{2v-1} h_i \Psi_i^{2v-1}(x)\, |I_i|^{-n-\frac{1}{q}} \parallel S_{i,1} - S_{i,2} \parallel_{L_{w,q}(\ell_{r,s})},$$

where $c_3 = c_1 c_2(v, d, m)$, $v > 0$, d is positive integer, $m \in N$, and

$$\parallel S_{i,1} - S_{i,2} \parallel_{L_{w,q}(\ell_{r,s})} \leq \parallel f - S_{i,1} \parallel_{L_{w,q}(\ell_{r,s})} + \parallel f - S_{i,2} \parallel_{L_{w,q}(\ell_{r,s})}\,.$$

By Whitney inequality (see [6]),

$$\parallel S_{i,1}(x) - S_{i,2}(x) \parallel_{L_{w,q}(\ell_{r,s})} \leq c_3 w_m^{\varphi}(f, |\ell_{r,s}|, \ell_{r,s})_{w,q}.$$

Hence, for $h_i = |I_i|$, we have

$$\begin{aligned} |(P_1(x) - P_2(x))\, w(x)| &\leq \sum_{i=0}^{n-1} \frac{c_3}{2v-1} h_i \Psi_i^{2v-1}(x) h_i^{-n-\frac{1}{q}} w_m^{\varphi}(f, |\ell_{r,s}|, \ell_{r,s})_{w,q} \\ &= \sum_{i=0}^{n-1} \frac{c_3}{2v-1} h_i^{-n+1-\frac{1}{q}} \Psi_i^{2v-1}(x)\, w_m^{\varphi}(f, |\ell_{r,s}|, \ell_{r,s})_{w,q}. \end{aligned}$$

Since $\ell_{r,s} \subset I_i,\ i = 0, \ldots, n-1,\ s = 1, \ldots, n$, then

$$|(P_1(x) - P_2(x))\, w(x)|^q \leq \sum_{i=0}^{n-1} \frac{c_3}{2v-1}\ h_i^{-n+1-\frac{1}{q}} \Psi_i^{2v-1}(x) w_m^{\varphi}(f, h_i, I_i)^q{}_{w,q}.$$

Now, by taking $L_{w,q}(I)$ of both sides, we get

$$\begin{aligned}\left(\int_I |(P_1(x) - P_2(x))\, w(x)|^q\, dx\right)^{\frac{1}{q}} &\le \sum_{i=0}^{n-1} \int_I \frac{c_3 h_i{}^{-(n-1+\frac{1}{q})}}{2v-1} \Psi_i^{2v-1}(x) w_m^{\varphi}(f, h_i, I_i) dx \\ &= \sum_{i=0}^{n-1} \frac{c_3}{2v-1} h_i{}^{-(n-1+\frac{1}{q})} w_m^{\varphi}(f, h_i, I_i)_{w,q}^q \int_I \Psi_i^{2v-1} dx.\end{aligned}$$

By using the fact

$$\int_I \Psi_i^{2v-1} dx = \int \frac{h_i^{2v-1}}{(|x-Y_i| + h_i)^{2v-1}} dx,$$

in ([11]) and since $x < Y_i$, we have

$$\begin{aligned}\int_I \Psi_i^{2v-1} dx &= \int_I h_i^{2v-1} (Y_i - x + h_i)^{-(2V-1)} dx \\ &= \frac{h_I^{2v-1}}{2v-2} [(Y_i - d + h_i)^{-2v+2} - (Y_i + d + h_i)^{-(2V-2)}].\end{aligned}$$

Let

$$c = \frac{c_3}{(2v-1)(2v-2)} \left[(Y_i - d + h_i)^{-2v-2} - (Y_i + d + h_i)^{-2v+2}\right].$$

Then

$$\begin{aligned}\| P_1 - P_2 \|_{L_{w,q}(I)} &\le c \sum_{i=0}^{n-1} h_i^{-\left(n-1+\frac{1}{q}\right)} h_i^{2v-1} w_m^{\varphi}\ (f, h_i, I_i)_{w,q} \\ &= c \sum_{i=0}^{n-1} h_i^{-n+1-\frac{1}{q}+2v-1} w_m^{\varphi}(f, h_i, I_i)_{w,q} \\ &= c \sum_{i=0}^{n-1} h_i^{-n+2v-\frac{1}{q}} w_m^{\varphi}(f, h_i, I_i)_{w,q}.\end{aligned}$$

By using Theorem 4.2, we get

$$\| P_1 - P_2 \|_{L_{w,q}(I)} \le c h_i^{-\left(n-2v+\frac{1}{q}\right)} w_m^{\varphi}(f, \delta)_{w,q}.$$

Let

$$t = \max_{0 \le i \le n-1} \{i : h_i < t,\ as\ t \to \infty\}.$$

Then from Definition 2.6, we get

$$\| P_1 - P_2 \|_{L_{w,q}(I)} \le c t^{-\left(n-2v+\frac{1}{q}\right)} \| w_m^{\varphi}\ (f,\ .,\delta) \|_{L_{w,q}(I)}.$$

Now, by using Definition 2.3, we get

$$\| P_1 - P_2 \|_{L_{w,q}(I)} \leq ct^{-\left(n-2v+\frac{1}{q}\right)} \tau_n(f,\delta)_{w,q}.$$

From the Definition 2.5 for the function $f \in L_{w,q}(I) \cap \Delta^1(A_s)$, for polynomials $P_1, P_2 \in L_{w,q}(I) \cap \Delta^1(A_s)$, we get

$$\begin{aligned} E_n^*(f, A_s)_{w,q} &\leq \| P_1 - P_2 \|_{L_{w,q}(I)} \\ &\leq ct^{-\left(n-2v+\frac{1}{q}\right)} \tau_n(f,\delta)_{w,q}. \end{aligned}$$

Let $n - 2v + \frac{1}{q} = c_4$ and $n + \frac{1}{q} > 2v$. Then taking $c_4 \to \infty$, we have

$$E_n^*(f, A_s)_{w,q} \leq ct^{-c_4} \tau_n(f,\delta)_{w,q},$$

where $c_4 = c_4(m, v, d, q)$, $m \in N$, $0 < q < 1, v \geq 2$, d is positive integer. □

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