Nonlinear Functional Analysis and Applications
Vol. 29, No. 4 (2024), pp. 1125-1136
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2024.29.04.12
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON FIXED POINT THEOREMS IN MR-METRIC SPACES

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Abstract. we explore a generalization of the contraction principle within the context of MR-metric spaces. The main objective is to establish results obtained by generalizing Rhoades' fixed point theorems. Furthermore, we focus on proving fixed point theorems specifically designed for MR-metric spaces developed by Malkawi. This research contributes to the understanding and application of fixed point theory in the field of MR-metric spaces. By extending existing principles and theorems, we aim to provide a broader perspective and deeper insights into the properties and behavior of fixed points in these spaces.

1. Introduction

Malkawi et al. [10] established the notion of MR-metric space, which is a generalization of a D-metric space [18], in a recent study and presented

[0]Received February 22, 2024. Revised May 24, 2024. Accepted May 27, 2024.
[0]2020 Mathematics Subject Classification: 54H25, 47H10, 34B15.
[0]Keywords: D-metric spaces, MR-metric spaces, fixed point.
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some fascinating work on MR-metric spaces. Dhage [6] showed the existence of a unique fixed point of a self-map satisfying a contractive condition in 1992, using a version of metric space called a generalized metric space or D-metric space. Rhoades [16, 17] generalized Dhage's contractive condition and came up with several fixed point theorems. Dhage also extended Rhoades' contractive condition to two D-metric space maps. Dhage discovered a unique common fixed point in a D-metric space by applying the concept of weak compatibility of self-mappings. For further information, please consult the following references ([1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15]).

2. Preliminaries

$\mathbb{N}$ stands for all natural numbers in this work, $(\mathbb{X}, M)$ for an $MR-$metric space and $\mathbb{R}^+$ for the set of all positive real numbers.

Definition 2.1. ([3]) Let $\mathbb{X} \neq \phi$ be a set. A function $D : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \to [0, \infty)$ is called a D-metric, if the following properties are satisfied for each $\zeta, \eta, \xi \in \mathbb{X}$.

(D1) : $D(\zeta, \eta, \xi) \geq 0$.
(D2) : $D(\zeta, \eta, \xi) = 0$ if and only if $\zeta = \eta = \xi$.
(D3) : $D(\zeta, \eta, \xi) = D(p(\zeta, \eta, \xi))$; for any permutation $p(\zeta, \eta, \xi)$ of ζ, η, ξ.
(D4) : $D(\zeta, \eta, \xi) \leq D(\zeta, \eta, \ell) + D(\zeta, \ell, \xi) + D(\ell, \eta, \xi)$.

A pair $(\mathbb{X}, D)$ is called a D-metric space.

The following is the definition of MR-metric space.

Definition 2.2. ([16]) Let $\mathbb{X} \neq \phi$ be a set and $R > 1$ be a real number. A function $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is called an MR-metric, if it satisfies the following properties for each $\zeta, \eta, \xi \in \mathbb{X}$.

(M1) : $M(\zeta, \eta, \xi) \geq 0$.
(M2) : $M(\zeta, \eta, \xi) = 0$ if and only if $\zeta = \eta = \xi$.
(M3) : $M(\zeta, \eta, \xi) = M(p(\zeta, \eta, \xi))$; for any permutation $p(\zeta, \eta, \xi)$ of ζ, η, ξ.
(M4) : $M(\zeta, \eta, \xi) \leq R\left[M(\zeta, \eta, \ell_1) + M(\zeta, \ell_1, \xi) + M(\ell_1, \eta, \xi)\right]$.

A pair $(\mathbb{X}, M)$ is called an MR-metric space.

In the following, we present two definitions of MR-convergence and MR-Cauchy defined by Malkawi et. al [16].

Definition 2.3. ([16]) A sequence $\{\zeta_{1_n}\}$ in an MR-metric space $(\mathbb{X}, M)$ is called an MR-convergence if there exists ζ_1 in $\mathbb{X}$ such that for $\epsilon > 0$, there exists a $N > 0$ integer number such that $M(\zeta_{1_n}, \zeta_{1_m}, \zeta_1) < \epsilon$ for all $m \geq N$, $n \geq N$. So $\{\zeta_{1_n}\}$ is called an MR-convergence to ζ_1 and ζ_1 is a limit of $\{\zeta_{1_n}\}$.

Definition 2.4. ([16]) A sequence $\{\zeta_{1_n}\}$ in MR-metric space $(\mathbb{X}, M)$ is called MR-Cauchy if for a given $\epsilon > 0$, there exists a positive integer N such that $M(\zeta_{1_n}, \zeta_{1_m}, \zeta_{1_p}) < \epsilon$ for all $m, n, p \geq N$.

The following theorem will be proved.

Theorem 2.5. *Let $\mathbb{X}$ be a complete and bounded MR-metric space, f be a self-map of $\mathbb{X}$ that is satisfying*

$$M(T_\varkappa, T_y, T_z) \leq q \max\{M(\varkappa, y, z), M(\varkappa, T_\varkappa, z), M(y, T_y, z), M(\varkappa, T_y, z), M(y, T_\varkappa, z)\} \quad (2.1)$$

for all $\varkappa, y, z \in \mathbb{X}$, $0 \leq q < 1$. Then T has a unique fixed point u in $\mathbb{X}$, and T is continuous at u.

Proof. Let $\varkappa_0 \in \mathbb{X}$ and define $\varkappa_{n+1} = T\varkappa_n$. If $\varkappa_{n+1} = \varkappa_n$ for some n, then T has a fixed point. Assume that $\varkappa_{n+1} \neq \varkappa_n$ for each n. In (2.1), setting $\varkappa = \varkappa_{n-1}$, $y = \varkappa_n$, $z = \varkappa_{n+p-1}$, we have

$$\begin{aligned} M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p}) \leq q \max\{&M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+p-1}), \\ &M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+p-1}), M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p-1}), \\ &M(\varkappa_{n-1}, \varkappa_{n+1}, \varkappa_{n+p-1}), M(\varkappa_n, \varkappa_n, \varkappa_{n+p-1})\}. \end{aligned} \quad (2.2)$$

But

$$\begin{aligned} M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+p-1}) \leq q \max\{&M(\varkappa_{n-2}, \varkappa_{n-1}, \varkappa_{n+p-2}), \\ &M(\varkappa_{n-2}, \varkappa_{n-1}, \varkappa_{n+p-2}), M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+p-2}), \\ &M(\varkappa_{n-2}, \varkappa_n, \varkappa_{n+p-2}), M(\varkappa_{n-1}, \varkappa_{n-1}, \varkappa_{n+p-2})\}, \end{aligned} \quad (2.3)$$

$$\begin{aligned} M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p-1}) \leq q \max\{&M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+p-2}), \\ &M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+p-2}), M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p-2}), \\ &M(\varkappa_{n-1}, \varkappa_{n+1}, \varkappa_{n+p-2}), M(\varkappa_n, \varkappa_n, \varkappa_{n+p-2})\}, \end{aligned} \quad (2.4)$$

$$\begin{aligned} M(\varkappa_{n-1}, \varkappa_{n+1}, \varkappa_{n+p-1}) \leq q \max\{&M(\varkappa_{n-2}, \varkappa_n, \varkappa_{n+p-2}), \\ &M(\varkappa_{n-2}, \varkappa_{n-1}, \varkappa_{n+p-2}), M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p-2}), \\ &M(\varkappa_{n-2}, \varkappa_{n+1}, \varkappa_{n+p-2}), M(\varkappa_n, \varkappa_{n-1}, \varkappa_{n+p-1})\} \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p-1}) \leq q \max\{&M(\varkappa_{n-1}, \varkappa_{n-1}, \varkappa_{n+p-2}), \\ &M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+p-2})\}. \end{aligned} \quad (2.6)$$

Substituting (2.3) − (2.6) into (2.2) gives

$$M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p-1}) \leq q^2 \max_{a,b,c} M(\varkappa_a, \varkappa_b, \varkappa_c),$$

where $n-2 \leq a \leq n$, $n-1 \leq b \leq n+1$ and $c = n+p-2$.

Continuing this process, it follows that

$$M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p-1}) \leq q^2 \max_{a,b,c} M(\varkappa_a, \varkappa_b, \varkappa_c), \tag{2.7}$$

where now $0 \leq a \leq n$, $1 \leq b \leq n+1$ and $c = p$.

Let $M := \sup_{\varkappa,y,z\in\mathbb{X}} M(\varkappa, y, z)$. Then, it follows from (2.7) that

$$M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p}) \leq q^n M. \tag{2.8}$$

Using $(M4)$ and (2.8),

$$\begin{aligned}
M(\varkappa_n, \varkappa_{n+p}, \varkappa_{n+p+t}) &\leq M(\varkappa_n, \varkappa_{n+p}, \varkappa_{n+1}) + M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+p+t}) \\
&\quad + M(\varkappa_{n+1}, \varkappa_{n+p}, \varkappa_{n+p+t}) \\
&\leq 2Mq^n + M(\varkappa_{n+1}, \varkappa_{n+p}, \varkappa_{n+p+t}) \\
&\leq 2Mq^n + M(\varkappa_{n+1}, \varkappa_{n+p}, \varkappa_{n+2}) \\
&\quad + M(\varkappa_{n+1}, \varkappa_{n+2}, \varkappa_{n+p+t}) \\
&\quad + M(\varkappa_{n+2}, \varkappa_{n+p}, \varkappa_{n+p+t}) \\
&\leq 2M(q^n + q^{n+1}) + M(\varkappa_{n+2}, \varkappa_{n+p}, \varkappa) \\
&\ \vdots \\
&\leq 2M(q^n + q^{n+1} + \cdots + q^{n+p-1}) \\
&\quad + M(\varkappa_{n+p-1}, \varkappa_{n+p}, \varkappa_{n+p+t}) \\
&\leq 2M \sum_{k=n}^{n+p} q^k \\
&\leq \frac{2Mq^n}{1-q} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, $\{\varkappa_n\}$ is M-Cauchy. Since $\mathbb{X}$ is complete, $\{\varkappa_n\}$ converges. Call the limit u. From (2.1),

$$\begin{aligned}
M(\varkappa_n, \varkappa_{n+1}, Tu) \leq q \max\{&M(\varkappa_{n-1}, \varkappa_n, u), M(\varkappa_n, \varkappa_{n+1}, u), \\
&M(\varkappa_{n-1}, \varkappa_{n+1}, u), M(\varkappa_n, \varkappa_n, u)\}.
\end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that M is continuous, yield $M(u, u, Tu) \leq 0$, which implies that $u = Tu$.

To prove uniqueness, assume that $w \neq u$ is also a fixed point of T. From (2.1),

$$\begin{aligned} M(u,w,u) &= M(Tu,Tw,Tu) \\ &\le q\max\{M(u,w,u), M(uTu,u), M(w,Tw,u), \\ &\qquad M(u,Tw,u), M(w,Tu,u)\} \\ &= q\max\{M(u,w,u), M(w,w,u)\} = qM(w,w,u). \end{aligned} \tag{2.9}$$

But

$$\begin{aligned} M(w,w,u) = M(w,u,w) &= M(Tw,Tu,Tw) \\ &\le q\max\{M(w,u,w), M(w,Tw,w), \\ &\qquad M(u,Tu,w), M(u,Tw,w)\} \\ &= q\max\{M(w,u,w), M(p,u,w)\} \\ &= qM(u,u,w). \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) yields $M(u,w,u) \le q^2 M(u,w,u)$, which is a contradiction. Therefore $u = w$.

To show that T is continuous at u, let $\{y_n\} \subseteq \mathbb{X}$ with $\lim y_n = u$. Then, substituting in (2.1), with $\varkappa = z = u$, $y = y_n$, we obtain

$$\begin{aligned} M(Tu,Ty_n,Tu) \le q\max\{&M(u,y_n,u), M)u,Tu,u), M(y_n,Ty_n,u), \\ &M(u,Ty_n,u), M(y_n,Tu,u)\}. \end{aligned} \tag{2.11}$$

Taking the lim sup of (2.11), we obtain

$$\limsup M(u,Ty_n,u) \le q\max\{0,0,\limsup M(u,Ty_n,u),0\},$$

which implies that $\lim Ty_n = u = Tu$, and T is continuous at u. □

3. Main results

All over this section $(\mathbb{X}, M)$ designates an MR-metric space and Φ denotes a family of mappings such that for each $\phi \in \Phi$, $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ is continuous and increasing in each co-ordinate variable. Also $\gamma(t) = \phi(t,t,t,t) < t$ for every $t \in \mathbb{R}^+$.

Example 3.1. Let $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ be defined by

$$\phi(t_1,t_2,t_3,t_4) = \frac{1}{5R}(t_1+t_2+t_3+t_4).$$

Then we have $\phi \in \Phi$.

The following is our main result for a complete MR-metric space on $\mathbb{X}$.

Definition 3.2. Let $(\mathbb{X}, M)$ be an MR-metric space. Then M is called the first type if for every $\wp, \varkappa \in \mathbb{X}$, we have

$$M(\wp, \wp, \Im) \leq M(\wp, \varkappa, \Im)$$

for every $\Im \in \mathbb{X}$.

Theorem 3.3. *Let A, B, C, S, T and Q be self-mappings of a complete MR-metric space $(\mathbb{X}, M)$ where M is first type with:*

(i) $A(\mathbb{X}) \subseteq T(\mathbb{X})$, $B(\mathbb{X}) \subseteq S(\mathbb{X})$, $C(\mathbb{X}) \subseteq Q(\mathbb{X})$ *and* $A(\mathbb{X})$ *or* $B(\mathbb{X})$ *or* $C(\mathbb{X})$ *is a closed subset of* $\mathbb{X}$,

(ii) $M(A\wp, B\varkappa, C\Im) \leq \frac{q}{R}\phi(RM(Q\wp, T\varkappa, S\Im), RM(Q\wp, T\varkappa, B\varkappa),$ $RM(T\varkappa, S\Im, C\Im), RM(S\Im, Q\wp, A\wp))$, *for every* $\wp, \varkappa, \Im \in \mathbb{X}$, *some* $0 < q < 1$ *and* $\phi \in \Phi$,

(iii) *the pair* (A, Q), (B, T) *and* (S, C) *are weak compatible.*

Then A, B, C, S, T and Q have a unique common fixed point in $\mathbb{X}$.

Proof. Let $\wp_0 \in \mathbb{X}$ be an arbitrary point. By (i), there exists $\wp_1, \wp_2, \wp_3 \in \mathbb{X}$ such that

$$A\wp_0 = T\wp_1 = \varkappa_0, \ B\wp_1 = S\wp_1 = \varkappa_1 \text{ and } C\wp_2 = Q\wp_3 = \varkappa_2.$$

Inductively, construct sequence $\{\varkappa_n\}$ in $\mathbb{X}$ such that

$$\varkappa_{3n} = A\wp_{3n} = T\wp_{3n+1}, \quad \varkappa_{3n+1} = B\wp_{3n+1} = S\wp_{3n+2}$$

and

$$\varkappa_{3n+2} = C\wp_{3n+2} = Q\wp_{3n+3}$$

for $n = 0, 1, ...$

Now, we prove $\{\varkappa_n\}$ is a Cauchy sequence. Let $M_m = M(\varkappa_m, \varkappa_{m+1}, \varkappa_{m+2})$. Then, we have

$$\begin{aligned}
M_{3n} &= M(\varkappa_{3n}, \varkappa_{3n+1}, \varkappa_{3n+2}) \\
&= M(A\wp_{3n}, B\wp_{3n+1}, C\wp_{3n+2}) \\
&\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\wp_{3n}, T\wp_{3n+1}, S\wp_{3n+2}), RM(Q\wp_{3n}, T\wp_{3n+1}, B\wp_{3n+1}), \\ RM(T\wp_{3n+1}, S\wp_{3n+2}, C\wp_{3n+2}), RM(S\wp_{3n+2}, Q\wp_{3n}, A\wp_{3n}) \end{pmatrix} \\
&= \frac{q}{R}\phi\begin{pmatrix} RM(\varkappa_{3n-1}, \varkappa_{3n}, \varkappa_{3n+1}), RM(\varkappa_{3n-1}, \varkappa_{3n}, \varkappa_{3n+1}), \\ RM(\varkappa_{3n}, \varkappa_{3n+1}, \varkappa_{3n+2}), RM(\varkappa_{3n+1}, \varkappa_{3n-1}, \varkappa_{3n}) \end{pmatrix} \\
&\leq q\phi\begin{pmatrix} M(\varkappa_{3n-1}, \varkappa_{3n}, \varkappa_{3n+1}), M(\varkappa_{3n-1}, \varkappa_{3n}, \varkappa_{3n+1}), \\ M(\varkappa_{3n}, \varkappa_{3n+1}, \varkappa_{3n+2}), M(\varkappa_{3n+1}, \varkappa_{3n-1}, \varkappa_{3n}) \end{pmatrix} \\
&= q\phi(M_{3n-1}, M_{3n-1}, M_{3n}, M_{3n-1}).
\end{aligned}$$

Then we prove that $L_{3n} \leq M_{3n-1}$, for every $n \in \mathbb{N}$. If $M_{3n} > M_{3n-1}$ for some $n \in \mathbb{N}$, by above inequality we have $M_{3n} < qM_{3n}$, which is a contradiction.

Now, if $m = 3n+1$, then

$$\begin{aligned}
M_{3n+1} &= M(\varkappa_{3n+1}, \varkappa_{3n+2}, \varkappa_{3n+3}) \\
&= M(\varkappa_{3n+3}, \varkappa_{3n+1}, \varkappa_{3n+2}) \\
&= M(A\wp_{3n+3}, B\wp_{3n+1}, C\wp_{3n+2}) \\
&\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\wp_{3n+3}, T\wp_{3n+1}, S\wp_{3n+2}), RM(Q\wp_{3n+3}, T\wp_{3n+1}, B\wp_{3n+1}), \\ RM(T\wp_{3n+1}, S\wp_{3n+2}, C\wp_{3n+2}), RM(S\wp_{3n+2}, Q\wp_{3n+3}, A\wp_{3n+3}) \end{pmatrix} \\
&= \frac{q}{R}\phi\begin{pmatrix} RM(\varkappa_{3n+2}, \varkappa_{3n}, \varkappa_{3n+1}), RM(\varkappa_{3n+2}, \varkappa_{3n}, \varkappa_{3n+1}), \\ RM(\varkappa_{3n}, \varkappa_{3n+1}, \varkappa_{3n+2}), RM(\varkappa_{3n+1}, \varkappa_{3n+2}, \varkappa_{3n+3}) \end{pmatrix} \\
&\leq q\phi\begin{pmatrix} M(\varkappa_{3n+2}, \varkappa_{3n}, \varkappa_{3n+1}), M(\varkappa_{3n+2}, \varkappa_{3n}, \varkappa_{3n+1}), \\ M(\varkappa_{3n}, \varkappa_{3n+1}, \varkappa_{3n+2}), M(\varkappa_{3n+1}, \varkappa_{3n+2}, \varkappa_{3n+3}) \end{pmatrix} \\
&= q\phi(M_{3n}, M_{3n}, M_{3n}, M_{3n+1}).
\end{aligned}$$

Similarly, if $M_{3n+1} > M_{3n}$ for some $n \in \mathbb{N}$, we have $M_{3n+1} < qM_{3n+1}$ which is a contradiction. If $m = 3n+2$, Then, we have

$$\begin{aligned}
M_{3n+2} &= M(\varkappa_{3n+2}, \varkappa_{3n+3}, \varkappa_{3n+4}) \\
&= M(\varkappa_{3n+3}, \varkappa_{3n+4}, \varkappa_{3n+2}) \\
&= M(A\wp_{3n+3}, B\wp_{3n+4}, C\wp_{3n+2}) \\
&\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\wp_{3n+3}, T\wp_{3n+4}, S\wp_{3n+2}), RM(Q\wp_{3n+3}, T\wp_{3n+4}, B\wp_{3n+4}), \\ RM(T\wp_{3n+4}, S\wp_{3n+2}, C\wp_{3n+2}), RM(S\wp_{3n+2}, Q\wp_{3n+3}, A\wp_{3n+3}) \end{pmatrix} \\
&= \frac{q}{R}\phi\begin{pmatrix} RM(\varkappa_{3n+2}, \varkappa_{3n+3}, \varkappa_{3n+1}), RM(\varkappa_{3n+2}, \varkappa_{3n+3}, \varkappa_{3n+4}), \\ RM(\varkappa_{3n+3}, \varkappa_{3n+1}, \varkappa_{3n+2}), RM(\varkappa_{3n+1}, \varkappa_{3n+2}, \varkappa_{3n+3}) \end{pmatrix} \\
&\leq q\phi\begin{pmatrix} M(\varkappa_{3n+2}, \varkappa_{3n+3}, \varkappa_{3n+1}), M(\varkappa_{3n+2}, \varkappa_{3n+3}, \varkappa_{3n+4}), \\ M(\varkappa_{3n+3}, \varkappa_{3n+1}, \varkappa_{3n+2}), M(\varkappa_{3n+1}, \varkappa_{3n+2}, \varkappa_{3n+3}) \end{pmatrix} \\
&= q\phi(M_{3n+1}, M_{3n+2}, M_{3n+1}, M_{3n+1}).
\end{aligned}$$

And also, similarly, if $M_{3n+2} > M_{3n+1}$ for some $n \in \mathbb{N}$, we have $M_{3n+2} < qM_{3n+2}$ which is a contradiction. Hence, for every $n \in \mathbb{N}$, we have $M_n \leq qM_{n-1}$. That is,

$$M_n = M(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+2}) \leq M(\varkappa_{n-1}, \varkappa_n, \varkappa_{n+1}) \leq \cdots \leq q^n M(\varkappa_0, \varkappa_1, \varkappa_2).$$

Since M is a first type, we have

$$M(\varkappa_n, \varkappa_n, \varkappa_{n+1}) \leq q^n M(\varkappa_0, \varkappa_1, \varkappa_2).$$

Therefore,

$$M(\varkappa_n, \varkappa_n, \varkappa_m) \le M(\varkappa_n, \varkappa_n, \varkappa_{n+1}) + M(\varkappa_{n+1}, \varkappa_{n+1}, \varkappa_{n+2}) + \cdots + M(\varkappa_{m-1}, \varkappa_{m-1}, \varkappa_m).$$

Hence,

$$\begin{aligned} M(\varkappa_n, \varkappa_n, \varkappa_m) &\le q^n M(\varkappa_0, \varkappa_1, \varkappa_2) + q^{n+1} M(\varkappa_0, \varkappa_1, \varkappa_2) \\ &\quad + \cdots + q^{m-1} M(\varkappa_0, \varkappa_1, \varkappa_2) \\ &= (q^n + q^{n+1} + \cdots + q^{m-1}) M(\varkappa_0, \varkappa_1, \varkappa_2) \\ &\le M(\varkappa_0, \varkappa_1, \varkappa_2) \frac{q^n}{1-q} \\ &\longrightarrow 0. \end{aligned}$$

Thus the sequence $\{\varkappa_n\}$ is Cauchy and by the completeness of $\mathbb{X}$, $\{\varkappa_n\}$ converges to $\varkappa$ in $\mathbb{X}$. That is, $\lim_{n\to\infty} \varkappa_n = \varkappa$,

$$\begin{aligned} \lim_{n\to\infty} \varkappa_n &= \lim_{n\to\infty} A\wp_{3n} = \lim_{n\to\infty} B\wp_{3n+1} = \lim_{n\to\infty} C\wp_{3n+2} \\ &= \lim_{n\to\infty} T\wp_{3n+1} = \lim_{n\to\infty} Q\wp_{3n+3} = \lim_{n\to\infty} S\wp_{3n+2} = \varkappa. \end{aligned}$$

Let $C(\mathbb{X})$ be a closed subset of $\mathbb{X}$, hence there exist $u \in \mathbb{X}$ such that $Qu = \varkappa$. We prove that $Au = \varkappa$. For

$$\begin{aligned} &M(Au, B\wp_{3n+1}, C\wp_{3n+2}) \\ &\le \frac{q}{R}\phi \begin{pmatrix} RM(Qu, T\wp_{3n+1}, S\wp_{3n+2}), RM(Qu, T\wp_{3n+1}, B\wp_{3n+1}), \\ RM(T\wp_{3n+1}, S\wp_{3n+2}, C\wp_{3n+2}), RM(S\wp_{3n+2}, Qu, Au) \end{pmatrix}. \end{aligned}$$

By letting $n \longrightarrow \infty$, we get

$$M(Au, \varkappa, \varkappa) \le \frac{q}{R}\phi \begin{pmatrix} RM(Qu, \varkappa, \varkappa), RM(Qu, \varkappa, \varkappa), \\ RM(\varkappa, \varkappa, \varkappa), RM(\varkappa, Qu, Au) \end{pmatrix}.$$

If $M(\varkappa, \varkappa, Au) > 0$, then we have $M(Au, \varkappa, \varkappa) < qM(\varkappa, \varkappa, Au)$ which is a contradiction. Thus $Au = \varkappa$. By the weak compatibility of the pair (Q, A), we have $AQu = QAu$. Hence $A\varkappa = Q\varkappa$.

We prove that $A\varkappa = \varkappa$, if $A\varkappa \neq \varkappa$, then

$$\begin{aligned} &M(A\varkappa, B\wp_{3n+1}, C\wp_{3n+2}) \\ &\le \frac{q}{R}\phi \begin{pmatrix} RM(Q\varkappa, T\wp_{3n+1}, S\wp_{3n+2}), RM(Q\varkappa, T\wp_{3n+1}, B\wp_{3n+1}), \\ RM(T\wp_{3n+1}, S\wp_{3n+2}, C\wp_{3n+2}), RM(S\wp_{3n+2}, Q\varkappa, A\varkappa) \end{pmatrix}. \end{aligned}$$

As $n \longrightarrow \infty$, we have

$$\begin{aligned} M(A\varkappa, \varkappa, \varkappa) &\le \frac{q}{R}\phi \begin{pmatrix} RM(Q\varkappa, \varkappa, \varkappa), RM(Q\varkappa, \varkappa, \varkappa), \\ RM(\varkappa, \varkappa, \varkappa), RM(\varkappa, Q\varkappa, A\varkappa) \end{pmatrix} \\ &\le qM(A\varkappa, \varkappa, \varkappa), \end{aligned}$$

which is a contradiction. Therefore, $Q\varkappa = A\varkappa = \varkappa$, that is, $\varkappa$ is a common fixed of Q, A.

Since $\varkappa = A\varkappa \in A(\mathbb{X}) \subseteq Q(\mathbb{X})$, there exist $v \in \mathbb{X}$ such that $Tv = \varkappa$. We prove that $Bv = \varkappa$. For

$$\begin{aligned} M(\varkappa, Bv, C\wp_{3n+2}) &= M(A\varkappa, Bv, C\wp_{3n+2}) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, Tv, S\wp_{3n+2}), RM(Q\varkappa, Tv, Bv), \\ RM(Tv, S\wp_{3n+2}, C\wp_{3n+2}), RM(S\wp_{3n+2}, Q\varkappa, A\varkappa) \end{pmatrix}. \end{aligned}$$

By letting $n \longrightarrow \infty$, we get

$$M(\varkappa, Bv, \varkappa) \leq \frac{q}{R}\phi\begin{pmatrix} RM(\varkappa, \varkappa, \varkappa), RM(\varkappa, \varkappa, Bv), \\ RM(\varkappa, \varkappa, \varkappa), RM(\varkappa, \varkappa, \varkappa) \end{pmatrix} \leq qM(\varkappa, \varkappa, Bv).$$

Thus, $Bv = \varkappa$. By the weak compatibility of the pair (B, T), we have $TBv = BTv$. Hence, $B\varkappa = T\varkappa$. We prove that $B\varkappa = \varkappa$, if $B\varkappa \neq \varkappa$, then

$$\begin{aligned} &M(A\varkappa, B\varkappa, C\wp_{3n+2}) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, S\wp_{3n+2}), RM(Q\varkappa, T\varkappa, B\varkappa), \\ RM(T\varkappa, S\wp_{3n+2}, C\wp_{3n+2}), RM(S\wp_{3n+2}, Q\varkappa, A\varkappa) \end{pmatrix}. \end{aligned}$$

As $n \longrightarrow \infty$, we have

$$\begin{aligned} M(\varkappa, B\varkappa, \varkappa) &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, \varkappa), RM(Q\varkappa, B\varkappa, B\varkappa), \\ RM(B\varkappa, \varkappa, \varkappa), RM(\varkappa, \varkappa, \varkappa) \end{pmatrix} \\ &\leq qM(\varkappa, B\varkappa, \varkappa), \end{aligned}$$

which a contradiction. Therefore, $B\varkappa = T\varkappa = \varkappa$, that is, $\varkappa$ is a common fixed of B, T.

Similarly, since $\varkappa = B\varkappa \in B(\mathbb{X}) \subseteq S(\mathbb{X})$, there exists $w \in \mathbb{X}$ such that $Sw = \varkappa$. We prove that $Cw = \varkappa$. For

$$\begin{aligned} M(\varkappa, \varkappa, Cw) &= M(A\varkappa, B\varkappa, Cw) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, w), RM(Q\varkappa, T\varkappa, B\varkappa), \\ RM(T\varkappa, Sw, Cw), RM(Sw, Q\varkappa, A\varkappa) \end{pmatrix} \\ &\leq qM(\varkappa, \varkappa, Cw). \end{aligned}$$

Thus, $Cw = \varkappa$. By the weak compatibility of the pair (C, S), we have $CSw = SCw$. Hence $C\varkappa = S\varkappa$. We prove that $C\varkappa = \varkappa$, if $C\varkappa \neq \varkappa$, then

$$\begin{aligned} M(\varkappa, \varkappa, C\varkappa) &= M(A\varkappa, B\varkappa, C\varkappa) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, S\varkappa), RM(Q\varkappa, T\varkappa, B\varkappa), \\ RM(T\varkappa, S\varkappa, C\varkappa), RM(S\varkappa, Q\varkappa, A\varkappa) \end{pmatrix} \\ &\leq qM(\varkappa, \varkappa, C\varkappa), \end{aligned}$$

which is a contradiction. Therefore, $C\varkappa = S\varkappa = \varkappa$, that is, $\varkappa$ is a common fixed of C, S. Thus

$$A\varkappa = S\varkappa = T\varkappa = B\varkappa = C\varkappa = Q\varkappa = \varkappa.$$

Next, to prove the uniqueness, let v be another common fixed point of T, A, B, C, Q, S.

If $M(\varkappa, \varkappa, v) > 0$, then

$$\begin{aligned} M(\varkappa, \varkappa, v) &= M(A\varkappa, B\varkappa, Cv) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, Sv), RM(Q\varkappa, T\varkappa, By), \\ RM(T\varkappa, Sv, Cv), RM(Sv, Q\varkappa, A\varkappa) \end{pmatrix} \\ &\leq qM(\varkappa, \varkappa, v), \end{aligned}$$

which is a contradiction, Therefore, $\varkappa = v$ is the unique common fixed point of self-maps T, A, B, C, Q, S. □

Corollary 3.4. *Let S, T, Q and $\{A_\alpha\}_{\alpha\in I}$, $\{B_\beta\}_{\beta\in J}$ and $\{C_\gamma\}_{\gamma\in K}$ be the set of all self-mappings of a complete $M^* -$metric space $(\mathbb{X}, M)$, where M is first type satisfying:*

(i) *there exists $\alpha_0 \in I$, $\beta_0 \in J$ and $\gamma_0 \in K$ such that $A_{\alpha_0}(\mathbb{X}) \subseteq T(\mathbb{X})$, $B_{\beta_0}(\mathbb{X}) \subseteq S(\mathbb{X})$ and $C_{\gamma_0}(\mathbb{X}) \subseteq Q(\mathbb{X})$,*

(ii) *A_{α_0} or B_{β_0} or $C_{\gamma_0}(\mathbb{X})$ is a closed subset of $\mathbb{X}$,*

(iii) *$M(A\wp, B\varkappa, C\Im) \leq \frac{q}{R}\phi(RM(Q\wp, T\varkappa, S\Im), RM^*(Q\wp, T\varkappa, B_\beta\varkappa), RM(T\varkappa, S\Im, C_\gamma\Im), RM(S\Im, Q\wp, A_\alpha\wp))$ for every $\wp, \varkappa, \Im \in \mathbb{X}$, some $0 < q < 1$, $\phi \in \Phi$, and every $\alpha \in I$, $\beta \in J$, $\gamma \in K$,*

(iv) *the pair (A_{α_0}, Q), (B_{β_0}, T) and (C_{γ_0}, S) are weak compatible.*

Then A, B, C, S, T and Q have a unique common fixed point in $\mathbb{X}$.

Proof. By Theorem 3.3, Q, S, T and $A_{\alpha_0}, B_{\beta_0}$ and C_{γ_0} for some $\alpha_0 \in I$, $\beta_0 \in J$, $\gamma_0 \in K$ have a unique common fixed point in $\mathbb{X}$. That is, there exist a unique $a \in \mathbb{X}$ such that $Q(a) = S(a) = T(a) = A_{\alpha_0}(a) = B_{\beta_0}(a) = C_{\gamma_0}(a) = a$. Let there exist $\lambda \in J$ such that $\lambda \neq \beta_0$ and $M^*(a, B_\lambda, a) > 0$ then we have

$$\begin{aligned} M(a, B_\lambda a, a) &= M(A_{\alpha_0}a, B_\lambda a, C_{\gamma_0}v) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Qa, Ta, Sa), RM(Qa, Ta, B_\lambda a), \\ RM(Ta, Sa, C_{\gamma_0}a), RM(Sa, Qa, A_{\alpha_0}a) \end{pmatrix} \\ &\leq qM(a, a, B_\lambda a), \end{aligned}$$

which is a contradiction. Hence, for every $\lambda \in J$, we have $B_\lambda(a) = a$. Similarly, for every $\delta \in I$ and $\varkappa \in K$, we get $A_\delta(a) = C_\varkappa(a) = a$. Therefore, for every $\delta \in I$, $\lambda \in J$ and $\varkappa \in K$, we have $A_\delta(a) = B_\lambda(a) = Q(a) = S(a) = T(a) = a$. □

Example 3.5. Let $\varkappa = B\varkappa \in B(\mathbb{X}) \subseteq S(\mathbb{X})$. This means that there exists $w \in \mathbb{X}$ such that $Sw = \varkappa$. We want to prove that $Cw = \varkappa$. For

$$\begin{aligned} M(\varkappa, \varkappa, Cw) &= M(A\varkappa, B\varkappa, Cw) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, w), RM(Q\varkappa, T\varkappa, B\varkappa), \\ RM(T\varkappa, Sw, Cw), RM(Sw, Q\varkappa, A\varkappa) \end{pmatrix} \\ &\leq qM(\varkappa, \varkappa, Cw). \end{aligned}$$

Therefore, we can conclude that $Cw = \varkappa$. Due to the weak compatibility of the pair (C, S), we have $CSw = SCw$.

So, we can say that $C\varkappa = S\varkappa$. Now, we need to prove that $C\varkappa = \varkappa$. If $C\varkappa \neq \varkappa$, then

$$\begin{aligned} M(\varkappa, \varkappa, C\varkappa) &= M(A\varkappa, B\varkappa, C\varkappa) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, S\varkappa), RM(Q\varkappa, T\varkappa, B\varkappa), \\ RM(T\varkappa, Sw, Cw), RM(Sw, Q\varkappa, A\varkappa) \end{pmatrix}. \end{aligned}$$

To prove the uniqueness, let's consider another common fixed point of T, A, B, C, Q, S, denoted as v. If $M(\varkappa, \varkappa, v) > 0$, then we have:

$$\begin{aligned} M(\varkappa, \varkappa, v) &= M(A\varkappa, B\varkappa, Cv) \\ &\leq \frac{q}{R}\phi\begin{pmatrix} RM(Q\varkappa, T\varkappa, Sv), RM(Q\varkappa, T\varkappa, By), \\ RM(T\varkappa, Sv, Cv), RM(Sv, Q\varkappa, A\varkappa) \end{pmatrix} \\ &\leq qM(\varkappa, \varkappa, v). \end{aligned}$$

This leads to a contradiction, which implies that $\varkappa = v$ is the unique common fixed point of the self-maps T, A, B, C, Q, S.

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