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A STUDY OF FIXED POINTS FOR FAMILY OF MAPPINGS

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Abstract. In this manuscript, we prove a fixed point theorem for a family of contractive self-mappings in a complete metric space or a complete b-metric space. We generalize the Caristi fixed point theorem, the Meir-Keeler type fixed point theorem, and the result of Pant. In our result, we use a generalization of the Meir-Keeler type contraction map for a family of self-mappings.

1. Introduction

Fixed point theory is a principal branch of Mathematics. The appearance or disappearance of a fixed point is an inherent characteristic of a map. However, many essential or abundant conditions for presence of such points comprises a mixture of algebraic, order theoretic, or topological properties of the mappings or its domain. The Banach contraction mapping theorem [1] itself does

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not characterize metric completeness [8]. Kannan [11, 12] proved that a self-mapping f of a complete metric space (X, d) satisfying the contractive condition have a unique fixed point. Kannan's results is marvellous for two purpose: that is, signalize the metric completeness [23], it was the beginning of the once open problem on the existence of contractive mappings which are discontinuous at fixed point [21]. After that many researchers explored in the concept of metric completeness and its properties e.g. [3, 5, 13, 19, 20, 22, 23, 24]. Browder and Petryshyn [4] defined the concept of asymptotically regular.

In 1969 Meir and Keeler [14] defined the contraction mapping. In 1976 Caristi [5] proved the fixed point theorem. In 1999, Pant [15] resolved the problem of continuity of contractive mappings at fixed points. In 2019, Pant et al. [17] gives the following result which is a generalization [5, 6, 7].

In this manuscript our results are the generalization of the result of Pant et al. [17, 18] because we are finding fixed points for family of mappings and we are using the contractive condition which is mentioned in the research paper of Pant [15] that condition is general than Meir-Keeler type contractive condition.

We now give some relevant definitions:

Definition 1.1. ([16]) A self-mapping f of a metric space (X, d) is called $k-$continuous, $k = 1, 2, 3, ...$, if $f^k x_n \to ft$ whenever $\{x_n\}$ in X such that $f^{k-1}x_n \to t$.

Definition 1.2. ([6, 7]) A self-mapping f of a metric space (X, d), then the set $O(x, f) = \{f^n x : n = 0, 1, 2, ...\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i ff^{m_i} x$.

Definition 1.3. ([17]) A self-mapping f of a metric space (X, d) is called weakly orbitally continuous, if the set $\{y \in X : \lim_i f^{m_i} y = u \Rightarrow \lim_i ff^{m_i} y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i} x = u\}$ is nonempty.

Example 1.4. Let $X = [0, 2]$ equipped with the Euclidean metric. Define $f : X \to X$ by

$$f(x) = \begin{cases} \frac{(1+x)}{2}, & \text{if } x < 1, \\ 0, & \text{if } 1 \leq x < 2, \\ 2, & \text{if } x = 2. \end{cases}$$

Then $f^n(0) \to 1$ and $f(f^n(0)) \to 1$. Therefore, f is not orbitally continuous. However, f is weakly orbitally continuous. If we consider the any sequence $f^n(0)$, then for any integer $k \geq 1$, we have $f^{k-1}(f^n(0)) \to 1 \neq f(1)$. This shows that f is not k-continuous.

Example 1.5. Let $X = [0, \infty)$ equipped with the Euclidean metric. Define $f : X \to X$ by

$$f(x) = \begin{cases} 1, & \text{if } \ x \leq 1, \\ \frac{x}{3}, & \text{if } \ x > 1. \end{cases}$$

Then it is easy to see that f is orbitally continuous. Let $k \geq 1$ be any integer. Consider the sequence $\{x_n\}$ given by $x_n = 3^{k-1} + \frac{1}{n}$. Then $f^{k-1}x_n = 1 + \frac{1}{n3^{k-1}}$, $f^k x_n = \frac{1}{3} + \frac{1}{(n3^k)}$. This implies $f^{k-1}x_n \to 1$, $f^k x_n \to \frac{1}{3} \neq f(1)$ as $n \to \infty$. Hence, f is not k-continuous.

Remark 1.6. It is shown in the research paper [16] that continuity of f^k and k-continuity of f are independent conditions when $k > 1$. It is also clear in the research paper [6, 7] that continuous mapping is orbitally continuous but not conversely. The concept of weakly orbitally continuous is more general than orbitally continuous is shown in the research paper [17].

Definition 1.7. ([9, 10]) Let X be a nonempty set and $s \leq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The triplet (X, d, s) is called a b-metric space.

Definition 1.8. ([9, 10]) Let (X, d, s) be a $b-$metric space. The sequence $\{x_n\}$ in X is called convergence if for all $\epsilon > 0$, there exists $k \in N$ such that $d(x_n, x) < \epsilon$ for all $n \geq k$. In this case, we write $\lim_{n\to\infty} x_n = x$ and x is called the limit of $\{x_n\}$.

Definition 1.9. ([9, 10]) Let (X, d, s) be a $b-$metric space. The sequence $\{x_n\}$ in X is called Cauchy in X if for all $\epsilon > 0$, there exists $k \in N$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq k$.

Definition 1.10. ([9, 10]) The $b-$metric space (X, d, s) is said to be complete if every Cauchy sequence is convergent to some x in X.

2. Main results

Theorem 2.1. *Let $\{f_r : 0 \leq r \leq 1\}$ be a family of self-mappings in a complete metric space (X, d) such that for any given mapping f_r, the following conditions are satisfied:*

$$d(f_r x, f_r y) \leq \max\{d(x, f_r x), d(y, f_r y)\} \tag{2.1}$$

for $\max\{d(x, f_rx), d(y, f_ry)\} > 0$, *and given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that,*

$$\epsilon < \max\{d(x, f_rx), d(y, f_ry)\} < \epsilon + \delta \quad \Rightarrow \quad d(f_rx, f_ry) < \epsilon. \tag{2.2}$$

Then f_r *possesses a fixed point if and only if* f_r *is weakly orbitally continuous.*

Moreover, the fixed point is unique and f_r *is continuous at the fixed point, say* z, *if and only if*

$$\lim_{x\to z} \max\{d(x, f_rx), d(z, f_rz)\} = 0$$

or equivalently,

$$\lim_{x\to z} \sup\, d(f_rz, f_rx) = 0.$$

Proof. Select any mapping f_r where $0 \le r \le 1$. It is obvious that f_r satisfies contractive condition (2.1),

$$d(f_rx, f_ry) < \max\{d(x, f_rx), d(y, f_ry)\} \tag{2.3}$$

for $\max\{d(x, f_rx), d(y, f_ry)\} > 0$.

Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by

$$\begin{aligned} x_n &= f_rx_{n-1}, \\ x_n &= f_r\{f_rx_{n-2}\}{=}f_r^2x_{n-2}, \\ &\vdots \\ x_n &= f_r^nx_0. \end{aligned}$$

If $x_n = x_{n+1}$ for some n then,

$$x_n = x_{n+1} = x_{n+2} = ...,$$

that is, $\{x_n\} = \{f_r^nx_0\}$ is a Cauchy sequence and x_n is a fixed point of f_r, we can, therefore assume that $x_n \ne x_{n+1}$ for each n. Then, using (2.3), we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f_rx_{n-1}, f_rx_n) \\ &< \max\{d(x_{n-1}, f_rx_{n-1}), d(x_n, f_rx_n)\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

Thus, $\{d(x_n, x_{n+1}\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $r \ge 0$. Suppose $r > 0$. Then there exist a positive integer N such that,

$$n \ge N \quad \Rightarrow \quad r < d(x_n, x_{n+1}) < r + \delta(r). \tag{2.4}$$

Now with reference of (2.2),

$$d(x_n, x_{n+1}) < r, \tag{2.5}$$

which is a contradiction of (2.4). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now if p is any positive integer then,

$$\begin{aligned} d(x_n, x_{n+p}) &= d(f_r x_{n-1}, f_r x_{n+p-1}) \\ &< \max\{d(x_{n-1}, f_r x_{n-1}), d(x_{n+p-1}, f_r x_{n+p-1})\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\} \\ &= d(x_{n-1}, x_n). \end{aligned}$$

This implies that $d(x_n, x_{n+p}) \to 0$. Therefore, $\{x_n\} = \{f_r^n x_0\}$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n \to z$. Moreover, for each integer $p \geq 1$, we have $f_r^p x_n \to z$ also using (2.3) it follows easily that $f_r^n y \to z$ for any y in X.

Suppose that f_r is weakly orbitally continuous. Since $f_r^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of f_r we get $f_r^n y_0 \to z$ and $f_r^{n+1} y_0 \to f_r z$ for some y_0 in X. This implies $z = f_r z$, since $f_r^{n+1} y_0 \to z$. Therefore, z is a fixed point of f_r.

Uniqueness of the fixed point follows easily. Conversely, suppose that the mapping f_r possesses a fixed point, say z. Then $\{f_r^n z = z\}$ is a constant sequence such that $\lim_n f_r^{n+1} z = z = f_r z$. Hence, f_r is weak orbitally continuous. It is also easy to verify that f_r is continuous at z if and only if $\lim_{x \to z} \max\{d(x, f_r x), d(z, f_r z)\} = 0$ or, equivalently, $\lim_{x \to z} \sup d(f_r z, f_r x) = 0$.

This can alternatively be stated as: f_r is discontinuous at z if and only if $\lim_{x \to z} \sup d(f_r z, f_r x) = 0$. This proves the theorem. □

Example 2.2. Let $X = [0, \infty)$ equipped with the usual metric and let $f_r : X \to X$ be defined by

$$f_r(x) = \frac{x}{3} \quad \text{for each } x \text{ in } X.$$

Then it is easy to verify that X is complete, f_r satisfies (2.2), f_r is continuous, and f_r has unique fixed point $x = 0$.

Example 2.3. Let $X = [0, 2]$ and d be the usual metric. Define $f_r : X \to X$ by

$$f_r(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1, \\ x - 1, & \text{if } 1 < x \leq 2. \end{cases}$$

Then f_r satisfy all the conditions of the above theorem and has a unique fixed point $z = 1$ at which f_r is discontinuous.

Theorem 2.4. *Let (X, d) be a complete metric space or a complete b-metric space and $\{f_r : 0 \leq r \leq 1\}$ be a family of asymptotically regular self-mappings* *of X satisfying*

$$d(f_r x, f_r y) \leq \lambda \max\{d(x, f_r x), d(y, f_r y)\}, \ \lambda > 0 \tag{2.6}$$

for each r. If f_r is weak orbitally continuous for some integer $k \geq 1$, then f_r has a unique fixed point. Moreover, if every pair of mappings (f_r, f_s) satisfies the condition:

$$d(f_r x_r, f_s y) \leq \lambda \max\{d(x, f_r x), d(y, f_s y)\}, \ \lambda > 0, \tag{2.7}$$

then the mappings have a unique common fixed point which is also the unique fixed of each f_r.

Proof. Select any mapping f_r. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = f_r x_{n-1}$. If $x_n = x_{n+1}$ for some n then x_n is a fixed point of f_r. If $x_n \neq x_{n+1}$ for each n, then using (2.4), for each positive integer p we get,

$$\begin{aligned} d(x_n, x_{n+p}) &= d(f_r x_{n-1}, f_r x_{n+p-1}) \\ &\leq \max\{d(x_{n-1}, f_r x_{n-1}), d(x_{n+p-1}, f_r x_{n+p-1})\} \\ &= \lambda \max\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p}\}. \end{aligned}$$

By asymptotic regularity of f_r, this implies that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. This further implies that $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$. That is, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists t in X such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f_r^p x_n = t, p = 1, 2, 3, ...$$

Suppose that f_r is weakly orbitally continuous. Since $f_r^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of f_r, we get $f_r^n y_0 \to z$ and $f_r^{n+1} y_0 \to f_r z$ for some y_0 in X. This implies $z = f_r z$ since $f_r^{n+1} y_0 \to z$. Therefore, z is a fixed point of f_r.

Uniqueness of the fixed point follows easily from (2.7). Let u, v are different fixed point of f_r. Then

$$d(u, v) = d(f_r u, f_s v) \leq \lambda \max\{d(u, f_r u), d(v, f_s v)\} = 0. \tag{2.8}$$

Hence $u = v$ and the family of mappings $\{f_r\}$ has a unique common fixed point which is also the unique fixed point of each f_r. □

Theorem 2.5. *Let $\{f_r : 0 \leq r \leq 1\}$ be a family of self-mapping in a complete metric space (X, d) such that for any given mapping f_r the following conditions are satisfied, for given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\epsilon < \max\{d(x, f_r x), d(y, f_r y)\} < \epsilon + \delta \quad \Rightarrow \quad d(f_r x, f_r y) < \epsilon. \tag{2.9}$$

Then there exists a point, say z in X such that for each x, y in X the sequence of iterates $\{f_r^n x\}$ is Cauchy and $\lim_{n\to\infty} f_r^n x = z$.

Example 2.6. Let $X = [0,2]$ equipped with the Euclidean metric d. Define $f_r : X \to X$ by

$$f_r x = \begin{cases} \frac{1+x}{2}, & \text{if } x < 1, \\ 0, & \text{if } 1 \leq x \leq 2. \end{cases}$$

Then X is complete metric space and f_r satisfies the contractive condition with $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$ and $\delta(\epsilon) = \epsilon$ for $\epsilon \geq 1$ but does not posses a fixed point. It is easy to verify that for each x in X, the sequence $\{f_r^n x\}$ is Cauchy and $f_r^n x \to 1$. It is easily seen that f_r is not weakly orbitally continuous.

Example 2.7. Let $X = [1,2] \cup \{1 - \frac{1}{2^n} : n = 0,1,2,...\}$ and d be the usual metric. Define $f_r : X \to X$ by

$$f_r x = 0 \quad \text{if} \quad 1 \leq x \leq 2, \quad f_r \left(1 - \frac{1}{2^n}\right) = \left\{1 - \frac{1}{2^{n+1}},\ n = 0,1,2,...\right\}.$$

Then range of f_r is the countable set $f_r(X) = 1 - \frac{1}{2^n} : n = 0,1,2,...$ and f_r has no fixed point. The mapping f_r in this example satisfies the contractive condition (2.9) with $\delta(\epsilon) = \epsilon$ if $\epsilon \geq 1$ and $\delta(\epsilon) = \frac{1}{2^n} - \epsilon$ if $\frac{1}{2^{n+1}} \leq \epsilon < \frac{1}{2^n}$, $n = 0,1,2,....$

Now we are using the condition of Bisht and Rakocevic [2] for family of mappings: Given x, y in X and $0 \leq a < 1$,

$$K(x,y) = \max[ad(x, f_r x) + (1-a)d(y, f_r y), (1-a)d(x, f_r x) + ad(y, f_r y)].$$

Theorem 2.8. Let $\{f_r : 0 \leq r \leq 1\}$ be a family of self-mappings in a complete metric space (X,d) such that for any given mapping f_r, the following conditions are satisfied: given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon < K(x,y) < \epsilon + \delta \quad \Rightarrow \quad d(f_r x, f_r y) < \epsilon. \tag{2.10}$$

If $a = 0$, then f_r has a unique fixed point whenever f_r is $k-$continuous or f_r^k is continuous for some $k \geq 1$, or f_r^k is weakly orbitally continuous. If $a > 0$, thev f_r^k possesses a unique fixed point at which f_r is continuous.

Proof. If we take $a = 0$ in $K(x,y)$ then condition (2.10) reduces to condition (2.1). When $a > 0$ the proof is similar to the case $a = 0$. As seen in Theorem 2.1, f_r need not be continuous at the fixed point if $a = 0$. We now show that f_r is continuous at the fixed point when $a > 0$. Suppose $a > 0$ and z is the fixed point of f. Let $\{x_n\}$ be any sequence in X such that $x_n \to z$ as $n \to \infty$.

For the sufficient large values of n, we get

$$\begin{aligned} d(z, f_r x_n) &= d(f_r z, f_r x_n) \\ &< \max[ad(z, f_r z)+(1-a)d(x_n, f_r x_n), (1-a)d(x, f_r z)+ad(x_n, f_r x_n)] \\ &= \max[(1-a)d(x_n, f_r x_n), ad(x_n, f_r x_n)] \\ &\leq \max[\epsilon_1 + (1-a)d(z, f_r x_n), \epsilon_2 + ad(z, f_r x_n], \end{aligned}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $n \to \infty$. This yields $d(z, f_r x_n) < \epsilon_1$ or $(1-a)d(z, f_r x_n) < \epsilon_2$. Taking $n \to \infty$, we get $\lim_{n\to\infty} f_r x_n = z = f_r z$. Hence, f_r is continuous at the fixed point. If f_r is $k-$continuous or f_r^k is continuous for some $k \geq 1$, then f_r^k is weakly orbitally continuous and the proof follows. This establishes the theorem. □

Theorem 2.9. *Let $\{f_r : 0 \leq r \leq 1\}$ be a family of self-mappings in a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is a function such that for each x, y in X, we have*

$$d(x, f_r x) \leq \phi(x) - \phi(fx). \tag{2.11}$$

If f_r is weakly orbitally continuous or f_r^k is continuous or f is $k-$continuous for some $k \geq 1$, then f_r has a fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_1 = f_r x_0$, $x_2 = f_r x_1, ...,$ that is $x_n = f_r^n x_0$. Then,

$$d(x_0, x_1) = d(x_0, f_r x_0) \leq \phi(x_0) - \phi(f_r x_0) = \phi(x_0) - \phi(x_1).$$

Similarly, we have

$$\begin{aligned} d(x_1, x_2) &\leq \phi(x_1) - \phi(x_2), \\ d(x_2, x_3) &\leq \phi(x_2) - \phi(x_3), \\ &\vdots \\ d(x_{n-1}, x_n) &\leq \phi(x_{n-1}) - \phi(x_n), \\ d(x_n, x_{n+1}) &\leq \phi(x_n) - \phi(x_{n+1}). \end{aligned}$$

Adding these inequalities, we get

$$\begin{aligned} d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_n, x_{n+1}) &\leq \phi(x_0) - \phi(x_{n+1}) \\ &\leq \phi(x_0). \end{aligned}$$

Taking $n \to \infty$, we get

$$\sum_{n=0}^{n=\infty} d(x_n, x_{n+1}) \leq \phi(x_0). \tag{2.12}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $t \in X$ such that $\lim_{n\to\infty} x_n = t$ and $\lim_{n\to\infty} f_r^n x_n = t$. Suppose that

f_r is weakly orbitally continuous. Since $\{f_n^r x_0\}$ converges for each x_0 in X, weak orbital continuity implies that there exists $y_0 \in X$ such that $f_r^n y_0 \to z$, $f_r^{n+1} y_0 \to f_r z$ for some z in X. This implies that $z = f_r z$, that is, z is a fixed point of f_r. □

Example 2.10. Let $X = [0, \infty)$ equipped with the Euclidean metric. Define $f : X \to X$ by,

$$f_r(x) = \begin{cases} 1, & \text{if } if\ x \leq 1, \\ \frac{x}{3}, & \text{if } if\ x > 1. \end{cases}$$

Then, it is easy to show that f_r is weakly orbitally continuous but not $k-$continuous or orbitally continuous. Let us define $\phi : X \to [0, \infty)$ by,

$$\phi(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ 1 + x, & \text{if } x \geq 1. \end{cases}$$

Then $d(x, f_r x) \leq \phi(x) - \phi(f_r x)$ for each x in X, f_r satisfies all the conditions of above theorem and has a fixed point $x = 2$.

From the above theorem, we can ger the following corollary.

Corollary 2.11. *Let $\{f_r : 0 \leq r \leq 1\}$ be a family of contractive type self-mappings in a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is a function such that for each x in X, we have*

$$d(x, f_r x) \leq \phi(x) - \phi(f_r x). \tag{2.13}$$

If f_r is weakly orbitally continuous or f_r^k is continuous or f_r is $k-$continuous for some $k \geq 1$, then f_r possesses a unique fixed point.

Theorem 2.12. *Suppose $\{f_r : 0 \leq r \leq 1\}$ is a family of self-mappings in a complete metric space (X, d) satisfies the Banach contraction condition:*

$$d(f_r x, f_r y) \leq a d(x, y), \quad 0 \leq a < 1. \tag{2.14}$$

Then there exists a function $\phi : X \to [0, \infty)$ such that for each x in X, we have

$$d(x, f_r x) \leq \phi(x) - \phi(f_r x). \tag{2.15}$$

Proof. For any x in X we have $d(f_r x, f_r^2 x)$, that is,

$$(\frac{1}{a}) d(f_r x, f_r^2 x) \leq d(x, f_r x).$$

By virtue of the inequality, we get

$$\begin{aligned} d(x, f_r x) &= \frac{1}{1-a} d(x, f_r x) - \frac{a}{1-a} d(x, f_r x) \\ &\leq \frac{1}{1-a} d(x, f_r x) - \frac{a}{1-a} \frac{1}{a} d(f_r x, f_r^2 x) \\ &= \frac{1}{1-a} d(x, f_r x) - \frac{1}{1-a} d(f_r x, f_r^2 x) \\ &= \phi(x) - \phi(f_r x), \end{aligned}$$

where $\phi : X \to [0, \infty)$ is defined by $\phi(x) = \frac{1}{1-a} d(x, f_r x)$. □

3. Conclusion

Since weak orbital continuity is general than $k-$continuity. Therefore, above results are generalized than the Pant et al. [18]. In the above theorem we are using family of mapping so this result is better than the result of Pant et al. [17].

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