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SOLVING SYSTEM OF SPLIT COMMON FIXED POINT AND MONOTONE VARIATIONAL INCLUSION PROBLEMS VIA DYNAMICAL STEP-SIZE IN BANACH SPACES

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Abstract. In this paper, we investigate a self adaptive accelerated method for solving split common fixed point problem for a finite family of firmly-nonexpansive type mappings (Type P) and monotone variational inclusion problem in p-uniformly convex and uniformly smooth Banach spaces. Using a modified Halpern method together with an inertial extrapolation method, we prove a strong convergence theorem for solving the aforementioned problems. The implementation of our iterative method does not require prior knowledge of the operator norm. We also provide some numerical examples to show better performance of our method. Our results extend and complement many related results existing in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space X. Given a nonlinear mapping $T : C \to C$, then we define a firmly nonexpansive

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type mapping (see [23]) (also known as type P mapping) as

$$\langle JTx - JTy, Tx - Ty\rangle \leq \langle Jx - Jy, Tx - Ty\rangle \tag{1.1}$$

for all $x, y \in C$, where $\langle ., .\rangle$ denotes the duality pairing between elements of X and X^* and J is the duality mapping of the Banach space X. It is obvious that if $J = I$, then the definition of a firmly nonexpansive mapping in real Hilbert spaces coincides with (1.1).

Let C and Q be nonempty, closed and convex subsets of two real Banach spaces X and Y with their dual spaces X^* and Y^*, respectively. Let $A : X \to Y$ be a bounded linear operator. Several optimization problems such as split variational inequality problem (SVIP), split variational inclusion problem (SVP), split minimization problem (SMP), split equilibrium problem (SEP), among others have been defined in terms of SFP, (see for example [1, 3, 19, 20, 27, 28, 31, 37] and the references therein).

If $C = F(T)$ and $Q = F(U)$, where $F(T)$ and $F(U)$ denote the set of fixed points of T and U, respectively, $T : X \to X$ and $U : Y \to Y$ form nonlinear mappings, then we obtain the split common fixed point problem (SCFPP).
Very recently Taiwo *et al.* [38] proposed an inertial-type shrinking projection algorithm for solving the two-set split common fixed point problems of a type (P) mapping in the framework of Banach spaces and proved a strong convergence theorem. Let $A : X \to X^*$ be a single-valued nonlinear mapping and $B : X \to 2^{X^*}$ be a multi-valued mapping. The monotone variational inclusion problem (MVIP) consists of finding a point $x^* \in X$ such that

$$0 \in A(x) + B(x). \tag{1.2}$$

The solution set of problem (1.2) is denoted by $MVI(A, B)$. Many nonlinear problems arising in applied sciences such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators, (see [15, 24]). MVIP has been an important tool for solving problems arising in mechanics, optimization, nonlinear programming, economics, finance, applied sciences, among others (see [2, 4, 6, 7, 8, 17] and the references therein).

If $A = 0$ in (1.2), then, we have the variational inclusion problem (VIP), which amounts to finding $x \in X$ such that

$$0 \in B(x). \tag{1.3}$$

In the settings of one Hilbert space and one Banach space, Takahashi and Yao [39] investigated the split common null point problem by using hybrid method and shrinking projection method. In the same year, Tang *et al.* [40]

introduced an iterative scheme, which does not involve the projection operator, to approximate a split common fixed point of a quasi pseudo-contractive mapping and an asymptotically nonexpansive mapping in the setting of two Banach spaces, and obtained a weak convergence theorem as well as a strong convergence theorem under the assumption of semi-compactness on the mapping.

In 2019, Ma *et al.* [25] introduced a shrinking algorithm for solving SFP and fixed point problem of quasi-ϕ-nonexpansive mapping in the setting of two Banach spaces, and proved that the sequence generated by the proposed algorithm converges strongly to a common solution of the SFP and fixed point problem. One of the best ways to speed up the convergence rate of iterative algorithms is to combine the iterative scheme with the inertial term. This term is represented by $\theta_n(x_n - x_{n-1})$ and is a remarkable tool for improving the performance of algorithms and it is known to have some nice convergence characteristics. Readers should consult [1, 3, 5, 9, 21, 22, 30, 41] for more information on inertial extrapolation technique.

In this article, we consider the following problem:

$$x^* \in F(Res_\sigma^B \circ A_\sigma^B) \cap \bigcap_{j=1}^{m} T^{-1}(F(V_j)). \tag{1.4}$$

The definitions of Res_σ^B, A_σ^B and V_j are given in Section 2 and Section 3 respectively.

Motivated by the results in [25], [39], [40] and other related results in literature, we introduce a self adaptive accelerated method for solving split common fixed point problem for a finite family of firmly-nonexpansive type mappings (Type P) and monotone variational inclusion problem in p-uniformly convex and uniformly smooth Banach spaces. Using a modified Halpern method together with an inertial extrapolation method, we prove a strong convergence theorem for solving the aforementioned problems.

Lastly, we provide some numerical examples to show better performance of our method in comparison with other related results. Our results extend and complement many related results in literature.

Our proposed method is endowed with the following features:

(1) We considered approximating the solution of problem (1.4) in a p-uniformly convex Banach space to get more general results than the ones in [9, 16, 30, 32].
(2) Our method uses self-adaptive stepsizes and the implementation of our method does not require the prior knowledge of the norm of the

bounded linear operator T, (see [32]). We emphasize that our strong convergence result is free of compactness condition.

(3) We were able to dispense with the condition $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$ which is often required for the inertial method.

(4) Our algorithm does not require at each step of the iteration process, the computation of subsets of C_n, Q_n and D_n (or C_{n+1}) as in the case in [38] and the computation of the projection of the initial point onto their intersection, which leads to a high computational cost of iteration processes.

The removal of all these restrictions makes our work applicable to more real world problems.

Remark 1.1. We emphasize that approximating a common solution of SMVIPs, SFPs and fixed point problem have some possible applications to mathematical models whose constraints can be expressed as SFPs and SMVIPs. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, among others (see [18]).

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively.

Let X be a real Banach space and $f : X \rightarrow \mathbb{R}$, then f is called:

(i) Gâteaux differentiable at $x \in X$, denoted by $f'(x)$ or $\nabla f(x)$, if there exists an element y of X, such that

$$\lim_{t\rightarrow 0} \frac{f(x+ty) - f(x)}{t} = \langle y, f'(x)\rangle, \quad y \in X,$$

f is Gâteaux differentiable on X if f is Gâteaux differentiable at every $x \in X$;

(ii) weakly lower semicontinuous at $x \in X$, if $x_k \rightharpoonup x$ implies $f(x) \leq \liminf_{k\rightarrow\infty} f(x_k)$. f is weakly lower semicontinuous on X, if f is weakly lower semicontinuous at every $x \in X$.

Let $K(X) := \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. The modulus of convexity is the function $\delta_X : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in K(X), \ \|x - y\| \geq \epsilon \right\}.$$

The space E is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let $p > 1$. Then X is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_X(\epsilon) \geq c_p \epsilon^p$ for all $\epsilon \in (0, 2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of X is the function $\rho_X : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in K(X)\right\}.$$

The space X is said to be uniformly smooth if $\frac{\rho_X(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$. Then a Banach space X is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_X(\tau) \leq \kappa_q \tau^q$ for all $\tau > 0$. It is known that X is p-uniformly convex if and only if X^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, (see [13]).

Let $p > 1$ be a real number, the generalized duality mapping $J_p^X : X \to 2^{X^*}$ is defined by

$$J_p^X(x) = \{\overline{x} \in X^* : \langle x, \overline{x}\rangle = \|x\|^p, \;\; \|\overline{x}\| = \|x\|^{p-1}\},$$

where $\langle ., . \rangle$ denotes the duality pairing between elements of X and X^*. In particular, if $p = 2$, then J_2^X is called the normalized duality mapping. If X is p-uniformly convex and uniformly smooth, then X^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_p^X is one-to-one, single-valued and satisfies $J_p^X = (J_q^{X^*})^{-1}$, where $J_q^{X^*}$ is the generalized duality mapping of X^*. Furthermore, if X is uniformly smooth, then the duality mapping J_p^X is norm-to-norm uniformly continuous on bounded subsets of X (see [14] for more details).

If $f : X \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then the Frenchel conjugate of f denoted by $f^* : X^* \to (-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X, \;\, x^* \in X^*\}.$$

Let the domain of f be denoted by $dom f = \{x \in X : f(x) < +\infty\}$. For any $x \in int(dom f)$ and $y \in X$, we denote and define the right-hand derivative of f at x in the direction of y by

$$f^0(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

Definition 2.1. ([10]) Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : X \times X \to [0, +\infty)$ defined by

$$\Delta_f(x, y) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{2.1}$$

is called the Bregman distance with respect of f, where $\langle \nabla f(x), y\rangle = f^0(x, y)$.

It is well known that Bregman distance Δ_f does not satisfy the properties of a metric, because Δ_f fails to satisfy the symmetric and triangular inequality properties. Moreover, it is well known that the duality mapping J_p^X is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$ (see [12]). Using (2.1), one can show that the following equality called three-point identity is satisfied:

$$\Delta_p(x, y) + \Delta_p(y, z) - \Delta_p(x, z) = \langle J_p^X(z) - J_p^X(y), x - y\rangle, \ \forall\ x, y, z \in X.$$

In addition, if $f(x) = \frac{1}{p}||x||^p$, where $\frac{1}{p} + \frac{1}{q} = 1$, then we obtain

$$\begin{aligned}\Delta_f(x, y) = \Delta_p(x, y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y - x, J_p^X(x)\rangle \\ &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y, J_p^X(x)\rangle + \langle x, J_p^X(x)\rangle \\ &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle y, J_p^X(x)\rangle + \|x\|^p \\ &= \frac{1}{p}\|y\|^p + \frac{1}{q}\|x\|^p - \langle y, J_p^X(x)\rangle. \qquad (2.2)\end{aligned}$$

Let $T : C \to C$ be a nonlinear mapping,

(i) a point $p \in C$ is called an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T;

(ii) T is said to be Bregman quasi-nonexpansive, if

$$F(T) \neq \emptyset \text{ and } \Delta_p(u, Tx) \leq \Delta_p(u, x),\ \forall\ x \in C,\ u \in F(T);$$

(iii) T is said to be Bregman relatively nonexpansive, if

$$\hat{F}(T) = F(T) \neq \emptyset \text{ and } \Delta_p(u, Tx) \leq \Delta_p(u, x),\ \forall\ x \in C,\ u \in F(T);$$

(iv) T is said to be Bregman firmly nonexpansive mapping (BFNE) if

$$\langle J_p^X(Tx) - J_p^X(Ty), Tx - Ty\rangle \leq \langle J_p^X(x) - J_p^X(y), Tx - Ty\rangle,\ \forall\ x, y \in C;$$

(v) T is said to be Bregman strongly nonexpansive mapping (BSNE) with $\hat{F}(T) \neq \emptyset$ if

$$\Delta_p(y, Tx) \leq \Delta_p(y, x),\ \forall\ y \in \hat{F}(T)$$

and for any bounded sequence $\{x_n\}_{n\geq 1} \subset C$,

$$\lim_{n\to\infty}(\Delta_p(y, x_n) - \Delta_p(y, Tx_n)) = 0$$

implies

$$\lim_{n\to\infty} \Delta_p(Tx_n, x_n) = 0.$$

Let C be a nonempty, closed and convex subset of X. The metric projection

$$P_C x := \arg\min_{y \in X} ||x - y||, \ x \in X$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_X^p(x - P_C x), z - P_C x \rangle \leq 0, \ \forall \ z \in C. \tag{2.3}$$

Also, the Bregman projection from X onto C denoted by Π_C satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y \in C} \Delta_p(x, y), \ \forall \ x \in X. \tag{2.4}$$

Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space X and $x \in X$. Then the following assertions hold: see [13], $z = \Pi_C x$ if and only if

$$\langle J_X^p(x) - J_X^p(z), \ y - z \rangle \leq 0, \ \forall \ y \in C; \tag{2.5}$$

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y), \ \forall \ y \in C. \tag{2.6}$$

Denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x^*, x \rangle$. Let $B : X \to X^*$ be a set-valued mapping. Then the graph of B is defined as $Gra(B) := \{(x, x^*) \in X \times X^* : x^* \in Bx\}$. A set-valued mapping B is said to be monotone if $\langle x^* - y^*, x - y \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in Gra(B)$ and B is said to be maximal monotone if its graph is not contained in the graph of any other monotone operator on X. Let $B : X \to 2^{X^*}$ be a mapping. Then, B is called a monotone mapping if for any $x, y \in \text{dom}B$, we have

$$u \in Bx \text{ and } v \in By \quad \Rightarrow \quad \langle u - v, x - y \rangle \geq 0. \tag{2.7}$$

The resolvent of B is the operator $\text{Res}_\sigma^B : X \to 2^X$ defined by

$$\text{Res}_\sigma^B = (J_X^p + \sigma B)^{-1} \circ J_X^p, \tag{2.8}$$

where $\circ$ stands for composition. Furthermore, let C be a nonempty, closed and convex subset of a real Banach space X. The mapping $B : X \to 2^{X^*}$ is called Bregman inverse strongly monotone (BISM) if

$$C \cap (\text{dom} f) \cap \text{int}(\ \text{dom} f) \neq \emptyset \tag{2.9}$$

for any $x, y \in C \cap \text{int}(\ \text{dom} g), \ u \in Bx$ and $v \in By$, we have

$$\langle u - v, J_{X^*}^q(J_X^p(x) - u) - J_{X^*}^q(J_X^p(y) - v) \rangle \geq 0. \tag{2.10}$$

Remark 2.2. ([11]) The BISM class of mappings is more general than the class of firmly nonexpansive mappings in Hilbert spaces. Indeed, if $J_X^p = J_{X^*}^q = I$, where I is the identity operator, then (2.10) becomes

$$\langle u - v, x - u - (y - v)\rangle \geq 0 \tag{2.11}$$

which implies that

$$\|u - v\|^2 \leq \langle x - y, u - v\rangle. \tag{2.12}$$

The reader may consult [11, 34] and the references therein for more details on BISM. The anti-resolvent $A_\sigma^B : X \to 2^X$ associated with a mapping $B : X \to 2^{X^*}$ and $\sigma > 0$ is defined by

$$A_\sigma^B := J_{X^*}^q \circ (J_X^p - \sigma B). \tag{2.13}$$

We now enlist some lemmas for the development of our main result.

Lemma 2.3. ([12]) *Let X be a Banach space and $x, y \in X$. If X is q-uniformly smooth. Then there exists $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle J_X^q(x), y\rangle + C_q\|y\|^q.$$

Lemma 2.4. ([35]) *Let X be a p-uniformly convex Banach space, the metric and Bregman distance have the following relation for all $x, y \in X$*

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_X^p(x) - J_X^p(y)\rangle, \tag{2.14}$$

where $\tau > 0$ is a fixed number.

Lemma 2.5. ([36]) *Let X be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : X^* \times X \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p, \ \forall\ x \in X, x^* \in X^*.$$

Then the following assertions hold:

(1) *V_p is nonnegative and convex in the first variable.*

(2) $\Delta_p(J_{X^*}^q(x^*), x) = V_p(x^*, x), \ \forall\ x \in X, \ x^* \in X^*.$

(3) $V_p(x^*, x) + \langle y^*, J_{X^*}^q(x^*) - x\rangle \leq V_p(x^* + y^*, x), \forall\ x \in X, \ x^*, y^* \in X^*.$ *Also for all $x^* \in E$, we have*

$$\Delta_p\left(x^*, J_{X^*}^q\big(\sum_{i=1}^N t_i J_X^p(x_i)\big)\right) \leq \sum_{i=1}^N t_i \Delta_p(x^*, x_i), \tag{2.15}$$

where $\{x_i\}_{i=1}^N \subset X$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum\limits_{i=1}^N t_1 = 1$.

Lemma 2.6. ([13]) *Let X be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in X. Then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ implies $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Definition 2.7. Let C be a nonempty, closed and convex subset of X. A mapping $U : C \to X$ is said to be of type (P) if

$$\langle J_p^X(x - Ux) - J_p^X(y - Uy), Ux - Uy\rangle \geq 0, \ \forall \ x, y \in C.$$

Examples of type (P) mappings can be found in [38].

Lemma 2.8. ([38]) *Let X be a p-uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of X and $U : C \to X$ be a mapping of type (P). Then the following hold:*

(1) *$y \in F(U)$ if and only if $\langle J_p^X(x - Ux), Ux - y\rangle \geq 0$ for every $x \in C$,*
(2) *$F(U)$ is closed and convex,*
(3) *$\hat{F}(U) = F(U)$.*

Lemma 2.9. ([33]) *Let E be a uniformly convex and uniformly smooth Banach space. If $x_0 \in E$ and the sequence $\{\Delta_p(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.10. ([29]) *Let X be a uniformly convex and uniformly smooth Banach space. Let $B : X \to 2^{X^*}$ be a maximal monotone operator and $A : X \to X^*$ be a BISM mapping such that $(A + B)^{-1}(0^*) \neq \emptyset$. Then,*

(1) *$(A + B)^{-1}(0^*) = F(Res_\sigma^B \circ A_\sigma^B)$;*
(2) *$(Res_\sigma^B \circ A_\sigma^B)$ is a BSNE operator with $F(Res_\sigma^B \circ A_\sigma^B) = \hat{F}(Res_\sigma^B \circ A_\sigma^B)$.*

Lemma 2.11. ([42]) *Let $\{a_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n a_{n-1} + t_n s_n + \delta_n, \ \forall n \geq 0,$$

where $\sum_{n=n_0}^{\infty} t_n = +\infty$, $\sum_{n=n_0}^{\infty} \delta_n < +\infty$ for each $n \geq n_0$ (where n_0 is a positive integer) and $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup_{n\to\infty} s_n \leq 0$. Then, the sequence $\{a_n\}$ converges strongly to zero.

Lemma 2.12. ([26]) *Let Υ_n be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Upsilon_{n_k}\}_{k\geq 0}$ of $\{\Upsilon_n\}$ which satisfies $\Upsilon_{n_k} \leq \Upsilon_{n_j+1}$ for all $j \geq 0$. Also, consider a sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) := \max\{k \leq n \mid \Upsilon_{n_k} \leq \Upsilon_{n_k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence satisfying* $\lim_{n\to\infty}\tau(n)=\infty$. *If it holds that* $\Upsilon_{\tau(n)}\leq\Upsilon_{\tau(n)+1}$ *for all* $n\geq n_0$, *then we have*

$$\Upsilon_\tau(n)\leq\Upsilon_{\tau(n)+1}.$$

3. Main result

Now, we present our main result.

Theorem 3.1. *Let* X *and* Y *be p-uniformly convex and uniformly smooth Banach spaces with duals* X^* *and* Y^* *respectively. For* $j=1,2,\cdots,m$, *let* $V_j: Y\to Y$ *be a finite family of type* (P) *mappings and* $T:X\to Y$ *be a bounded linear operator with its adjoint* $T^*:Y^*\to X^*$. *Suppose that* $A:X\to X^*$ *is a BISM mapping and* $B:X\to 2^{X^*}$ *is a maximal monotone mapping such that*

$$\Theta := F(Res_\sigma^B\circ A_\sigma^B)\cap\bigcap_{j=1}^{m}T^{-1}(F(V_j))\neq\emptyset.$$

Assume that $\{\mu_n\}\subset[0,\frac{1}{2}],\{\alpha_n\},\{\beta_n\}$ *and* $\{\theta_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n+\beta_n+\theta_n=1$, $\alpha_n\leq d<1$, $(1-\alpha_n)c<\mu_n$, $c\in(0,\frac{1}{2})$. *Let* $x_0,x_1\in X$ *and* $\{x_n\}$ *be a sequence generated as follows:*

$$\begin{cases} u_n = J_q^{X^*}\Big[J_p^X(x_n)+\mu_n\big(J_p^X(x_{n-1})-J_p^X(x_n)\big)\Big],\\ w_{n,1} = J_q^{X^*}\Big[J_p^X(u_n)-\gamma_{n,1}T^*J_p^Y\big(Tu_n-V_1(Tu_n)\big)\Big],\\ w_{n,2} = J_q^{X^*}\Big[J_p^X(w_{n,1})-\gamma_{n,2}T^*J_p^Y\big(Tw_{n,1}-V_2(Tw_{n,1})\big)\Big],\\ \qquad\vdots\\ w_{n,m} = J_q^{X^*}\Big[J_p^X(w_{n,m-1})-\gamma_{n,m}T^*J_p^Y\big(Tw_{n,m-1}-V_m(Tw_{n,m-1})\big)\Big],\\ x_{n+1} = J_q^{X^*}\Big[\alpha_nJ_p^X(u)+\beta_nJ_p^X(x_n)+\theta_nJ_p^X(Res_\sigma^B\circ A_\sigma^B)w_{n,m}\Big]. \end{cases} \tag{3.1}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon>0$

$$\epsilon\leq\gamma_{n,j}\leq\left(\frac{q\|Tw_{n,j}-V_j(Tw_{n,j-1})\|^p}{C_q\|T^*J_p^Y(Tw_{n,j-1}-V_j(Tw_{n,j-1}))\|^q}-\epsilon\right)^{\frac{1}{q-1}},$$

where $w_{n,0}=u_n$. *Assume that the sequences* $\{\alpha_n\},\{\rho_{n,j}\},\{\beta_n\},\{\theta_n\}$ *and* $\{\mu_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $0<e<\mu_n<\theta_n\leq\frac{1}{2}$, $\forall\ n\geq1$,

(iii) $0 < \liminf_{n\to\infty} \theta_n, \beta_n \leq \limsup_{n\to\infty} \theta_n, \beta_n < 1$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $z \in \Theta$, *where* $z = \Pi_\Theta u$.

Proof. Let $z \in \Theta$. Then using Lemma 2.8, it is obvious that

$$\langle J_p^Y(Tu_n - V_1(Tu_n)), V_1Tu_n - Tz\rangle \geq 0.$$

Hence, we have

$$\begin{aligned}
&\langle J_p^Y(Tu_n - V_1(Tu_n)), Tu_n - Tz\rangle \\
&\quad = \langle J_p^Y(Tu_n - V_1(Tu_n)), Tu_n - V_1Tu_n + V_1Tu_n - Tz\rangle \\
&\quad = \|Tu_n - V_1Tu_n\|^p + \langle J_p^Y(Tu_n - V_1(Tu_n)), V_1Tu_n - Tz\rangle \\
&\quad \geq \|Tu_n - V_1Tu_n\|^p. \qquad (3.2)
\end{aligned}$$

Now, since $V_j, j = 1, 2, \cdots, m$ is a of type (P) mapping, we obtain from (3.1), (3.2) and Lemma 2.3 that

$$\begin{aligned}
\Delta_p(z, w_{n,1}) &= \Delta_p(z, J_q^{X^*}(J_p^X(u_n) - \gamma_{n,1}T^*J_p^Y(Tu_n - V_1(Tu_n)))) \\
&= \frac{1}{p}\|z\|^p - \langle J_p^X(u_n) - \gamma_{n,1}T^*J_p^Y(Tu_n - V_1(Tu_n)), z\rangle \\
&\quad + \frac{1}{q}\|J_p^X(u_n) - \gamma_{n,1}T^*(Tu_n - V_1(Tu_n))\|^q \\
&\leq \frac{1}{p}\|z\|^p - \langle J_p^X(u_n) - \gamma_{n,1}T^*J_p^Y(Tu_n - V_1(Tu - n)), z\rangle \\
&\quad + \frac{\|J_p^X u_n\|^q}{q} - \gamma_{n,1}\langle J_p^Y(Tu_n - V_1(Tu_n)), Tu_n\rangle \\
&\quad + \frac{C_q(\gamma_{n,1})^q}{q}\|T^*J_p^Y(Tu_n - V_1(Tu_n))\|^q \\
&= \frac{1}{p}\|z\|^p - \langle J_p^X(u_n), z\rangle - \gamma_{n,1}\langle J_p^Y(Tu_n - V_1(Tu_n)), Tu_n - Tz\rangle \\
&\quad + \frac{\|J_p^X u_n\|^q}{q} + \frac{C_q(\gamma_{n,1})^q}{q}\|T^*J_p^Y(Tu_n - V_1(Tu_n))\|^q \\
&= V_p(z, J_p^X u_n) - \gamma_{n,1}\langle J_p^Y(Tu_n - V_1(Tu_n)), Tu_n - Tz\rangle \\
&\quad + \frac{C_q(\gamma_{n,1})^q}{q}\|T^*J_p^Y(Tu_n - V_1(Tu_n))\|^q \\
&\leq \Delta_p(z, u_n) - \gamma_{n,1}\|Tu_n - V_1(Tu_n)\|^p \\
&\quad + \frac{C_q(\gamma_{n,1})^q}{q}\|T^*J_p^Y(Tu_n - V_1(Tu_n))\|^q. \qquad (3.3)
\end{aligned}$$

Using our step-size, we have $\epsilon \leq \gamma_{n,j}$ and

$$\gamma_{n,j} \leq \left(\frac{q\|Tw_{n,j} - V_j(Tw_{n,j-1})\|^p}{C_q\|T^* J_p^Y (Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|^q} - \epsilon \right)^{\frac{1}{q-1}}.$$

This implies that

$$\gamma_{n,j}^{q-1} \leq \frac{q\|Tw_{n,j} - V_j(Tw_{n,j-1})\|^p}{C_q\|T^* J_p^Y (Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|^q} - \epsilon.$$

Hence,

$$\begin{aligned} &\frac{C_q \epsilon}{q} \|T^* J_p^Y (Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|^q \\ &\quad \leq \|Tw_{n,j} - V_j(Tw_{n,j-1})\|^p - \frac{C_q(\gamma_{n,1})^q}{q} \|T^* J_p^Y (Tu_n - V_1(Tu_n))\|^q. \end{aligned}$$

For $\epsilon \leq \gamma_{n_j}$, we have

$$\begin{aligned} &\frac{C_q \epsilon^2}{q} \|T^* J_p^Y (Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|^q \\ &\quad \leq \gamma_{n,j} \frac{C_q \epsilon}{q} \|T^* J_p^Y (Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|^q \\ &\quad \leq \gamma_{n,j} \left(\|Tw_{n,j} - V_j(Tw_{n,j-1})\|^p - \frac{C_q(\gamma_{n,1})^{q-1}}{q} \|T^* J_p^Y (Tu_n - V_1(Tu_n))\|^q \right). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned} \Delta_p(z, w_{n,1}) &\leq \Delta_p(z, u_n) - \gamma_{n,1} \Big(\|Tu_n - V_1(Tu_n)\|^p \\ &\qquad - \frac{C_q(\gamma_{n,1})^{q-1}}{q} \|T^* J_p^Y (Tu_n - V_1(Tu_n))\|^q \Big) \\ &\leq \Delta_p(z, u_n) - \frac{C_q \epsilon^2}{q} \|T^* J_p^Y (Tu_n - V_1(Tu_n))\|. \end{aligned} \tag{3.5}$$

Following a similar approach for $j = 2, \cdots, m$ yields

$$\Delta_p(z, w_{n,j}) \leq \Delta_p(z, w_{n,j-1}) - \frac{C_q \epsilon^2}{q} \|T^* J_p^Y (Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|. \tag{3.6}$$

Substituting $w_{n,0} = u_n$, we have from the above inequality that

$$\Delta_p(z, w_{n,m}) \leq \Delta_p(z, u_n) - \frac{C_q \epsilon^2}{q} \|T^* J_p^Y (Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|. \tag{3.7}$$

By applying condition (iv) of (3.1) for all $n \in \mathbb{N}$, we get

$$\Delta_p(z, w_{n,j}) \leq \Delta_p(z, w_{n,j-1}) \ \ (j = 1, 2, \cdots, m). \tag{3.8}$$

From (3.1), (3.7) and (2.15), we get

$$\begin{aligned}
\Delta_p(z, x_{n+1}) &= \Delta_p(z, J_q^{X^*}(\alpha_n J_p^X(u) + \beta_n J_p^X(x_n) + \theta_n J_p^X(Res_\sigma^B \circ A_\sigma^B)w_{n,m}) \\
&\le \alpha_n \Delta_p(z, u) + \beta_n \Delta_p(z, x_n) + \theta_n \Delta_p(z, (Res_\sigma^B \circ A_\sigma^B)w_{n,m}) \\
&\le \alpha_n \Delta_p(z, u) + \beta_n \Delta_p(z, x_n) + \theta_n \Delta_p(z, w_{n,m}) \\
&\le \alpha_n \Delta_p(z, u) + \beta_n \Delta_p(z, x_n) + \theta_n \Delta_p(z, u_n) \\
&\le \alpha_n \Delta_p(z, u) + \beta_n \Delta_p(z, x_n) \\
&\quad + \theta_n((1-\mu_n)\Delta_p(z, x_n) + \mu_n \Delta_p(z, x_{n-1})) \\
&= \alpha_n \Delta_p(z, u) + (\beta_n + \theta_n)\Delta_p(z, x_n) \\
&\quad - \theta_n \mu_n \Delta_p(z, x_n) + \theta_n \mu_n \Delta_p(z, x_{n-1}) \\
&= \alpha_n \Delta_p(z, u) + (1-\alpha_n)\Delta_p(z, x_n) \\
&\quad - \theta_n \mu_n \Delta_p(z, x_n) + \theta_n \mu_n \Delta_p(z, x_{n-1}) \\
&= \alpha_n \Delta_p(z, u) + (1-\alpha_n - \theta_n \mu_n)\Delta_p(z, x_n) + \theta_n \mu_n \Delta_p(z, x_{n-1}) \\
&\le \max\{\Delta_p(z, u), \Delta_p(z, x_n), \Delta_p(z, x_{n-1})\}, \ \forall\ n \ge 1.
\end{aligned}$$

By induction,

$$\Delta_p(z, x_n) \le \max\{\Delta_p(z, u), \Delta_p(z, x_1), \Delta_p(z, x_0)\}.$$

Hence, $\{\Delta_p(z, x_n)\}$ is bounded. Consequently, $\{\Delta_p(z, w_{n,m})\}$ and $\{\Delta_p(x^*, u_n)\}$ are bounded. In view of Lemma 2.9, we conclude that $\{x_n\}$, $\{u_n\}$ $\{w_{n_m}\}$ are bounded. Using (3.1), (3.7) and Lemma 2.5 (iii), we get

$$\begin{aligned}
\Delta_p(z, x_{n+1}) &= \Delta_p(z, J_q^{X^*}(\alpha_n J_p^X(u) + \beta_n J_p^X(x_n) + \theta_n J_p^X(Res_\sigma^B \circ A_\sigma^B)w_{n,m}) \\
&= V_p(z, \alpha_n J_p^X(u) + \beta_n J_p^X(x_n) + \theta_n J_p^X(w_{n,m})) \\
&\le V_p(z, \alpha_n J_p^X(u) + \beta_n J_p^X(x_n) + \theta_n J_p^X(w_{n,m})) \\
&\quad - \alpha_n(J_p^X(u) - J_p^X(z)) - \langle -\alpha_n(J_p^X(u) - J_p^X(z)), J_q^{X^*}(\alpha_n J_p^X(u) \\
&\quad + \beta_n J_p^X(x_n) + \theta_n J_p^X(w_{n,m})) - z\rangle \\
&= V_p(z, \alpha_n J_p^X(u) + \beta_n J_p^X(x_n) + \theta_n J_p^X(w_{n,m}) \\
&\quad + \alpha_n \langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle \\
&= \Delta_p(z, J_q^{X^*}(\alpha_n J_p^X(z) + \beta_n J_p^X(x_n) + \theta_n J_p^X(w_{n,m})) \\
&\quad + \alpha_n \langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle \\
&= \alpha_n \Delta_p(z, z) + \beta_n \Delta_p(z, x_n) + \theta_n \Delta_p(z, w_{n,m}) \\
&\quad + \alpha_n \langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\leq \alpha_n\Delta_p(z,z) + \beta_n\Delta_p(z,x_n) + \theta_n\Delta_p(z,u_n)\\
&\quad + \alpha_n\langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle\\
&\leq \alpha_n\Delta_p(z,z) + \beta_n\Delta_p(z,x_n)\\
&\quad + \theta_n[(1-\mu_n)\Delta_p(z,x_n) + \mu_n\Delta_p(z,x_{n-1})]\\
&\quad + \alpha_n\langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle\\
&= (1-\alpha_n-\theta_n\mu_n)\Delta_p(z,x_n) + \theta_n\mu_n\Delta_p(z,x_{n-1})\\
&\quad + \alpha_n\langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle.
\end{aligned}$$

Case 1: Assume that $\{\Delta_p(z,x_n)\}$ is monotone decreasing, that is,

$$\Delta_p(z,x_{n+1}) \leq \Delta_p(z,x_n),$$

since $\Delta_p(x^*,x_n) \leq M$ for all $n \geq 1$, where

$$M := \max\{\Delta_p(z,u), \Delta_p(z,x_1), \Delta_p(z,x_0)\},$$

which means $\{\Delta_p(z,x_n)\}$ is bounded. Then $\{\Delta_p(z,x_n)\}$ is convergent. Thus,

$$\begin{aligned}
\lim_{n\to\infty}\big(\Delta_p(x^*,x_n) - \Delta_p(x^*,x_{n+1})\big) &= \lim_{n\to\infty}\big(\Delta_p(x^*,x_{n-1}) - \Delta_p(x^*,x_n)\big)\\
&= 0. \qquad (3.10)
\end{aligned}$$

From (3.7) and (3.9), we obtain that

$$\begin{aligned}
\Delta_p(z,x_{n+1}) &\leq \beta_n\Delta_p(z,x_n) + \theta_n\Delta_p(z,w_{n,m})\\
&\quad + \alpha_n\langle J_E^p(u) - J_E^p(z), x_{n+1} - z\rangle\\
&\leq \beta_n\Delta_p(z,x_n) + \theta_n\Delta_p(z,u_n)\\
&\quad - \theta_n\frac{C_q\epsilon^2}{q}\|T^*J_p^Y(Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|\\
&\quad + \alpha_n\langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle\\
&\leq (1-\alpha_n)\Delta_p(z,x_n) + \theta_n\mu_n(\Delta_p(z,x_{n-1}) - \Delta_p(z,x_n))\\
&\quad - \theta_n\frac{C_q\epsilon^2}{q}\|T^*J_p^Y(Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|. \qquad (3.11)
\end{aligned}$$

Using (3.10) and condition (iv) of (3.1) in (3.11), we get

$$\begin{aligned}
&\theta_n\frac{C_q\epsilon^2}{q}\|T^*J_p^Y(Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|\\
&\quad \leq (1-\alpha_n)\Delta_p(z,x_n) - \Delta_p(z,x_{n+1}) + \theta_n\mu_n(\Delta_p(z,x_{n-1}) - \Delta_p(z,x_n))\\
&\qquad + \alpha_n\langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle. \qquad (3.12)
\end{aligned}$$

By passing the limit on (3.12), we obtain that

$$\|Tw_{n,j-1} - V_j(Tw_{n,j-1})\| = 0 \ \ (j = 1, 2, \cdots, m). \qquad (3.13)$$

More so, for $j = 1, 2, \cdots, m$, we have

$$\begin{aligned}\|J_p^X(w_{n,j}) - J_p^X(w_{n,j-1})\| &\leq \gamma_{n,j}\|T^* J_p^Y(Tw_{n,j-1} - V_j(Tw_{n,j-1}))\| \\ &\leq \gamma_{n,j}\|T^*\|\|(Tw_{n,j-1} - V_j(Tw_{n,j-1}))\|^{p-1} \\ &\to 0. \end{aligned} \tag{3.14}$$

Thus, we have

$$\lim_{n\to\infty} \|J_p^X(w_{n,j}) - J_p^X(w_{n,j-1})\| = 0, \ j = 1, 2, 3, \cdots, m.$$

By uniform continuity of $J_p^{X^*}$ on bounded subsets of X^*, we conclude that

$$\lim_{n\to\infty} \|w_{n,j} - w_{n,j-1}\| = 0, \ j = 1, 2, \cdots, m. \tag{3.15}$$

Let $u_n = J_q^{X^*}\big(J_p^X x_n + \mu_n(J_p^X(x_{n-1}) - J_p^X(x_n)\big)$. It then follows that

$$J_p^X u_n - J_p^X x_n = \mu_n\big(J_p^X(x_{n-1}) - J_p^X(x_n)\big).$$

Now by the uniform continuity of J_p^X on bounded subsets of X, we get that

$$\begin{aligned} &||J_p^X u_n - J_p^X x_n||_* \\ &\quad = ||\mu_n\big(J_p^X(x_{n-1}) - J_p^X(x_n)\big)||_* \\ &\quad \leq \mu_n||J_p^X(x_{n-1}) - J_p^X(x_n)||_* \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.16}$$

By the uniform continuity of $J_q^{X^*}$ on bounded subsets of X^* and (3.16), we obtain that

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.17}$$

Using (3.15) and (3.17), it is obvious for $j = 1, 2, \cdots, m$ that

$$\lim_{n\to\infty} \|w_{n,j} - x_n\| = 0. \tag{3.18}$$

Let

$$v_n := J_q^{X^*}\big(\frac{\beta_n}{1-\alpha_n} J_p^X x_n + \frac{\theta_n}{1-\alpha_n} J_p^X(Res_\sigma^B \circ A_\sigma^B)w_{n,m})\big).$$

Then we obtain from (2.2) that

$$\begin{aligned}
\Delta_p(z, v_n) &= \Delta_p\big(z, J_q^{X^*}(\frac{\beta_n}{1-\alpha_n} J_p^X x_n + \frac{\theta_n}{1-\alpha_n} J_p^X (Res_\sigma^B \circ A_\sigma^B) w_{n,m})\big) \\
&\leq \frac{\beta_n}{1-\alpha_n} \Delta_p(z, x_n) + \frac{\theta_n}{1-\alpha_n} \Delta_p(z, (Res_\sigma^B \circ A_\sigma^B) w_{n,m}) \\
&\leq \frac{\beta_n}{1-\alpha_n} \Delta_p(z, x_n) + \frac{\theta_n}{1-\alpha_n} \Delta_p(z, w_{n,m}) \\
&\leq \frac{\beta_n}{1-\alpha_n} \Delta_p(z, x_n) + \frac{\theta_n}{1-\alpha_n} \Delta_p(z, u_n) \\
&\leq \frac{\beta_n}{1-\alpha_n} \Delta_p(z, x_n) + \frac{\theta_n}{1-\alpha_n} \Delta_p(z, x_n) \\
&\quad + \frac{\theta_n \mu_n}{1-\alpha_n} (\Delta_p(z, x_{n-1}) - \Delta_p(z, x_n)) \\
&= \Delta_p(z, x_n) + \frac{\theta_n \mu_n}{1-\alpha_n} (\Delta_p(z, x_{n-1}) - \Delta_p(z, x_n)).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
0 &\leq \Delta_p(z, x_n) - \Delta_p(z, x_{n+1}) + \Delta_p(z, x_{n+1}) \\
&\quad + \frac{\theta_n \mu_n}{1-\alpha_n} (\Delta_p(z, x_{n-1}) - \Delta_p(z, x_n)) - \Delta_p(z, v_n) \\
&\leq \Delta_p(z, x_n) - \Delta_p(z, x_{n+1}) + \alpha_n \Delta_p(z, u) + (1-\alpha_n) \Delta_p(z, v_n) \\
&\quad + \frac{\theta_n \mu_n}{1-\alpha_n} (\Delta_p(z, x_{n-1}) - \Delta_p(z, x_n)) - \Delta_p(z, v_n) \\
&= \Delta_p(z, x_n) - \Delta_p(z, x_{n+1}) + \alpha_n (\Delta_p(z, u) - \Delta_p(z, v_n)) \\
&\quad + \frac{\theta_n \mu_n}{1-\alpha_n} (\Delta_p(z, x_{n-1}) - \Delta_p(z, x_n)) - \Delta_p(z, v_n).
\end{aligned}$$

Hence,

$$\Delta_p(z, x_n) - \Delta_p(z, v_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.19}$$

Also,

$$\begin{aligned}
\Delta_p(z, v_n) &\leq \frac{\beta_n}{1-\alpha_n} \Delta_p(z, x_n) + \frac{\theta_n}{1-\alpha_n} \Delta_p(z, (Res_\sigma^B \circ A_\sigma^B) w_{n,m}) \\
&= (1 - \frac{\theta_n}{1-\alpha_n}) \Delta_p(z, x_n) + \frac{\theta_n}{1-\alpha_n} \Delta_p(z, (Res_\sigma^B \circ A_\sigma^B) w_{n,m}) \\
&= \Delta_p(z, x_n) + \frac{\theta_n}{1-\alpha_n} \big(\Delta_p(z, (Res_\sigma^B \circ A_\sigma^B) w_{n,m}) - \Delta_p(z, x_n)\big).
\end{aligned}$$

Since $\alpha_n + \theta_n \leq 1$, then from (3.19), we obtain

$$\begin{aligned}&\big(\Delta_p(z, x_n) - \Delta_p(z, (Res_\sigma^B \circ A_\sigma^B)w_{n,m})\big)\\ &\quad \leq \frac{\theta_n}{1-\alpha_n}\big(\Delta_p(z, x_n) - \Delta_p(z, (Res_\sigma^B \circ A_\sigma^B)w_{n,m})\big)\\ &\quad \leq \Delta_p(z, x_n) - \Delta_p(z, v_n) \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence,

$$\Delta_p(z, x_n) - \Delta_p(z, (Res_\sigma^B \circ A_\sigma^B)w_{n,m}) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $Res_\sigma^B \circ A_\sigma^B$ is a BSNE operator, then it is strongly nonexpansive and thus

$$\lim_{n\to\infty} \Delta_p(x_n, (Res_\sigma^B \circ A_\sigma^B)w_{n,m}) = 0, \tag{3.20}$$

which implies from Lemma 2.6 that

$$\lim_{n\to\infty} \left|\left|x_n - (Res_\sigma^B \circ A_\sigma^B)w_{n,m}\right|\right| = 0. \tag{3.21}$$

Also, from (3.1) and (3.20), we obtain that

$$\begin{aligned}\Delta_p(x_n, x_{n+1}) &\leq \alpha_n\Delta_p(x_n, u) + \beta_n\Delta_p(x_n, x_n) + \theta_n\Delta_p(x_n, (Res_\sigma^B \circ A_\sigma^B)w_{n,m}))\\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence, by Lemma 2.6, we obtain that

$$\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0. \tag{3.22}$$

Finally, using (3.18) and (3.21), we get

$$\lim_{n\to\infty} \|w_{n,j} - Res_\sigma^B \circ A_\sigma^B)w_{n,m}\| = 0,\ j = 1, 2, \cdots, m. \tag{3.23}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ which converges weakly to $x^* \in X$. From (3.17) and (3.23), there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$ and $\{w_{n_k,m}\}$ of $\{w_{n,m}\}$ which converge weakly to $x^* \in X$. More so, since $\|w_{n,j-1} - x_n\| \to 0$, we have that $w_{n,j-1} \rightharpoonup x^*, j = 1, 2, \cdots, m$. Also, using the fact that T is a bounded linear operator, we obtain that $Tw_{n_k,j-1} \rightharpoonup Tx^* \in Y$ as $k \to \infty$. Thus $Tx^* \in \bigcap_{j=1}^{m} F(V_j)$. In addition, applying Lemma 2.10 and (3.23), we obtain that $x^* \in \hat{F}(Res_\sigma^B \circ A_\sigma^B) = F(Res_\sigma^B \circ A_\sigma^B)$. Hence, we conclude that $x^* \in \Theta$.

Next, as $x_{n_k} \rightharpoonup x^* \in \Theta$, so for any $x^* = \Pi_\Theta u$, we get from (2.5) that

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^X(u) - J_p^X(z), x_n - z\rangle &= \lim_{k\to\infty}\langle J_p^X(u) - J_p^X(z), x_{n_k} - z\rangle \\ &= \langle J_p^X(u) - J_p^X(z), x^* - z\rangle \\ &\leq 0. \end{aligned} \tag{3.24}$$

Furthermore,

$$\begin{aligned}\langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle &= \langle J_p^X(u) - J_p^X(z), x_{n+1} - x_n\rangle \\ &\quad + \langle J_p^X(u) - J_p^X(z), x_n - z\rangle.\end{aligned}$$

Hence, from (3.22) and (3.24), we obtain that

$$\limsup_{n\to\infty}\langle J_p^X(u) - J_p^X(z), x_{n+1} - z\rangle \leq 0. \tag{3.25}$$

By applying Lemma 2.11, (3.9) and (3.25), we obtain that $\{x_n\}$ converges strongly to z.

Case 2: Assume that $\{\Delta_p(z, x_n)\}$ is non-monotone. Set $\Upsilon_n = \Delta_p(z, x_n)$ as stated in Lemma 2.12 and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$, (for some n_0 large enough) defined by $\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Upsilon_k \leq \Upsilon_{k+1}\}$. Then $\{\tau(n)\}$ is non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$. Thus

$$0 \leq \Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1},\ \forall\ n \geq n_0,$$

this implies that

$$\Delta_p(z, x_{\tau(n)}) \leq \Delta_p(z, x_{\tau(n)+1}),\ n \geq n_0.$$

Now following the estimation process of Case 1, we have that

$$\begin{cases}\lim\limits_{\tau(n)\to\infty} \left|\left|x_\tau(n) - (Res_\sigma^B \circ A_\sigma^B)w_{\tau(n),m}\right|\right| = 0, \\ \lim\limits_{\tau(n)\to\infty} ||u_{\tau(n)} - x_{\tau(n)}|| = 0, \\ \lim\limits_{\tau(n)\to\infty} ||w_{\tau(n),j-1} - x_{\tau(n)}|| = 0, \\ \lim\limits_{\tau(n)\to\infty} \|Tw_{\tau(n),j-1} - V_j(Tw_{\tau(n),j-1})\| = 0,\ (j = 1, 2, \cdots, m), \\ \lim\limits_{\tau(n)\to\infty} \langle J_p^X(u) - J_p^X(z), x_{\tau(n)+1} - z\rangle \leq 0.\end{cases} \tag{3.26}$$

From (3.1) and $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$, we have

$$\begin{aligned}\Delta_p(z, x_{\tau(n)+1}) &\leq (1 - \alpha_n - \theta_{\tau(n)}\mu_{\tau(n)})\Delta_p(z, x_{\tau(n)}) + \mu_{\tau(n)}\theta_{\tau(n)}\Delta_p(z, x_{\tau(n)-1}) \\ &\quad + \alpha_{\tau(n)}\langle J_{E_1}^p(u) - J_E^p(z), x_{\tau(n)+1} - z\rangle.\end{aligned}$$

So, we obtain

$$\Delta_p(z, x_{\tau(n)}) \leq \Delta_p(z, x_{\tau(n)+1}) \leq \langle J_E^p(u) - J_E^p(z), x_{\tau(n)+1} - z\rangle.$$

Hence, from (3.26), we get

$$\lim_{\tau(n)\to\infty} \Delta_p(z, x_{\tau(n)}) = 0$$

and

$$\Delta_p(z, x_{\tau(n)+1}) = 0.$$

Thus,

$$\lim_{\tau(n)\to\infty} \Upsilon_{\tau(n)} = \lim_{\tau(n)\to\infty} \Upsilon_{\tau(n)+1} = 0 \tag{3.27}$$

for all $n \geq n_0$. So we have that $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$, if $n \neq \tau(n)$ (that is, $\tau(n) < n$), since $\Upsilon_{k+1} \leq \Upsilon_k$ for some $\tau(n) \leq k \leq n$. Hence, we obtain for all $n \geq n_0$,

$$0 \leq \Upsilon_n \leq \max\left\{\Upsilon_{\tau(n)}, \Upsilon_{\tau(n)+1}\right\} = \Upsilon_{\tau(n)+1}.$$

This implies that $\lim\limits_{n\to\infty} \Upsilon_n = 0$ which implies that $\lim\limits_{n\to\infty} \Delta_p(z, x_n) = 0$ $n \to \infty$. Hence $x_n \to z = \Pi_\Theta u$ as $n \to \infty$. □

By taking $j = 1$ in Theorem 3.1, then (3.1) becomes

Corollary 3.2.

$$\begin{cases} u_n = J_q^{X^*}\Big[J_p^X(x_n) + \mu_n\big(J_p^X(x_{n-1}) - J_p^X(x_n)\big)\Big], \\ w_n = J_q^{X^*}\Big[J_p^X(u_n) - \gamma_{n,1}T^*J_p^Y\big(Tu_n - V(Tu_n)\big)\Big], \\ x_{n+1} = J_q^{X^*}\Big[\alpha_n J_p^X(u) + \beta_n J_p^X(x_n) + \theta_n J_p^X(Res_\sigma^B \circ A_\sigma^B)w_n\Big]. \end{cases} \tag{3.28}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$

$$\epsilon \leq \gamma_n \leq \left(\frac{q\|Tw_n - V(Tw_n)\|^p}{C_q\|T^*J_p^Y(Tw_n - V(Tw_n))\|^q} - \epsilon\right)^{\frac{1}{q-1}},$$

where $w_{n,0} = u_n$. Assume that the sequences $\{\alpha_n\}, \{\rho_{n,j}\}, \{\beta_n\}, \{\theta_n\}$ and $\{\mu_n\}$ satisfy the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < e < \mu_n < \theta_n \leq \frac{1}{2}$, $\forall\ n \geq 1$,
(iii) $0 < \liminf\limits_{n\to\infty} \theta_n, \beta_n \leq \limsup\limits_{n\to\infty} \theta_n, \beta_n < 1$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $z \in \Theta$, where $z = \Pi_\Theta u$.

By taking $j = 1$ and $A_\sigma^B = I$ in Theorem 3.1, where I is an identity operator, then (3.1) becomes

Corollary 3.3.

$$\begin{cases} u_n = J_q^{X^*}\Big[J_p^X(x_n) + \mu_n\big(J_p^X(x_{n-1}) - J_p^X(x_n)\big)\Big], \\ w_n = J_q^{X^*}\Big[J_p^X(u_n) - \gamma_{n,1}T^*J_p^Y\big(Tu_n - V(Tu_n)\big)\Big], \\ x_{n+1} = J_q^{X^*}\Big[\alpha_n J_p^X(u) + \beta_n J_p^X(x_n) + \theta_n J_p^X(Res_\sigma^B)w_n\Big]. \end{cases} \tag{3.29}$$

Suppose the stepsizes are chosen in such a way that for small enough $\epsilon > 0$

$$\epsilon \le \gamma_n \le \left(\frac{q\|Tw_n - V(Tw_n)\|^p}{C_q\|T^*J_p^Y(Tw_n - V(Tw_n))\|^q} - \epsilon\right)^{\frac{1}{q-1}},$$

where $w_{n,0} = u_n$. Assume that the sequences $\{\alpha_n\}, \{\rho_{n,j}\}, \{\beta_n\}, \{\theta_n\}$ and $\{\mu_n\}$ satisfy the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < e < \mu_n < \theta_n \le \frac{1}{2}, \ \forall\ n \ge 1$,

(iii) $0 < \liminf\limits_{n\to\infty} \theta_n, \beta_n \le \limsup\limits_{n\to\infty} \theta_n, \beta_n < 1$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $z \in \Theta$, where $z = \Pi_\Theta u$.

4. Numerical example

In this section, we present a numerical example to illustrate the performance of our method in ℓ_3 space which is a uniformly convex and 2-uniformly smooth Banach space but not a Hilbert space.

Example 4.1. Let $X = Y = \ell_3$. We denote the closed ball in ℓ_3 centred at $b_j \in \ell_3$ with radius $r > 0$, by $Q_j = \{y \in \ell_3 : \|y - b_j\| \le r\}$. We define the mapping $A : \ell_3 \to \ell_3$ by $Ax = \frac{x}{3}$. Then A is $\sigma-$ BISM for $0 < \sigma \le 3$, for each $x = (x_1, x_2, x_3, \cdots) \in \ell_3$. Define $Bx = \frac{x}{\sigma}$, for each $x \in \ell_3$ where $x = (x_1, x_2, x_3, \cdots)$ and $T : X \to Y$ by $Tx = \frac{5}{4}x$ for each $x \in \ell_3$ where $x = (x_1, x_2, x_3, \cdots)$. By some simple calculations, we obtain the following for some $\sigma > 0$:

$$Res_\sigma^B \circ A_\sigma^B x := (I + \sigma B)^{-1} \circ (I - \sigma B)x = \frac{x}{2}(1 - \frac{\sigma}{3}).$$

Now, for $j = 1, \cdots, m$, let $V_j : \ell_3 \to \ell_3$ be defined by

$$V_j(y) = P_{Q_j}(y) = \begin{cases} b_j + r\frac{y - b_i}{\|y - b_j\|}, & \text{if } \|y - b_j\| < r, \\ y, & \text{otherwise.} \end{cases}$$

Then it is easy to see that P_{Q_j} is firmly nonexpansive, hence nonexpansive. For this experiment, let $\alpha_n = \frac{1}{150n+1}$, $\beta_n = \frac{1}{2n+13}$, $\theta_n = 1 - \alpha_n - \beta_n$, and $\mu_n = \frac{1}{4}$. Let $E_n = \|x_{n+1} - x_n\|^2 \leq 10^{-4}$, be the stopping criterion.

Case 1. $x_0 = (0.5, 0.35, \cdots)$ and $x_1 = (0.78, 1.25, \cdots)$;
Case 2. $x_0 = (1.5, 2.35, \cdots)$ and $x_1 = (3.78, 1.25, \cdots)$;
Case 3. $x_0 = (0, 3, \cdots)$ and $x_1 = (4, 2, \cdots)$;
Csae 4. $x_0 = (-4, -4, \cdots)$ and $x_1 = (-10, -20, \cdots)$.
The results of this experiment are reported in Figure 1.

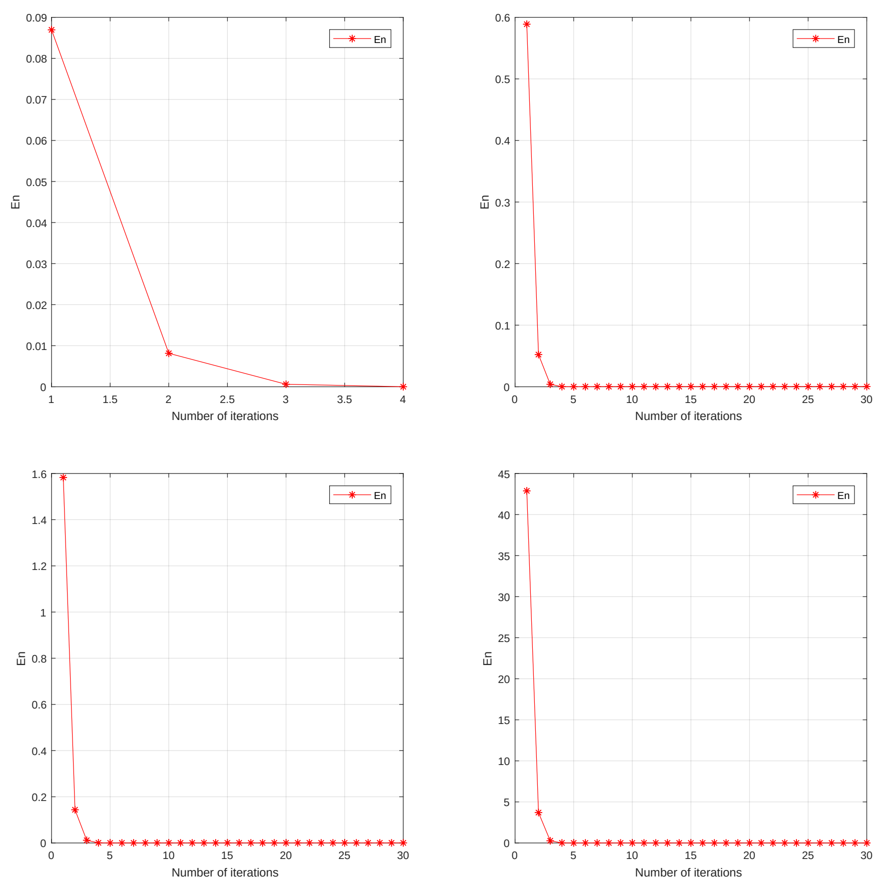

FIGURE 1. Example 4.1. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

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