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LINEARIZATION OF LAPLACIAN OF A HYPERHOLOMORPHIC FUNCTION OF EIGHT REAL VARIABLES

Ji Eun Kim

Department of Mathematics, Dongguk University,
Gyeongju-si 38066, Republic of Korea
e-mail: jeunkim@pusan.ac.kr

Abstract. When defining a hyperholomorphic function by defining a number system as non-commutative base for products, the proposed differential operator is investigated, and differences with variables in the existing number system are discovered in Octonions. We define algebraic properties and operations for elements of Cayley algebra called octonion and we give a function using octonion as a variable. We propose a corresponding differential operator and Investigate that the defined differential operator is the linearization of Laplacian on real multivariate. By defining the hyperholomorphic function of the octonion variable, the utilization and properties of differential operators are investigated.

1. Introduction

In 1845, Cayley [2] discovered a certain algebra, which generally does not hold the associative law for products, but has a characteristic that satisfies

$$(zz)w = z(zw) \quad \text{and} \quad w(zz) = (wz)z$$

for any element z and w of the algebra (referenced [13, 15]). This algebra is called Cayley algebra after Cayley and denoted by $\mathcal{A}$. Cayley algebra extends the definition of the multiplicative 'norm', denoted by $n(z)$, in the complex number system to define 'norm' on Cayley algebra. From this it satisfies

$$n(zw) = n(z)n(w).$$

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Only complex, quaternion, and Cayley algebras (or octonion) are algebras of real coefficients that deal with this norm (referenced [1, 3, 5]). They investigated whether the theories and properties of the analytic function established in the complex number also hold in the octonions. Or whether the modification of the form or the transformation of the expression is necessary due to the intrinsic properties of the octonions. In particular, the expression of differential operators focusing on the shape and structure of variables used in algebra is investigated. When the differential operator is well defined, it is useful to define the analytic function and derive the solution of the differential equation.

We propose the differential operator on the octonion, just as we give the properties of the differential operator D using the coefficient α_j $(j = 0, 1, \ldots, n)$ on the real Clifford algebra. All solutions of the differential equation $Df = 0$ are always solutions of the Laplace equation $\Delta f = 0$, where Δ is Laplacian of $(n+1)$-real variables. Many studies also pay attention to the expression and application of differential operators. In the case of quaternions, researchers have investigated the definitions and properties related to the solutions of differential equations. This work is derived according to the definition of differential operators, starting with Fueter's work [6]. Naser [10] defined a function with Clifford algebra as a variable on Clifford algebra and defined the hyperholomorphic function by citing the hyperholomorphicity of the quaternionic function.

Nôno [11] focused on and applied the hyperholomorphicity of the quaternion function based on the content mentioned in Naser's research, and confirmed the properties of the quaternion hyperholomorphic function. Nôno [12] proposes a quaternion hyperholomorphic function by defining partial differential operators on quaternions and determining the possibility of linearization, and defines properties.

To define the hyperholomorphic function by defining the regularity of the quaternion, Sudbey [14] proposed and tested various interpretation theories in the quaternion. Based on Sudbey's research, Kim et. al. [8, 9, 7] investigated the form and properties of the hyperholomorphic function defined on quaternions. Also, according to the expression defining the hyperholomorphic function, the application and operation of the hyperholomorphic function considering the properties of the quaternion were confirmed. Dentoni and Sce [4] gave a construction of octonionic regular functions from the theory of holomorphic functions of one complex variable and two Cauchy type integral theorem by using the differential operator composed of partial derivatives for each real components.

In this paper, we give a characterization of a differential operator $D = \sum_{j=0}^{7} \frac{\partial \alpha_j}{\partial x_j}$ with coefficients α_j and a hyperholomorphic function of an octonion variable. In Section 2, the algebraic properties of octonion are presented, the differential operators to be used in octonion are expressed using partial derivatives and matrices, respectively, and the properties of each expression are investigated. Section 3 investigates a characterization of a differential operator $D = \sum_{j=0}^{n} \frac{\partial \alpha_j}{\partial x_j}$ with coefficients α_j of Cayley numbers $(j = 0, 1, \ldots, n, m \geq 1)$. Any solution of the differential equation $Df = 0$ is a solution of Laplace's equation $\Delta f = 0$, where Δ is Laplacian of $(n+1)$-real variables and f is a function with values in the Cayley algebra. Also, the hyperholomorphic function is defined using the definition and properties of the differential operator investigated in Section 2 . The relation between the hyperholomorphicity and harmonicity in the octonion is defined. In Section 4, we describe and explain how the results relate to the hypothesis presented as the basis of the study and provide a concise explanation of the implications of the findings, particularly about previous related studies and potential future directions for research.

2. Preliminaries and notations

Cayley algebra $\mathcal{A}$ is an 8-dimensional real field generated by eight bases e_j $(j = 0, 1, \ldots, 7)$ with multiplication rules:

$$e_j^2 = -1,\ e_j e_k = -e_k e_j\ (j, k = 1, 2, \ldots, 7),$$

$$e_1 e_2 = e_3,\ e_1 e_4 = e_5,\ e_6 e_7 = e_1,$$

$$e_2 e_4 = e_6,\ e_3 e_4 = e_7,\ e_3 e_5 = e_6,\ e_2 e_5 = e_7,$$

where e_0 is the identity of $\mathcal{A}$. An element in Cayley algebra is called an octonion and has the form:

$$z = \sum_{j=0}^{7} e_j x_j,$$

where x_j $(j = 0, 1, ..., 7)$ are real numbers. For two octonions $z = \sum_{j=0}^{7} e_j x_j$ and $w = \sum_{j=0}^{7} e_j y_j$, the inner product $< z, w >$ is given by

$$< z, w >= \sum_{j=0}^{7} x_j y_j. \tag{2.1}$$

Based on the defined inner product, the conjugate octonion is given by $z^* = x_0 - \sum_{j=1}^{7} e_j x_j$. By the definition of the conjugate octonion, we have the following properties:

$$(z^*)^* = z,\ (z + w)^* = z^* + w^*,\ (zw)^* = w^* z^*$$

and

$$z(z^*w) = (zz^*)w, \; z(w^*a) + w(z^*a) = (zw^* + wz^*)a,$$

where a is octonionic constants. And then, the absolute value and inverse of an octonion in $\mathcal{A}$ are denoted by

$$|z| = \sqrt{< z, z >} \quad \text{and} \quad z^{-1} = \frac{z^*}{|z|^2},$$

respectively. Some properties derived by defining the inner product for octonions as the equation (2.1) are presented: For any octonions z and w

(1) $|zw| = |z||w|$,
(2) $< z, w >=< w, z >$,
(3) $< az, aw >=< za, wa >$,
(4) $< az, bw > + < bz, aw >= 2 < a, b >< z, w >$,
(5) $< az, w >=< z, a^*w >, \; < za, w >=< z, wa^* >$,

where a and b are octonionic constants.

Let Ω be a set in $\mathbb{R}^8$. Consider a function $f : \Omega \to \mathcal{A}$ as for $z = (x_0, x_1, \ldots, x_7) \in \Omega$

$$f(z) = \sum_{j=0}^{7} e_j f_j(x_0, x_1, \ldots, x_7),$$

where $f_j (j = 0, 1, \ldots, 7)$ are real-valued functions. The function f is said to an octonionic function defined on Ω. For an octonionic function f, let the differential operators over $\mathcal{A}$ be denoted by

$$D = \sum_{j=0}^{7} \alpha_j \frac{\partial}{\partial x_j}, \tag{2.2}$$

where $\frac{\partial}{\partial x_j} (j = 0, 1, \ldots, 7)$ are the usual real partial differential derivatives and

$$\alpha_j = \sum_{j=0}^{7} e_j a_{jk} \; (k = 0, 1, \ldots, 7) \tag{2.3}$$

are nonzero octonionic constants.

To confirm in detail the structure of the differential operator and the role of the octonionic constant included in the differential operator, a more specific form of the differential operator is presented:

$$D = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_7 \end{pmatrix}^T \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{07} \\ a_{10} & a_{11} & \cdots & a_{17} \\ \vdots & \vdots & \ddots & \vdots \\ a_{60} & a_{61} & \cdots & a_{67} \\ a_{70} & a_{71} & \cdots & a_{77} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_0} \\ \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_7} \end{pmatrix} \tag{2.4}$$

and

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{07} \\ a_{10} & a_{11} & \cdots & & a_{17} \\ \vdots & & \ddots & & \vdots \\ a_{70} & & \cdots & & a_{77} \end{pmatrix}. \tag{2.5}$$

The matrix A, which expresses the real constants constituting the octonionic constant α_j as a matrix, is said to be the matrix of the operator D. Given the differential operator D, the conjugate differential operator is given as follows:

$$D^* = \sum_{j=0}^{7} \alpha_j^* \frac{\partial}{\partial x_j},$$

where $\alpha_j^*(j = 0, 1, \ldots, 7)$ is the conjugate octonion of α_j. Investigating the properties of matrices A and D based on the characteristics of matrix A from D.

Proposition 2.1. *Let $A = (a_{ij})$ be a matrix of D. Then the followings are satisfied:*

(1) $(A^*)^* = A$,
(2) $(A^T)^* A = (A^T A)^*$,
(3) $(D^*)^* = D$,

where A^T is the transposed matrix of A.

Proof. (1) From the expression (2.5) and by using the definition of the conjugate of octonions, A^* can be expressed as

$$A^* = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{07} \\ -a_{10} & -a_{11} & \cdots & & -a_{17} \\ \vdots & & \ddots & & \vdots \\ -a_{70} & & \cdots & & -a_{77} \end{pmatrix}.$$

Then, we have

$$(A^*)^* = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{07} \\ -a_{10} & -a_{11} & \cdots & & -a_{17} \\ \vdots & & \ddots & & \vdots \\ -a_{70} & & \cdots & & -a_{77} \end{pmatrix}^* = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{07} \\ a_{10} & a_{11} & \cdots & & a_{17} \\ \vdots & & \ddots & & \vdots \\ a_{70} & & \cdots & & a_{77} \end{pmatrix}.$$

Thus, the equation $(A^*)^* = A$ is satisfied.

(2) From the expression (2.5), we obtain $(A^T)^*A$ such that

$$(A^T)^*A = \begin{pmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{07} \\ b_{10} & b_{11} & \cdots & & b_{17} \\ \vdots & & \ddots & & \vdots \\ b_{70} & & \cdots & & b_{77} \end{pmatrix}, \tag{2.6}$$

where

$$b_{kj} = \sum_{i=0}^{7} a_{ik}a_{ij} \ (k, j = 0, 1, ..., 7).$$

Thus, the matrix expression in the expression (2.6) is the same as the matrix expression of $(A^TA)^*$.

(3) The conjugate differential operator D^* over octonions is expressed as

$$D^* = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_7 \end{pmatrix}^T \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{07} \\ -a_{10} & -a_{11} & \cdots & -a_{17} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{60} & -a_{61} & \cdots & -a_{67} \\ -a_{70} & -a_{71} & \cdots & -a_{77} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_0} \\ \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_7} \end{pmatrix}.$$

Then the operator $(D^*)^*$ is written as

$$(D^*)^* = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_7 \end{pmatrix}^T \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{07} \\ a_{10} & a_{11} & \cdots & a_{17} \\ \vdots & \vdots & \ddots & \vdots \\ a_{60} & a_{61} & \cdots & a_{67} \\ a_{70} & a_{71} & \cdots & a_{77} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_0} \\ \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_7} \end{pmatrix}.$$

$\square$

3. Linearization of Laplacian over octonions

Let $\Delta = \sum_{j=0}^{7} \frac{\partial^2}{\partial x_j^2}$ be Laplacian for eight real variables and f be an octonionic function. The differential operator D in the equation (2.2) is said to be a linearization of Laplacian Δ, if any solution of the differential equation $Df = 0$ is a solution of Laplace's equation $\Delta f = 0$. In this section, we see that the differential operator D in the octonions is a linearization of the Laplacian Δ. We also examine the theorems derived from the result that the differential operator D is a linearization of Laplacian Δ. Let $\mathbb{O}(8)$ be the orthogonal group with an 8×8 matrix as its elements.

Lemma 3.1. *Let $A = (a_{ij})$ be a matrix expressed as the equation* (2.5) *of the differential operator D over octonions expressed as the equation* (2.4)*. Suppose that A is an element of $\mathbb{O}(8)$. Then, the equation $\frac{1}{2}(\alpha_j^* \alpha_k + \alpha_k^* \alpha_j) = \delta_{jk}$ is satisfied if and only if the inner product satisfies $< \alpha_j, \alpha_k >= \delta_{jk}$ $(j, k = 0, 1, \ldots, 7)$, where δ_{jk} is Kronecker's delta.*

Proof. The inner product $< \alpha_j, \alpha_k >$ in the equation (2.1) can be induced by

$$< \alpha_j, \alpha_k >= \sum_{l=0}^{7} a_{li} a_{lk} \quad (j, k = 0, 1, \ldots, 7).$$

It follows that the equation $< \alpha_j, \alpha_k >= \delta_{jk}$ $(j, k = 0, 1, ..., 7)$ if and only if $A \in \mathbb{O}(8)$. By the definitions of the inner product and the conjugate, the equation

$$< \alpha_j, \alpha_k >= \frac{1}{2}(\alpha_j^* \alpha_k + \alpha_k^* \alpha_j) \quad (j, k = 0, 1, \ldots, 7)$$

is satisfied. Hence, we have $\frac{1}{2}(\alpha_j^* \alpha_k + \alpha_k^* \alpha_j) = \delta_{jk}$ $(j, k = 0, 1, \ldots, 7)$ and then the result is obtained. □

Lemma 3.2. *Let f_{jk} be a function such as*

$$f_{jk}(x) = (\alpha_j^{-1} \alpha_k)^2 x_j^2 - 2(\alpha_j^{-1} \alpha_k) x_j x_k + x_k^2,$$

where $x = (x_0, x_1, \ldots, x_7)$ and $(j, k = 0, 1, \ldots, 7)$. Hence, the function f_{jk} are solutions of $Df_{jk}(x) = 0$ $(j, k = 0, 1, \ldots, 7)$.

Proof. By the expression (2.2) of the differential operator D, we can calculus as follows:

$$\begin{aligned} Df_{jk}(x) &= \alpha_j \frac{\partial f_{jk}}{\partial x_j} + \alpha_k \frac{\partial f_{jk}}{\partial x_k} \\ &= \alpha_j \{2(\alpha_j^{-1} \alpha_k)^2 x_j - 2(\alpha_j^{-1} \alpha_k) x_k\} + \alpha_k \{-2(\alpha_j^{-1} \alpha_k) x_j + 2x_k\} \\ &= 2(\alpha_j \alpha_j^{-1}) \alpha_k (\alpha_j^{-1} \alpha_k) x_j - 2(\alpha_j \alpha_j^{-1}) \alpha_k x_k - 2(\alpha_j^{-1} \alpha_k) x_j + 2\alpha_k x_k \\ &= 2\alpha_k (\alpha_j^{-1} \alpha_k) x_j - 2\alpha_k x_k - 2\alpha_k (\alpha_j^{-1} \alpha_k) x_j + 2\alpha_k x_k = 0. \end{aligned}$$

□

Theorem 3.3. *Let D be a differential operator as the equation* (2.2) *and A be the matrix of the differential operator D. If the differential operator D in the equation* (2.2) *is a linearization of Laplacian Δ, then for $\mathcal{O} \in \mathbb{O}(8)$, A can be expressed by $A = \lambda\mathcal{O}$ if and only if the equation $D^*D = \lambda^2\Delta$ $(\lambda > 0)$ is satisfied.*

Proof. Let f_{jk} $(j, k = 0, 1, \ldots, 7)$ be the functions in Lemma 3.2. Given that D is a linearization of Δ, we have

$$\begin{aligned}\Delta f_{jk} &= \frac{\partial^2 f_{jk}}{\partial x_j^2} + \frac{\partial^2 f_{jk}}{\partial x_k^2} \\ &= \frac{\partial\{2(\alpha_j^{-1}\alpha_k)^2 x_j - 2(\alpha_j^{-1}\alpha_k)x_k\}}{\partial x_j} + \frac{\partial\{-2(\alpha_j^{-1}\alpha_k)x_j + 2x_k\}}{\partial x_k} \\ &= 2(\alpha_j^{-1}\alpha_k)^2 + 2 = 0.\end{aligned}$$

That is, each f_{jk} satisfies

$$\Delta f_{jk} = (\alpha_j^{-1}\alpha_k)^2 + 1 = 0 \quad (j, k = 0, 1, \ldots, 7). \tag{3.1}$$

It follows that the equation $\alpha_j^{-1}\alpha_k\alpha_j^{-1}\alpha_k = -1$ and then, we obtain that $\alpha_k\alpha_j^{-1}\alpha_k = -\alpha_j$. It can be expressed as

$$\begin{cases} \alpha_k\alpha_j^{-1} = -\alpha_j\alpha_k^{-1}, \\ \alpha_j^{-1}\alpha_k = -\alpha_k^{-1}\alpha_j. \end{cases}$$

Hence, these equations can be induced as $|\alpha_k| = |\alpha_j|$. For $\lambda > 0$, we put

$$\lambda := |\alpha_0| = |\alpha_2| = \cdots = |\alpha_7|. \tag{3.2}$$

By the equations (3.1) and (3.2), the following equations are derived

$$\begin{aligned}&\alpha_j^{-1}\alpha_k + \alpha_k^{-1}\alpha_j = 0, \ \alpha_j^{-1} = \frac{\alpha_j^*}{\lambda^2}, \\ &\alpha_j^*\alpha_k + \alpha_k^*\alpha_j = 0 \quad (j, k = 0, 1, \ldots, 7).\end{aligned}$$

Then, we have $A = \lambda\mathcal{O}$ for $\mathcal{O} \in \mathbb{O}(8)$. The differential operator D can be calculated as follows:

$$
\begin{aligned}
D^*D &= \alpha_j^*\alpha_j\frac{\partial^2}{\partial x_j^2} + \alpha_k^*\alpha_j\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j} + \alpha_j^*\alpha_k\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} + \alpha_k^*\alpha_k\frac{\partial^2}{\partial x_k^2} \\
&= \alpha_j^*\alpha_j\frac{\partial^2}{\partial x_j^2} + \alpha_k^*\alpha_k\frac{\partial^2}{\partial x_k^2} + \sum_{j,k=0}^{7}(\alpha_j^*\alpha_k + \alpha_k^*\alpha_j)\frac{\partial^2}{\partial x_j \partial x_k} \\
&= \lambda^2\frac{\partial^2}{\partial x_j^2} + \lambda^2\frac{\partial^2}{\partial x_k^2} + \sum_{j,k=0}^{7}(\alpha_j^*\alpha_k + \alpha_k^*\alpha_j)\frac{\partial^2}{\partial x_j \partial x_k} \\
&= \lambda^2\Delta + \sum_{j,k=0}^{7}(\alpha_j^*\alpha_k + \alpha_k^*\alpha_j)\frac{\partial^2}{\partial x_j \partial x_k} \\
&= \lambda^2\Delta.
\end{aligned}
$$

Conversely, since f satisfies $D^*D = \lambda^2\Delta$, the function f is any solution of the differential equation $Df = 0$. Then, we obtain

$$0 = D^*Df = \alpha^2\Delta f.$$

Thus, D is a linearization of Laplacian Δ over octonions. □

The statements in Theorem 3.3 are properties for D, which holds for D^*. By applying the respective definitions of A^* and D^* in the same way as the proof in Theorem 3.3, the statement can be proven.

Corollary 3.4. *Let D^* be the conjugate differential operator of D in the equation* (2.2)*. Let A^* be the matrix of D^*. If the conjugate differential operator D^* is a linearization of Laplacian Δ, then $A^* = \lambda\mathcal{O}$ for $\mathcal{O} \in \mathbb{O}(8)$ and $\lambda > 0$ if and only if $DD^* = \lambda^2\Delta$ for $\lambda > 0$.*

Referred by [5] and [15], we give the following definitions:

Definition 3.5. Let G be an open set in $\mathbb{R}^8$ and $f = \sum_{j=0}^{7} e_j u_j$ be an octonionic function defined in G. The function f is said to be continuously differentiable in G if real functions u_j $(j = 0, 1, \ldots, 7)$ are continuously differentiable in G. Also, f is said to be harmonic in G if real functions $u_j (j = 0, 1, \ldots, 7)$ are harmonic in G.

Definition 3.6. Let G be an open set in $\mathbb{R}^8$ and $f = \sum_{j=0}^{7} e_j u_j$ be an octonionic function defined in G. Let D be an octonionic linearization of Laplacian Δ. The octonionic function f is said to be $\mathcal{O}_l$-hyperholomorphic($\mathcal{O}_r$-hyperholomorphic) in G if

(1) f is continuously differentiable in G,
(2) Df=0 in G (or fD=0 in G).

By Definition 3.5 and 3.6, the operator D can be calculated to f such that

$$Df = \sum \alpha_j \frac{\partial f}{\partial x_j} \quad \text{and} \quad fD = \sum \frac{\partial f}{\partial x_j} \alpha_j. \tag{3.3}$$

The symbol $\mathcal{O}_l$ indicates the case where hyperholomorphicity is determined by applying the differential operator D to the left of the function f. The symbol $\mathcal{O}_r$ indicates the case where hyperholomorphicity is determined by applying the differential operator D to the right of the function f. The theory of an $\mathcal{O}_r$-hyperholomorphic functions can be obtained equivalently to the theory of that. We only consider $\mathcal{O}_l$-hyperholomorphic functions, which we call simply $\mathcal{O}$-hyperholomorphic.

We give some fundamental properties of hyperholomorphic functions obtained from Definition 3.6.

Proposition 3.7. *Let G be an open set in $\mathbb{R}^8$ and f, g be $\mathcal{O}$-hyperholomorphic functions in G. Then, $f + g$ is $\mathcal{O}$-hyperholomorphic in G and βf is $\mathcal{O}$-hyperholomorphic in G if β is a real constant.*

Proof. Since f and g are $\mathcal{O}$-hyperholomorphic in G, we have $Df = 0$ and $Dg = 0$. Thus, the equation $Df + Dg = D(f + g) = 0$ is satisfied. Suppose that βf is $\mathcal{O}$-hyperholomorphic in G. Then $D\beta f = 0$. We put

$$\beta = \beta_0 + e_1\beta_1,$$

where β_1 and β_2 are real numbers. Then we obtain

$$\begin{aligned} D\beta f &= D(\beta_0 + e_1\beta_1)\big(\sum_{j=0}^{7} e_j u_j\big) \\ &= D\big(\beta_0 \sum_{j=0}^{7} e_j u_j + e_1\beta_1 \sum_{j=0}^{7} e_j u_j\big) \\ &= D\big(\beta_0 \sum_{j=0}^{7} e_j u_j\big) + D\big(e_1\beta_1 \sum_{j=0}^{7} e_j u_j\big). \end{aligned}$$

Since f is $\mathcal{O}$-hyperholomorphic in G, the equation Df is vanish, by the equations (3.3) and Definition 3.6. Hence, we have

$$D\big(\beta_0 \sum_{j=0}^{7} e_j u_j\big) = \beta_0 D\big(\sum_{j=0}^{7} e_j u_j\big) = 0.$$

It is sufficient to show $D\big(e_1\beta_1 \sum_{j=0}^{7} e_j u_j\big) = 0$. Then we can calculate such that

$$D\big(e_1\beta_1\sum_{j=0}^{7} e_ju_j\big)=\beta_1 D(e_1u_0 - u_1 + e_3u_2 - e_2u_3 + e_5u_4 - e_4u_5 + e_7u_6 - e_6u_7).$$

Since β is a real constant, $D\big(e_1\beta_1\sum_{j=0}^{7} e_ju_j\big) = 0$ is obtained. Thus, βf is $\mathcal{O}$-hyperholomorphic in G. □

Example 3.8. Let $(y_0, y_1, \ldots, y_7)$ be a point in $\mathbb{R}^8$. For two octonions $\zeta = \sum_{j=0}^{7} e_jy_j$ and $z = \sum_{j=0}^{7} e_jx_j$, the function $H(z;\zeta)$ of variables $z = (x_0, x_1, \ldots, x_7)$ which is expressed by

$$H(z;\zeta) = \sum \frac{\alpha_j^*(y_j - x_j)}{|\zeta - z|^8}$$

is $\mathcal{O}$-hyperholomorphic in $\mathbb{R}^8 \setminus \{(y_0, y_1, \ldots, y_7)\}$, where α_j^* is the conjugate octonion of the coefficient α_j of D $(j = 0, 1, \ldots, 7)$.

Now we consider the relation between a harmonic function and $\mathcal{O}$-hyperholomorphic function in octonions.

Proposition 3.9. *Let G be an open set in $\mathbb{R}^8$ and D^* be the conjugate differential operator of D. If an octonionic function f is harmonic in G, then the function D^*f is $\mathcal{O}$-hyperholomorphic in G.*

Proof. By the definitions of the differential operators D and D^*, we have $DD^* = \lambda^2\Delta$. Since the function f is harmonic in G, the equation

$$D(D^*) = \lambda^2\Delta f = 0$$

is satisfied in G. Thus, D^*f is $\mathcal{O}$-hyperholomorphic in G. □

4. Conclusion

This paper defines a function for a variable as eight non-commutative bases for the product, and a differential operator applicable to the function. The differential operator is defined and the term of the partial derivative is used to apply it to the operation. Or it is defined in the form of a matrix for the convenience of operation. The expression of the differential operator is defined in two ways. It is used to perform the operation according to the form and definition of the function. We investigate the properties induced by the form of the differential operator.

In this paper, it is discovered that the differential operator is a linearization of the real multivariate Laplacian operator and enables the definition of a hyperholomorphic function. From the definition of the differential operator, the hyperholomorphic function is defined, its properties are defined. The relation between harmonicity and hyperholomorphicity is defined.

By defining the hyperholomorphicity of the differential operator, it is an opportunity to study the various uses and applications of the hyperholomorphic function. It can be investigated the relation between the Laplacian operator and the hyperholomorphicity.

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