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CONVERGENCE THEOREMS FOR SEQUENTIALLY ADMISSIBLE PERTURBATIONS OF ASYMPTOTICALLY DEMICONTRACTIVE AND HEMICONTRACTIVE MAPPINGS IN $CAT(0)$ SPACES

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Abstract. In this paper, we introduce a new concept of sequentially admissible mapping and sequentially admissible perturbation. Also we construct iteration process corresponding to sequentially admissible mappings. Moreover, we establish theorems of strong convergence for the Mann type iterative method(called G_*M-algorithm) defined as an uniformly L-Lipschitzian, sequentially admissible perturbation of asymptotically demicontractive mappings and for the Ishikawa type iterative method(called G_*I-algorithm) defined as an uniformly L-Lipschitzian, sequentially admissible perturbation of asymptotically hemicontractive mappings to a fixed point in $CAT(0)$ spaces. Finally, we propose an open problem.

1. Introduction

Let (X, d) be a metric space. One of the most interesting aspects of metric fixed point theory is to extend a linear version of known result to the nonlinear case in metric spaces. To achieve this, Takahashi [29] introduced a convex structure in a metric space (X, d). A mapping $W : X \times X \times [0, 1] \to X$ is a *convex structure* in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

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for all $x, y \in X$ and $\lambda \in [0,1]$. A metric space together with a convex structure W is known as a convex metric space. A nonempty subset K of a convex metric space is said to be *convex* if

$$W(x, y, \lambda) \in K$$

for all $x, y \in K$ and $\lambda \in [0,1]$. In fact, every normed space and its convex subsets are convex metric spaces but the converse is not true, in general (see [29]).

Example 1.1. ([14]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\ ,\ x_2 > 0\}$. For all $x = (x_1, x_2), y = (y_1, y_2) \in X$ and $\lambda \in [0,1]$. We define a mapping $W : X \times X \times [0,1] \to X$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1-\lambda)y_1, \frac{\lambda x_1 x_2 + (1-\lambda)y_1 y_2}{\lambda x_1 + (1-\lambda)y_1}\right)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (X, d, W) is a convex metric space, but it is not a normed linear space.

In 2012, Rus [27] introduced the theory of admissible perturbation of an operator. This theory opened a new direction of research and unified the most important aspects on the iterative approximation of fixed point for single valued self or nonself operators (see [1, 2, 3, 18]).

Definition 1.2. ([27]) Let X be a nonempty set. A mapping $G : X \times X \to X$ is called *admissible* if it satisfies the following two conditions:

(A1) $G(x, x) = x$ for all $x \in X$;
(A2) $G(x, y) = x$ implies $y = x$.

Definition 1.3. ([27]) Let X be a nonempty set. If $f : X \to X$ is a given mapping and $G : X \times X \to X$ is an admissible mapping, then the mapping $f_G : X \to X$ defined by

$$f_G(x) = G(x, f(x)), \quad \forall\, x \in X$$

is called the *admissible perturbation* of f with respect to G.

Remark 1.4. The following property of admissible perturbation is fundamental in the iterative approximation of fixed points: if $f : X \to X$ is a given mapping and $f_G : X \to X$ denotes its admissible perturbation, then

$$\mathcal{F}(f_G) = \mathcal{F}(f) = \{x \in X : x = f(x)\},$$

that is, the admissible perturbation f_G of f has the same set of fixed points as the mapping f itself. Note that, in general,

$$\mathcal{F}(f_G^n) \neq \mathcal{F}(f^n), \quad n \geq 2.$$

Example 1.5. ([27]) Let (X,d) be a metric space endowed with a W-convex structure of Takahashi ([29]). Then $W : X \times X \times [0,1] \to X$ is an operator with the following property

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y), \quad \forall\, x,y,u \in X, \lambda \in [0,1].$$

We additionally suppose that $\lambda \in (0,1)$, $W(x,y,\lambda) = x$ implies $y = x$.

Especially, given $\lambda \in (0,1)$, $Y \subset X$, a W-convex set, $f : Y \to Y$, and $G(x,y) = W(x,y,\lambda)$, the operator f_G is known as admissible perturbation of the operator f.

For other important examples of admissible mappings and admissible perturbations of nonlinear mappings, see [27] for the case of self mappings and [3] for the case of nonself mappings.

Definition 1.6. ([2]) Let $G : X \times X \to X$ be an admissible mapping on a normed space X. We say that G is *affine Lipschitzian* if there exists a constant $\lambda \in [0,1]$ such that

$$\|G(x_1,y_1) - G(x_2,y_2)\| \leq \|\lambda(x_1 - x_2) + (1-\lambda)(y_1 - y_2)\| \tag{1.1}$$

for all $x_1, x_2, y_1, y_2 \in X$.

A metric space X is a $CAT(0)$ space(the term is due to Gromov [10] and it is an acronym for Cartan, Aleksandrov and Toponogov) if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane (see *e.g.*, [4, p.159]). It is well known that any complete, simply connected Riemannian manifold nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [4] or Burago *et al.* [6].

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or, *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [4]).

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [22]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [5]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (*cf.* [4, p.163]). The above inequality has been extended by Khamsi and Kirk [13] as

$$d^2(z, \alpha x \oplus (1-\alpha) y) \leq \alpha d^2(z, x) + (1-\alpha) d^2(z, y) - \alpha(1-\alpha) d^2(x, y), \quad (\mathrm{CN}^*)$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$. The inequality (CN*) was also appeared in [7].

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [4, p.163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha) y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha) y) \leq \alpha d(z, x) + (1-\alpha) d(z, y)$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha) y : \alpha \in [0, 1]\}$. In view of the above inequality, $CAT(0)$ space have Takahashi's convex structure $W(x, y, \alpha) = \alpha x \oplus (1-\alpha) y$. It is easy to see that for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\begin{aligned} d(x, (1-\lambda) x \oplus \lambda y) &= \lambda d(x, y), \\ d(y, (1-\lambda) x \oplus \lambda y) &= (1-\lambda) d(x, y). \end{aligned} \tag{1.2}$$

As a consequence,

$$1 \cdot x \oplus 0 \cdot y = x,$$
$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Moreover, a subset K of $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

Definition 1.7. ([15]) Let C be a nonempty subset of a metric space (X, d). Let $F(T)$ denote the fixed point set of T. Let $F(T) \neq \emptyset$.

(1) A mapping $T : C \to C$ is said to be *k-strictly asymptotically pseudocontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ and for some constant k with $0 \leq k < 1$,

$$d^2(T^n x, T^n y) \leq a_n^2 d^2(x, y) + k(d(x, T^n x) - d(y, T^n y))^2$$

for all $x, y \in C$, $n \in \mathbb{N}$. If $k = 0$, then T is said to be *asymptotically nonexpansive* with sequence $\{a_n\}$, *i.e.*,

$$d(T^n x, T^n y) \leq a_n d(x, y), \quad \forall\, x, y \in C.$$

(2) A mapping $T : C \to C$ is said to be *asymptotically demicontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ and for some constant k with $0 \leq k < 1$,

$$d^2(T^n x, p) \leq a_n^2 d^2(x, p) + k \cdot d^2(x, T^n x), \quad \forall\, p \in F(T)$$

for all $x \in C$, $n \in \mathbb{N}$. If $k = 0$, then T is said to be *asymptotically quasi-nonexpansive* with sequence $\{a_n\}$, *i.e.*,

$$d(T^n x, p) \leq a_n d(x, p), \quad \forall\, x \in C, \ \forall\, p \in F(T).$$

(3) A mapping $T : C \to C$ is said to be *asymptotically pseudocontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ and

$$d^2(T^n x, T^n y) \leq a_n d^2(x, y) + [d(x, T^n x) - d(y, T^n y)]^2$$

for all $x, y \in C$, $n \in \mathbb{N}$.

(4) A mapping $T : C \to C$ is said to be *asymptotically hemicontractive* with sequence $\{a_n\}$ if $\lim_{n\to\infty} a_n = 1$ and

$$d^2(T^n x, p) \leq a_n d^2(x, p) + d^2(x, T^n x), \quad \forall\, p \in F(T)$$

for all $x \in C$, $n \in \mathbb{N}$.

(5) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitzian* if for some constant $L > 0$,

$$d(T^n x, T^n y) \leq L \cdot d(x, y), \quad \forall\, x, y \in C$$

for all $n \in \mathbb{N}$.

Liu [24] has proved the convergence of Mann and Ishikawa iterative sequence for uniformly L-Lipschitzian asymptotically demicontractive and hemicontractive mappings in Hilbert space (*cf.* [28]). The existence of (common) fixed points of one mapping(or two mappings or family of mappings) is not known in many situations. So the approximation of fixed points of one or more nonexpansive, asymptotically nonexpansive or asymptotically quasi-nonexpansive mappings by various iterations have been extensively studied in Banach spaces, convex metric spaces, $CAT(0)$ spaces and so on (see [7, 8, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 27]).

In this paper, we introduce a new concept of sequentially admissible mapping and sequentially admissible perturbation. Also we construct iteration process corresponding to sequentially admissible mappings. Moreover, we establish theorems of strong convergence for the Mann type iterative method (called G_*M-algorithm) defined as an uniformly L-Lipschitzian, sequentially admissible perturbation of asymptotically demicontractive mappings and for the Ishikawa type iterative method(called G_*I-algorithm) defined as an uniformly L-Lipschitzian, sequentially admissible perturbation of asymptotically hemicontractive mappings to a fixed point in $CAT(0)$ spaces. Finally, we propose an open problem.

2. Preliminaries

Definition 2.1. Let X be a nonempty set. A mapping $G_* : X \times X \to X$ is called *sequentially admissible* if it satisfies the following two conditions:

(SA1) $G_*(x, x) = x$ for all $x \in X$;
(SA2) $G_*(x, y) = x$ implies $y = x$,

where $\{*\}$ is an arbitrary sequence in [0,1].

Assumption 2.2. Let X be a nonempty set, $G_* : X \times X \to X$ be a sequentially admissible mapping and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, 1]$. We assume that

$$G_{\alpha_n}(x, x) = x = G_{\beta_n}(x, x), \quad \forall\, x \in X. \tag{2.1}$$

Definition 2.3. Let X be a nonempty set. If $f : X \to X$ is a given mapping and $G_* : X \times X \to X$ is a sequentially admissible mapping, then the mapping $f_{G_*} : X \to X$ defined by

$$f_{G_*}(x) = G_*(x, f(x)), \quad \forall\, x \in X$$

is called a *sequentially admissible perturbation* of f with respect to G_*.

Definition 2.4. Let $G_* : X \times X \to X$ be a sequentially admissible mapping on a normed space X. We say that G_* is *sequentially admissible Lipschitzian* if

$$\begin{aligned}\|G_{\alpha_n}(x_1, y_1) - G_{\beta_n}(x_2, y_2)\| \leq \max\{1 - \alpha_n, 1 - \beta_n\}\|x_1 - x_2\| \\ + \max\{\alpha_n, \beta_n\}\|y_1 - y_2\|\end{aligned} \tag{2.2}$$

for all sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ and $x_1, x_2, y_1, y_2 \in X$.

Assumption 2.5. Let $G_* : X \times X \to X$ be a sequentially admissible Lipschitzian mapping on a normed space X. We assume that

$$\begin{aligned}\|G_{\alpha_n}(x_1, y_1) - G_{\beta_n}(x_2, y_2)\|^2 \leq \max\{1 - \alpha_n, 1 - \beta_n\}\|x_1 - x_2\|^2 \\ + \max\{\alpha_n, \beta_n\}\|y_1 - y_2\|^2 \\ - (\min\{\alpha_n, \beta_n\} - \alpha_n\beta_n)(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2)\end{aligned} \tag{2.3}$$

for all sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ and $x_1, x_2, y_1, y_2 \in X$.

Remark 2.6.
(1) If we take the sequence $\{*\} = \lambda$ in Definition 2.1 and Definition 2.3, it reduces to Definition 1.2 and Definition 1.3, respectively.
(2) If we take the sequences $\{\alpha_n\} = \{\beta_n\} = \lambda$ in (2.2), it reduces to (1.1).
(3) If we take the sequences $\{\alpha_n\} = \{\beta_n\} = \lambda$, $G_*(x, y) = (1 - \lambda)x \oplus \lambda y$ and $x_1 = y_1$ in (2.3) on X, it reduces to (CN^*) inequality.
(4) From (2.2), it is easy to see that for any $x, y \in X$ and sequence $\{\alpha_n\}$ in $[0, 1]$,

$$\begin{aligned}d(G_{\alpha_n}(x, x), G_{\alpha_n}(x, y)) &\leq (1 - \alpha_n)d(x, x) + \alpha_n d(x, y) \\ &= \alpha_n d(x, y), \\ d(G_{\alpha_n}(y, y), G_{\alpha_n}(x, y)) &\leq (1 - \alpha_n)d(x, y) + \alpha_n d(y, y) \\ &= (1 - \alpha_n)d(x, y).\end{aligned} \tag{2.4}$$

If we take $\alpha_n = \lambda$ and $G_{\alpha_n}(x, y) = (1 - \lambda)x \oplus \lambda y$ in (2.4), it reduces to (1.2).

We introduce the following iteration process.

Let C be a nonempty convex subset of a $CAT(0)$ space (X, d) and $T : C \to C$ be a given mapping. Let $x_1 \in C$ be a given point.

Algorithm 2.7. Let C be a nonempty subset of a metric space (X, d), $T : C \to C$ be a nonlinear mapping and $G_* : C \times C \to C$ be a sequentially admissible Lipschitzian mapping. The sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative algorithm

$$\begin{aligned} x_{n+1} &= G_{\alpha_n}(x_n, T^n y_n), \\ y_n &= G_{\beta_n}(x_n, T^n x_n), \quad n \geq 1, \end{aligned} \tag{2.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, is called the *Ishikawa-type* algorithm corresponding to G_* or G_*I-algorithm (*cf.* [12]).

Algorithm 2.8. Let C be a nonempty subset of a metric space (X,d), $T : C \to C$ be a nonlinear mapping and $G_* : C \times C \to C$ be sequentially admissible Lipschitzian mapping. The sequence $\{x_n\}$ defined by the iterative algorithm

$$x_{n+1} = G_{\alpha_n}(x_n, T^n x_n), \quad n \geq 1, \tag{2.6}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, is called a *Mann-type* algorithm corresponding to G_* or G_*M-algorithm (*cf.* [25]).

Lemma 2.9. ([24]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences satisfying*

$$a_{n+1} \leq a_n + b_n,$$

where $a_n \geq 0$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n$ is convergent and $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to 0. Then we must have

$$\lim_{n\to\infty} a_n = 0.$$

3. Convergence theorems

Lemma 3.1. *Let (X,d) be a $CAT(0)$ space and C be a nonempty convex subset of X. Let $T : C \to C$ be a uniformly L-Lipschitzian mapping and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$. Define the G_*I-algorithm $\{x_n\}$ as (2.5) in Algorithm 2.7. Then*

$$d(x_n, Tx_n) \leq d(x_n, T^n x_n) + L(1 + 2L + L^2) d(x_{n-1}, T^{n-1} x_{n-1})$$

for all $n \geq 1$.

Proof. Let $C_n = d(x_n, T^n x_n)$. From (2.4) and (2.5), we have

$$\begin{aligned} d(x_{n-1}, y_{n-1}) &= d(x_{n-1}, G_{\beta_{n-1}}(x_{n-1}, T^{n-1} x_{n-1})) \\ &= d(G_{\beta_{n-1}}(x_{n-1}, x_{n-1}), G_{\beta_{n-1}}(x_{n-1}, T^{n-1} x_{n-1})) \\ &\leq (1 - \beta_{n-1}) d(x_{n-1}, x_{n-1}) + \beta_{n-1} \cdot d(x_{n-1}, T^{n-1} x_{n-1}) \\ &= \beta_{n-1} C_{n-1}. \end{aligned} \tag{3.1}$$

From (3.1), we get

$$\begin{aligned} d(x_{n-1},T^{n-1}y_{n-1}) &\le d(x_{n-1},T^{n-1}x_{n-1}) + d(T^{n-1}x_{n-1},T^{n-1}y_{n-1}) \\ &\le C_{n-1} + L\cdot d(x_{n-1},y_{n-1}) \\ &\le C_{n-1} + \beta_{n-1}\cdot L\cdot C_{n-1}. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} d(x_n,Tx_n) &\le d(x_n,T^n x_n) + d(T^n x_n, Tx_n) \\ &\le C_n + L\cdot d(T^{n-1}x_n, x_n) \\ &\le C_n + L\{d(T^{n-1}x_n, T^{n-1}x_{n-1}) + d(T^{n-1}x_{n-1}, x_n)\} \\ &\le C_n + L^2\cdot d(x_n,x_{n-1}) + L\cdot d(T^{n-1}x_{n-1},x_n) \\ &\le C_n + L^2\cdot d(G_{\alpha_{n-1}}(x_{n-1},T^{n-1}y_{n-1}), G_{\alpha_{n-1}}(x_{n-1},x_{n-1})) \\ &\quad + L\cdot d(G_{\alpha_{n-1}}(T^{n-1}x_{n-1},T^{n-1}x_{n-1}), G_{\alpha_{n-1}}(x_{n-1},T^{n-1}y_{n-1})) \\ &\le C_n + L^2[(1-\alpha_{n-1})d(x_{n-1},x_{n-1}) + \alpha_{n-1}\cdot d(T^{n-1}y_{n-1},x_{n-1})] \\ &\quad + L[(1-\alpha_{n-1})d(T^{n-1}x_{n-1},x_{n-1}) \\ &\quad + \alpha_{n-1}\cdot d(T^{n-1}x_{n-1},T^{n-1}y_{n-1})] \\ &\le C_n + L^2\cdot\alpha_{n-1}(C_{n-1}+\beta_{n-1}\cdot L\cdot C_{n-1}) \\ &\quad + L(1-\alpha_{n-1})C_{n-1} + L^2\cdot\alpha_{n-1}\cdot\beta_{n-1}\cdot C_{n-1} \\ &\le C_n + L(1+2L+L^2)C_{n-1}, \quad n\ge 1. \end{aligned}$$

This completes the proof of Lemma 3.1. □

We remind that a mapping $T : D \subset X \to Y$ is said to be *completely continuous* if it is continuous and maps any bounded subset of D into a relatively compact subset of Y.

Theorem 3.2. *Let (X,d) be a complete $CAT(0)$ space, C be a nonempty bounded closed convex subset of X, $T : C \to C$ be completely continuous and uniformly L-Lipschitzian and asymptotically demicontractive with sequence $\{a_n\}$, $a_n \in [1,\infty)$, $\sum_{n=1}^{\infty}(a_n^2-1) < \infty$, $\varepsilon \le \alpha_n \le 1-k-\varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Given $x_0 \in C$, defined the G_*M-algorithm $\{x_n\}$ as* (2.6) *in Algorithm* 2.8. *Then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. Since T is a completely continuous mapping in a bounded closed convex subset C of complete metric space, from Schauder's theorem, $F(T)$ is nonempty. It follows from (2.3) that

$$\begin{aligned} d^2(x_{n+1},p) &= d^2(G_{\alpha_n}(x_n,T^nx_n),G_{\alpha_n}(p,p)) \\ &\le (1-\alpha_n)d^2(x_n,p)+\alpha_n d^2(T^nx_n,p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(x_n,T^nx_n) \end{aligned} \tag{3.3}$$

for all $p \in F(T)$. Since T is an asymptotically demicontractive, from (3.3), we get

$$\begin{aligned} d^2(x_{n+1},p) &\le (1-\alpha_n)d^2(x_n,p)+\alpha_n\{a_n^2d^2(x_n,p)+k\cdot d^2(x_n,T^nx_n)\} \\ &\quad -\alpha_n(1-\alpha_n)d^2(x_n,T^nx_n) \\ &= d^2(x_n,p)+\alpha_n(a_n^2-1)d^2(x_n,p) \\ &\quad -\alpha_n(1-\alpha_n-k)d^2(x_n,T^nx_n), \quad \forall\, p\in F(T). \end{aligned} \tag{3.4}$$

Since $0<\varepsilon\le\alpha_n\le 1-k-\varepsilon$, we have $1-k-\alpha_n\ge\varepsilon$. Thus,

$$\alpha_n(1-k-\alpha_n)\ge\varepsilon^2.$$

From (3.4), we have

$$d^2(x_{n+1},p)\le d^2(x_n,p)+\alpha_n(a_n^2-1)d^2(x_n,p)-\varepsilon^2\cdot d^2(x_n,T^nx_n) \tag{3.5}$$

for all $p\in F(T)$. Since C is bounded and T is a self-mapping on C, there exists $M>0$ such that $d^2(x_n,p)\le M$ for all $n\in\mathbb{N}$. Since $0\le\alpha_n\le 1$, it follows from (3.5) that

$$d^2(x_{n+1},p)\le d^2(x_n,p)+(a_n^2-1)M-\varepsilon^2\cdot d^2(x_n,T^nx_n), \quad \forall\, p\in F(T). \tag{3.6}$$

Therefore,

$$\varepsilon^2\cdot d^2(x_n,T^nx_n)\le d^2(x_n,p)-d^2(x_{n+1},p)+(a_n^2-1)M.$$

So

$$\begin{aligned} \sum_{n=1}^{m}\varepsilon^2\cdot d^2(x_n,T^nx_n) &\le d^2(x_1,p)-d^2(x_{m+1},p)+M\sum_{n=1}^{m}(a_n^2-1) \\ &\le d^2(x_1,p)+M\sum_{n=1}^{\infty}(a_n^2-1) \end{aligned}$$

for all $m\in\mathbb{N}$. Since $\sum_{n=1}^{\infty}(a_n^2-1)<\infty$, we get

$$\sum_{n=1}^{\infty}\varepsilon^2\cdot d^2(x_n,T^nx_n)<\infty.$$

Therefore,

$$\lim_{n\to\infty}d^2(x_n,T^nx_n)=0 \quad \text{and} \quad \lim_{n\to\infty}d(x_n,T^nx_n)=0. \tag{3.7}$$

Since T is uniformly L-Lipschitzian, it follows from (3.7) and Lemma 3.1 that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.8}$$

Since $\{x_n\}$ is a bounded sequence and T is completely continuous, there exists a convergent subsequence $\{Tx_{n_j}\}$ of $\{Tx_n\}$. Therefore, from (3.8), $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$. Let $\lim_{j\to\infty} x_{n_j} = q$. It follows from the continuity of T and (3.8) that $q = Tq$. Therefore, $\{x_n\}$ has a subsequence which converges to the fixed point q of T. Take $p = q$ in the inequality (3.6), then

$$d^2(x_{n+1}, q) \leq d^2(x_n, q) + (a_n^2 - 1)M, \quad \forall\, n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$ and $\{d(x_n, q)\}$ has a subsequence which converges to 0, it follows from Lemma 2.9 that

$$\lim_{n\to\infty} d^2(x_n, q) = 0.$$

Therefore,

$$\lim_{n\to\infty} x_n = q.$$

This completes the proof. □

Corollary 3.3. *Let (X, d) be a complete $CAT(0)$ space, C be a nonempty bounded closed convex subset of X, $T : C \to C$ be completely continuous and uniformly L-Lipschitzian and asymptotically demicontractive with sequence $\{a_n\}$, $a_n \in [1, \infty)$, $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, $\varepsilon \leq \alpha_n \leq 1 - k - \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Given $x_0 \in C$, defined the iteration process $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T^n x_n, \quad n \geq 1.$$

Then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. If we take

$$G_{\alpha_n}(x_n, T^n x_n) = (1 - \alpha_n)x_n \oplus \alpha_n T^n x_n$$

in (2.6), then we get Corollary 3.3 from Theorem 3.2. □

Corollary 3.4. *Let (X, d) be a complete $CAT(0)$ space, C be a nonempty bounded closed convex subset of X, $T : C \to C$ be completely continuous and uniformly L-Lipschitzian and k-strictly asymptotically pseudocontractive with sequence $\{a_n\}$, $a_n \in [1, \infty)$, $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$, $\varepsilon \leq \alpha_n \leq 1 - k - \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Given $x_0 \in C$, defined the G_*M-algorithm $\{x_n\}$ as (2.6) in Algorithm 2.8. Then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. By Definition 1.7, it is clear that T is a k-strictly asymptotically pseudocontractive mapping with a nonempty fixed point set and so T is an asymptotically demicontractive mapping. Therefore, Corollary 3.4 can be proved by using Theorem 3.2. □

Lemma 3.5. *Let (X, d) be a $CAT(0)$ space and C be a nonempty convex subset of X. Let $T : C \to C$ be uniformly L-Lipschitzian and asymptotically hemicontractive with sequence $\{a_n\} \subset [1, \infty)$ for all $n \in \mathbb{N}$ and $F(T)$ be nonempty. Define the G_*I-algorithm $\{x_n\}$ as (2.5) in Algorithm 2.7. Then the following inequality holds:*

$$\begin{aligned} d^2(x_{n+1}, p) \leq\ & [1 + \alpha_n(a_n - 1)(1 + a_n\beta_n)]d^2(x_n, p) \\ & - \alpha_n\beta_n(1 - \beta_n - a_n\beta_n - L^2\beta_n^2)d^2(x_n, T^n x_n) \\ & - \alpha_n(\beta_n - \alpha_n)d^2(x_n, T^n y_n) \end{aligned}$$

for all $p \in F(T)$.

Proof. It follows from (2.3) that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(G_{\alpha_n}(x_n, T^n y_n), G_{\alpha_n}(p, p)) \\ &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(T^n y_n, p) \\ &\quad - \alpha_n(1 - \alpha_n)d^2(x_n, T^n y_n) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} d^2(y_n, p) &= d^2(G_{\beta_n}(x_n, T^n x_n), G_{\beta_n}(p, p)) \\ &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n d^2(T^n x_n, p) \\ &\quad - \beta_n(1 - \beta_n)d^2(x_n, T^n x_n) \end{aligned} \tag{3.10}$$

for all $p \in F(T)$. Since T is asymptotically hemicontractive, we get

$$d^2(T^n y_n, p) \leq a_n d^2(y_n, p) + d^2(y_n, T^n y_n) \tag{3.11}$$

and

$$d^2(T^n x_n, p) \leq a_n d^2(x_n, p) + d^2(x_n, T^n x_n) \tag{3.12}$$

for all $p \in F(T)$. From (3.10) and (3.12), we have

$$\begin{aligned} d^2(y_n, p) &\leq (1 - \beta_n)d^2(x_n, p) + \beta_n[a_n d^2(x_n, p) + d^2(x_n, T^n x_n)] \\ &\quad - (1 - \beta_n)\beta_n d^2(x_n, T^n x_n) \\ &= [1 + (a_n - 1)\beta_n]d^2(x_n, p) + \beta_n^2 d^2(x_n, T^n x_n). \end{aligned} \tag{3.13}$$

From (2.3), we have

$$\begin{aligned} d^2(y_n, T^n y_n) &= d^2(G_{\beta_n}(x_n, T^n x_n), G_{\beta_n}(T^n y_n, T^n y_n)) \\ &\leq (1-\beta_n) d^2(x_n, T^n y_n) + \beta_n d^2(T^n x_n, T^n y_n) \\ &\quad - \beta_n(1-\beta_n) d^2(x_n, T^n x_n). \end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.11), we get

$$\begin{aligned} d^2(T^n y_n, p) \leq\ & a_n[1 + (a_n - 1)\beta_n] d^2(x_n, p) + a_n \beta_n^2 d^2(x_n, T^n x_n) \\ & + (1-\beta_n) d^2(x_n, T^n y_n) + \beta_n d^2(T^n x_n, T^n y_n) \\ & - \beta_n(1-\beta_n) d^2(x_n, T^n x_n). \end{aligned} \tag{3.15}$$

From (3.9) and (3.15), we obtain

$$\begin{aligned} d^2(x_{n+1}, p) \leq\ & (1-\alpha_n) d^2(x_n, p) + \alpha_n a_n (1 + (a_n - 1)\beta_n) d^2(x_n, p) \\ & + \alpha_n a_n \beta_n^2 d^2(x_n, T^n x_n) + \alpha_n (1-\beta_n) d^2(x_n, T^n y_n) \\ & + \alpha_n \beta_n d^2(T^n x_n, T^n y_n) - \alpha_n \beta_n (1-\beta_n) d^2(x_n, T^n x_n) \\ & - \alpha_n (1-\alpha_n) d^2(x_n, T^n y_n) \\ =\ & [1 + \alpha_n\{(a_n - 1) + a_n(a_n - 1)\beta_n\}] d^2(x_n, p) \\ & - \alpha_n \beta_n (1 - \beta_n - a_n \beta_n) d^2(x_n, T^n x_n) \\ & - \alpha_n (\beta_n - \alpha_n) d^2(x_n, T^n y_n) + \alpha_n \beta_n d^2(T^n x_n, T^n y_n). \end{aligned} \tag{3.16}$$

Since T is uniformly L-Lipschitzian and from (2.3), we have

$$\begin{aligned} d^2(T^n x_n, T^n y_n) &\leq L^2 \cdot d^2(x_n, y_n) \\ &= L^2 \cdot d^2(G_{\beta_n}(x_n, x_n), G_{\beta_n}(x_n, T^n x_n)) \\ &\leq L^2[\beta_n d^2(x_n, T^n x_n) - \beta_n(1-\beta_n) d^2(x_n, T^n x_n)] \\ &\leq L^2 \beta_n^2 \cdot d^2(x_n, T^n x_n). \end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.16), we obtain

$$\begin{aligned} d^2(x_{n+1}, p) \leq\ & [1 + \alpha_n(a_n - 1)(1 + a_n\beta_n)] d^2(x_n, p) \\ & - \alpha_n \beta_n (1 - \beta_n - a_n \beta_n - L^2 \beta_n^2) d^2(x_n, T^n x_n) \\ & - \alpha_n (\beta_n - \alpha_n) d^2(x_n, T^n y_n), \quad \forall\, p \in F(T). \end{aligned}$$

This completes the proof. □

Lemma 3.6. *Let (X, d) be a $CAT(0)$ space and C be a nonempty bounded convex subset of X. Let $T : C \to C$ be uniformly L-Lipschitzian and asymptotically hemicontractive with sequence $\{a_n\} \subset [1, \infty)$ for all $n \in \mathbb{N}$ and*

$\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Let $F(T)$ be nonempty. Given $x_1 \in C$, define the G_*I-algorithm $\{x_n\}$ as (2.5) in Algorithm 2.7. If $\varepsilon \leq \alpha_n \leq \beta_n \leq b$ for some $\varepsilon > 0$ and $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$, then

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Proof. First, we will prove $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. From Lemma 3.5 and $0 \leq \alpha_n \leq \beta_n$, we have

$$\begin{aligned} d^2(x_{n+1}, p) \leq\ & [1 + \alpha_n(a_n - 1)(1 + a_n\beta_n)]d^2(x_n, p) \\ & - \alpha_n\beta_n(1 - \beta_n - a_n\beta_n - L^2\beta_n^2)d^2(x_n, T^n x_n). \end{aligned}$$

Thus

$$\begin{aligned} d^2(x_{n+1}, p) - d^2(x_n, p) \leq\ & \alpha_n(a_n - 1)(1 + a_n\beta_n)d^2(x_n, p) \\ & - \alpha_n\beta_n(1 - \beta_n - a_n\beta_n - L^2\beta_n^2)d^2(x_n, T^n x_n). \quad (3.18) \end{aligned}$$

Since $\sum_{n=1}^{\infty}(a_n - 1) < \infty$, we have $\lim_{n\to\infty}(a_n - 1) = 0$. Hence $\{a_n\}$ is bounded. By boundedness of C and $0 \leq \alpha_n \leq \beta_n \leq 1$, we obtain that $\{\alpha_n(1 + a_n\beta_n)d^2(x_n, p)\}$ is bounded. Therefore, there exists a constant $M > 0$ such that

$$0 \leq \alpha_n(1 + a_n\beta_n)d^2(x_n, p) \leq M. \quad (3.19)$$

From (3.18) and (3.19), we get

$$\begin{aligned} & d^2(x_{n+1}, p) - d^2(x_n, p) \\ & \leq (a_n - 1)M - \alpha_n\beta_n(1 - \beta_n - a_n\beta_n - L^2\beta_n^2)d^2(x_n, T^n x_n). \quad (3.20) \end{aligned}$$

Let $D = 1 - 2b - L^2b^2 > 0$. Since $\lim_{n\to\infty} a_n = 1$, there exists $N \in \mathbb{N}$ such that

$$1 - \beta_n - a_n\beta_n - L^2\beta_n^2 \geq 1 - b - a_n b - L^2b^2 \geq \frac{D}{2} > 0 \quad (3.21)$$

for all $n \geq N$. Suppose that $\lim_{n\to\infty} d(x_n, T^n x_n) \neq 0$. Then there exist a $\varepsilon_0 > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$d^2(x_{n_i}, T^{n_i} x_{n_i}) \geq \varepsilon_0. \quad (3.22)$$

Without loss of generality, we let $n_1 \geq N$. From (3.20), (3.21) and $\varepsilon \leq \alpha_n \leq \beta_n$, we have

$$\begin{aligned} & \varepsilon^2(1 - \beta_n - a_n\beta_n - L^2\beta_n^2)d^2(x_n, T^n x_n) \\ & \leq \alpha_n\beta_n(1 - \beta_n - a_n\beta_n - L^2\beta_n^2)d^2(x_n, T^n x_n) \\ & \leq (a_n - 1)M + d^2(x_n, p) - d^2(x_{n+1}, p), \end{aligned}$$

so,

$$\begin{aligned}
&\varepsilon^2 \sum_{m=n_1}^{n_i} (1-\beta_m - a_m\beta_m - L^2\beta_m^2) d^2(x_m, T^m x_m) \\
&= \varepsilon^2 \sum_{l=1}^{i} (1-\beta_{n_l} - a_{n_l}\beta_{n_l} - L^2\beta_{n_l}^2) d^2(x_{n_l}, T^{n_l} x_{n_l}) \\
&\leq d^2(x_{n_1}, p) - d^2(x_{n_i+1}, p) + M \sum_{m=n_1}^{n_i} (a_m - 1). \qquad (3.23)
\end{aligned}$$

From (3.21), (3.22) and (3.23), we obtain

$$\varepsilon^2 \cdot i \cdot \frac{D}{2} \cdot \varepsilon_0 \leq d^2(x_{n_1}, p) - d^2(x_{n_i+1}, p) + M \sum_{m=n_1}^{n_i} (a_m - 1). \qquad (3.24)$$

From $\sum_{n=1}^{\infty}(a_n - 1) < \infty$ and the boundedness of C, the right side of (3.24) is bounded. However, if we have $i \to \infty$, then the left side of (3.24) is unbounded. This is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0.$$

Since T is uniformly L-Lipschitzaian, from Lemma 3.1, we get

$$\lim_{n\to\infty} d(x_n, T x_n) = 0.$$

This completes the proof. □

Theorem 3.7. *Let (X, d) be a complete $CAT(0)$ space, C be a nonempty bounded closed convex subset of X, $T : C \to C$ be completely continuous and uniformly L-Lipschitzian and asymptotically hemicontractive with sequence $\{a_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$ for all $n \in \mathbb{N}$. Given $x_1 \in C$, define the G_*I-algorithm $\{x_n\}$ as (2.5) in Algorithm 2.7. If $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ with $\varepsilon \leq \alpha_n \leq \beta_n \leq b$ for some $\varepsilon > 0$ and $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$, then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. Since T is a completely continuous mapping in a bounded closed convex subset C of complete metric space, from Schauder's theorem, $F(T)$ is nonempty. Since T is completely continuous, there exists a convergent subset $\{Tx_{n_i}\}$ of $\{Tx_n\}$. Let

$$\lim_{i\to\infty} Tx_{n_i} = p.$$

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, from Lemma 3.6, we have

$$\lim_{i\to\infty} x_{n_i} = p. \qquad (3.25)$$

On the other hand, from the continuity of T, (3.25) and Lemma 3.6, we have

$$d(p, Tp) = \lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0.$$

This means that p is a fixed point of T. From (3.19), (3.21), $\alpha_n \le \beta_n$ and Lemma 3.5, it follows that

$$d^2(x_{n+1}, p) \le d^2(x_n, p) + (a_n - 1)M. \tag{3.26}$$

From (3.25), there exists a subsequence $\{d^2(x_{n_i}, p)\}$ of $\{d^2(x_n, p)\}$ which converges to 0. Therefore, from Lemma 2.9 and (3.26),

$$\lim_{n\to\infty} d^2(x_n, p) = 0.$$

Hence,

$$\lim_{n\to\infty} x_n = p.$$

This completes the proof. □

Corollary 3.8. *Let (X, d) be a complete $CAT(0)$ space, C be a nonempty bounded closed convex subset of X, $T : C \to C$ be completely continuous and uniformly L-Lipschitzian and asymptotically hemicontractive with sequence $\{a_n\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$ for all $n \in \mathbb{N}$. Given $x_1 \in C$, define the iterative process $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \ge 1. \end{aligned}$$

If $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\varepsilon \le \alpha_n \le \beta_n \le b$ for some $\varepsilon > 0$ and $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$, then $\{x_n\}$ converges strongly to some fixed point of T.

Proof. If we take

$$\begin{aligned} G_{\alpha_n}(x_n, T^n y_n) &= (1-\alpha_n)x_n \oplus \alpha_n T^n y_n, \\ G_{\beta_n}(x_n, T^n x_n) &= (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \ge 1 \end{aligned}$$

in (2.5), then we get Corollary 3.8 from Theorem 3.7. □

Corollary 3.9. *Let (X, d) be a complete $CAT(0)$ space, C be a nonempty bounded closed convex subset of X, $T : C \to C$ be completely continuous and uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{a_n\} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$ for all $n \in \mathbb{N}$. Given $x_1 \in C$, define the G_*I-algorithm $\{x_n\}$ as (2.5) in Algorithm 2.7. If $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\varepsilon \le \alpha_n \le \beta_n \le b$ for some $\varepsilon > 0$ and $b \in \left(0, \frac{\sqrt{1+L^2}-1}{L^2}\right)$, then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof. By Definition 1.7, T is an asymptotically pseudocontractive mapping, and so T is an asymptotically hemicontractive mapping. Since $a_n \in [1, \infty)$, we have $a_n^2 - 1 \geq a_n - 1 \geq 0$. Obviously, $\sum_{n=1}^{\infty}(a_n - 1) \leq \sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$. Therefore, Corollary 3.9 can be proved by using Theorem 3.7. □

4. Some remarks and open problem

For a real number κ, a $CAT(\kappa)$ space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding triangle in a model space with curvature κ. For $\kappa = 0$, the 2-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the 2-dimensional sphere $\frac{1}{\sqrt{\kappa}}\mathbb{S}^2$ whose metric is length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the 2-dimensional hyperbolic space $\frac{1}{\sqrt{-\kappa}}\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance. For more details about the properties of $CAT(\kappa)$ spaces, see [4, 9, 20, 21].

Open Problem: It will be interesting to obtain a generalization of both Theorem 3.2 and Theorem 3.7 to $CAT(\kappa)$ space.

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