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(ψ, φ)-COMMON FIXED POINT THEOREMS IN WEAK NON-ARCHIMEDEAN FUZZY METRIC SPACES AND APPLICATIONS

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Abstract. In this paper, we prove a common fixed point theorem in weak non-Archimedean fuzzy metric spaces and we give an application for (ψ, φ)-contractive maps satisfying inequality of integral type. The result extends the main theorems of V. Sihag *et al.*: Fixed point theorems for (ψ, φ)-contractive maps in weak non-archimedean fuzzy metric spaces and application[Int. J. Comp. Appl., 27(2) (2011), 8975–8875.]

1. Introduction and Preliminaries

In 1965, the concept of fuzzy sets was introduced by Zadeh [18]. Since then many authors have expansively developed the theory of fuzzy sets and applications. In 1975, Kramosil and Michalek [9] first introduced the concept of a fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space and it provides an important basis for the construction of fixed point theory in fuzzy metric spaces. After that, Wenzhi [17] and many others initiated the study of probabilistic 2-metric spaces which is a real valued function of a point triples on a set X, whose abstract properties were suggested by the area function in Euclidean spaces. Afterwards, Grabiec [7] de ned the completeness of the fuzzy metric space or what is known as a G-complete fuzzy metric space in [8], and extended the Banach contraction

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theorem to G-complete fuzzy metric spaces. Following Grabiec's work, Fang [3] further established some new fixed point theorems for contractive type mappings in G-complete fuzzy metric spaces. Soon after, Mishra *et al.* [10] also obtained several common fixed point theorems for asymptotically commuting maps in the same space, which generalize several fixed point theorems in metric, fuzzy, Menger and uniform spaces. Besides these works based on the G-complete fuzzy metric space, George and Veeramani [16] modifed the definition of the Cauchy sequence introduced by Grabiec [7] because even R is not complete with Grabiec's completeness definition. George and Veeramani [5] slightly modifed the notion of a fuzzy metric space introduced by Kramosil and Michalek [9] and then defined a Hausdorff and first countable topology on this fuzzy metric space which has important applications in quantum particle physics in connection with string and E-infinity theory. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani [6], which is now known as an M-complete fuzzy metric space (as in [16] has emerged as another characterization of completeness, and some fixed point theorems have also been constructed on the basis of this metric space. Recently, Fang [4] gave some common fixed point theorems under φ-contractions for compatible and weakly compatible mappings in Menger probabilistic metric spaces. Moreover, Rao *et al.* [11] have proved two unique common coupled fixed point theorems for self maps in symmetric G-fuzzy metric spaces. Recently, Shen *et al.* [15] have proposed a new class of selfmaps by altering the distance between two points in fuzzy environment, in which the φ-function was used, and on the basis of this kind of self-map, they have proved some fixed point theorems in M-complete fuzzy metric spaces and compact fuzzy metric spaces. From the above analysis, we can see that there are many studies related to fixed point theory based on the above two kinds of complete fuzzy metric spaces, namely: G-complete and M-complete fuzzy metric spaces. Note that every G-complete fuzzy metric space is M-complete and the construction of fixed point theorems in M-complete fuzzy metric spaces seems to be more valuable.

The purpose of this work is to prove a common fixed point theorems in weak non-archimedean fuzzy metric spaces and we give an application for (ψ, φ)-contractive maps satisfying inequality of integral type. The result obtained extends the main theorems of V. Sihag *et al.* [14] and some results in the literature.

For the reader's convenience, we restate some definitions as follows:

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative,
(b) $*$ is continuous,

(c) $a * 1 = a$ for all $a \in [0,1]$,
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

The following are classical examples of continuous t-norm:

$$a * b = \min\{a, b\}$$

Minimum t-norm.

$$a * b = a.b$$

Product t-norm.

$$a * b = \begin{cases} 0\,, & \text{if} \quad a = b = 0, \\ \frac{ab}{a+b+ab}\,, & \text{otherwise} \end{cases}$$

Hamacher product.

$$a * b = \begin{cases} \min\{a, b\}\,, & \text{if} \quad a + b > 1, \\ 0\,, & \text{otherwise} \end{cases}$$

Nilpotent minimum.

$$a * b = \min\{a + b - 1, 0\}$$

Likasiewicz t-norm.

$$a * b = \begin{cases} b\,, & \text{if} \quad a = 1, \\ a\,, & \text{if} \quad b = 1, \\ 0\,, & \text{otherwise} \end{cases}$$

Drastic t-norm.

The minimum t-norm is the pointwise largest t-norm and the drastic t-norm is the pointwise smallest t-norm.

Definition 1.2. ([5]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a a continuous $t-$ norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(e) $M(x, y, t) > 0$,
(f) $M(x, y, t) = 1$ if and only if $x = y$,
(g) $M(x, y, t) = M(y, x, t)$,
(h) $M(x, y, t) * M(y, z, t) \leq M(x, z, t + s)$,
(m) $M(x, y, \cdot) : (0, \infty) \to (0, 1)$ is continuous.

Remark 1.3. ([14])

(1) If the triangular inequality (h) is remplaced by the following:
(n) $M(x,y,t)*M(y,z,t)\leq M(x,z,\max\{t,s\})$ for all $x,y,z\in X$ and $s,t>0$, then the triple $(X,M,*)$ is called a non-archimedean fuzzy metric space. It is obvious to see that every non-archimedean fuzzy metric space is itself a fuzzy metric space.

(2) If the triangular inequality (h) is remplaced by the following:

$$\text{(p)}\ \max\left\{M(x,y,t)*M(y,z,\frac{t}{2}),M(x,y,\frac{t}{2})*M(y,z,t)\right\}\leq M(x,z,t)$$

for all $x,y,z\in X$ and $t>0$, then the triple $(X,M,*)$ is called a weak non-archimedean fuzzy metric space.

Remark 1.4. ([14]) A weak non-Archimedean fuzzy metric space is not necessarly a fuzzy metric space and if $M(x,y,\cdot)$ is nondecreasing, then a weak non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 1.5. ([5]) Let $(X,M,*)$ be a fuzzy metric space. Then

(1) A sequence $\{x_n\}$ is said to converge to x in X, denoted by $x_n\to x$, if and only if $\lim_{n\to\infty}M(x_n,x,t)=1$ for all $t>0$, i.e. for each $\varepsilon\in(0,1)$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n,x,t)>1-\varepsilon$ for all $n\geq n_0$.

(2) A sequence $\{x_n\}$ in X is an M-Cauchy sequence if and only if for each $\varepsilon\in(0,1)\,,t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for all $n,m\geq n_0$.

(3) The fuzzy metric space $(X,M,*)$ is called M-complete if every M-Cauchy sequence is convergent.

V. Sihag *et al.* [14] introduced the topology induced by a weak non-archimedean fuzzy metric space and obtained the same result given by [5] in the case of fuzzy metric spaces, see for details [14].

2. Main Result

In this section, we prove a fixed point theorem for (ψ,φ)-contractive maps in weak non-archimedean fuzzy metric spaces.

Let $\psi,\varphi:[0,1]\to[0,1]$ be such that

(a_1) ψ is continuous strictly monotone decreasing,
(b_1) φ is lower semi-continuous,
(c_1) $\psi\left(t\right)=\varphi\left(t\right)=0$ if and only if $t=1$.

Let X be a nonempty set, M be a fuzzy set on $X^2\times(0,\infty)$ and $f,g:X\to X$, we assume that there exist two functions ψ and φ defined as above, such that for every $t > 0$, and $x, y \in X$ with $x \neq y$ and $M(x, y, t) > 0$, the following condition holds:

(d_1) $\psi(M(fx, gy, t)) \leq \psi(m(x, y, t)) - \varphi(m(x, y, t))$,
(e_1) $m(x, y, t) = \min\{M(x, y, t), M(fx, x, t), M(gy, y, t)\}$.

Theorem 2.1. *Let $(X, M, *)$ be a complete weak non-archimedean fuzzy metric space and let ψ, φ, f and g four maps such that the forgoing conditions (a_1), (b_1), (c_1), (d_1) and (e_1) are satisfied. If for all $x, y \in X$ with $x \neq y$, $0 < M(x, y, t) < 1$ and if there exists $x_0 \in X$ such that $M(x_0, fx_0, t) > 0$ for all $t > 0$, then f and g have a unique common fixed point.*

Proof. Fix $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, we prove the theorem in several steps.

Step-1: Let $x_0 \in X$ such that $M(x_0, fx_0, t) > 0$ for all $t > 0$. We show that $M(x_n, x_{n-1}, t) \to 1$ as $n \to \infty$.

Suppose that n is an even number. Substituting $x = x_n$ and $y = x_{n-1}$ in (d_1) and using the properties of the functions ψ and φ we obtain

$$\begin{aligned}\psi(M(x_{n+1}, x_n, t)) &= \psi(M(fx_n, gx_{n-1}, t)) \\ &\leq \psi(m(x_n, x_{n-1}, t)) - \varphi(m(x_n, x_{n-1}, t)) \\ &\leq \psi(m(x_n, x_{n-1}, t)),\end{aligned}$$

using the monotone strictly decreasing of ψ, we get

$$M(x_{n+1}, x_n, t) \geq m(x_n, x_{n-1}, t), \tag{2.1}$$

setting $\tau_n(t) = M(x_{n+1}, x_n, t)$ and $\theta_{n-1}(t) = m(x_{n+1}, x_n, t)$. Now from (e_1) we have

$$\begin{aligned}\theta_{n-1}(t) &= \min\{\tau_{n-1}(t), M(fx_n, x_n, t), M(gx_{n-1}, x_{n-1}, t)\} \\ &= \min\{\tau_{n-1}(t), \tau_n(t), \tau_{n-1}(t)\}.\end{aligned}$$

If $\tau_n(t) < \tau_{n-1}(t)$, then $\theta_{n-1}(t) = \tau_n(t)$, it furthermore implies that

$$\psi(\tau_n(t)) \leq \psi(\tau_n(t)) - \varphi(\tau_n(t))$$

which leads to a contradiction. So we have

$$\tau_n(t) \geq \tau_{n-1}(t) = \theta_{n-1}(t) \geq M(x_0, fx_0, t) > 0. \tag{2.2}$$

Similarly, we can obtain the inequality (2.2) also in the case that n is odd number. Therefore the sequence $\{\tau_n(t)\}$ is monotone non-decreasing and bounded and so

$$\lim_{n\to\infty} \tau_n(t) = \lim_{n\to\infty} \theta_{n-1}(t) = r, \text{ where } 0 < r \leq 1.$$

We claim that $\lim_{n\to\infty} \tau_n (t) = r = 1$. In fact, if $0 < r < 1$, then, as φ is lower semi-continuous, from

$$\psi (\tau_n (t)) \leq \psi (\theta_{n-1} (t)) - \varphi (\theta_{n-1} (t)) ,$$

letting n goes to infinity, we get

$$\psi (r) \leq \psi (r) - \varphi (r) ,$$

which is a contradiction since $\varphi (r) > 0$. Hence

$$\lim_{n\to\infty} \tau_n (t) = 1. \tag{2.3}$$

Step-2: Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence. For this, it is sufficient to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $0 < \varepsilon < 1$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$. Setting $S_k(t) = M(x_{2m(k)}, x_{2n(k)}, t)$.

$$S_k(t) \leq 1 - \varepsilon,$$

this implies that $M(x_{2m(k)}, x_{2n(k)-2}, t) > 1 - \varepsilon$. Therefore

$$\begin{aligned} 1 - \varepsilon \geq S_k(t) &\geq M(x_{2m(k)}, x_{2n(k)-2}, t) * \tau_{2n-1} \left(\frac{t}{2}\right) * \tau_{2n} \left(\frac{t}{4}\right) \\ &\geq (1 - \varepsilon) * \tau_{2n(k)-1} \left(\frac{t}{2}\right) * \tau_{2n(k)} \left(\frac{t}{4}\right) \end{aligned}$$

letting $k \to \infty$ and using (2.3) we conclude that

$$\lim_{k\to\infty} S_k(t) = 1 - \varepsilon. \tag{2.4}$$

Moreover, from

$$M(x_{2m(k)}, x_{2n(k)+1}, t). \geq M(x_{2m(k)}, x_{2n(k)}, t) * \tau_{2n(k)} \left(\frac{t}{2}\right)$$

and

$$M(x_{2m(k)}, x_{2n(k)}, t). \geq M(x_{2m(k)}, x_{2n(k)+1}, t) * \tau_{2n(k)} \left(\frac{t}{2}\right)$$

letting $k \to \infty$, we obtain

$$\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1}, t) \geq 1 - \varepsilon$$

and

$$1 - \varepsilon \geq \limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1}, t)$$

which implies that

$$\lim_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1}, t) = 1-\varepsilon. \tag{2.5}$$

Analously, one can show that

$$\lim_{k\to\infty} M(x_{2m(k)-1}, x_{2n(k)+1}, t) = M(x_{2n(k)}, x_{2m(k)+1}, t) = 1-\varepsilon, \tag{2.6}$$

$$\begin{aligned}\psi\Big(M(x_{2m(k)}, x_{2n(k)+1}, t)\Big) &= \psi\left(M(fx_{2m(k)-1}, gx_{2n(k)}, t)\right)\\ &\le \psi\left(m(x_{2m(k)-1}, x_{2n(k)}, t)\right)\\ &\quad - \varphi\left(m(x_{2m(k)-1}, x_{2n(k)}, t)\right).\end{aligned}$$

Finally, as φ is a lower semi continuous function, letting $k\to\infty$, we get

$$\psi(1-\varepsilon) \le \psi(1-\varepsilon) - \varphi(1-\varepsilon),$$

which is a contradiction with $\varphi(1-\varepsilon) > 0$. Thus the sequence $\{x_{2n}\}$ is a Cauchy sequence and hence also the sequence $\{x_n\}$ is a Cauchy sequence. Since the weak non-Archimedean fuzzy metric space X is complete, therefore, there exists $x \in X$ such that $x_n \to x$ as $n\to\infty$.

Step-3: Let us now prove that x is a common fixed point of f and g, that is $x = fx = gx$. If $fx \neq x$, then there exists $t > 0$ such that $0 < M(x, fx, t) < 1$, from (e_1) we have

$$m(x, x_{2n}, t) = \min\{M(x, x_{2n-1}, t), M(fx, x, t), M(gx_{2n-1}, x_{2n-1}, t)\}.$$

Letting the limit as $n\to\infty$, we obtain

$$\lim_{n\to\infty} m(x, x_{2n}, t) = M(fx, x, t).$$

Now from

$$\begin{aligned}\psi\left(M(fx,, x_{2n}, t)\right) &= \psi\left(M(fx,, gx_{2n-1}, t)\right)\\ &\le \psi\left(m(x, x_{2n-1}, t)\right) - \varphi\left(m(x, x_{2n-1}, t)\right)\end{aligned}$$

taking the limit as $n\to\infty$, we obtain

$$\psi\left(M(fx,, x, t)\right) \le \psi\left(M(fx, x, t)\right) - \varphi\left(M(fx, x, t)\right),$$

which is a contradiction and therefore $x = fx$.

Analogously, we obtain that $x = gx$ and thus x is a common fixed point of f and g.

Step-4: We prove the uniqueness of the common fixed point of f and g.

Assume that $x, y \in X$ are two fixed points of f and g, then

$$\begin{aligned}\psi\left(M(x,, y, t)\right) &= \psi\left(M(fx,, gy, t)\right)\\ &\le \psi\left(m(x, y, t)\right) - \varphi\left(m(x, y, t)\right),\end{aligned}$$

which is a contradiction as $M(x,, y, t) = m(x,, y, t)$ and therefore $x = y$. □

As a consequence of Theorem 2.1, we state the following result.

Corollary 2.2. *Let $(X, M, *)$ be a complete weak non-archimedean fuzzy metric space and let ψ, φ, f three maps such that the forgoing conditions (a), $(b), (c), (d)$ and (e) are satisfied. If for all $x, y \in X$, with $x \neq y$ and $0 < M(x, y, t) < 1$ and if there exists $x_0 \in X$ such that $M(x_0, fx_0, t) > 0$ for all $t > 0$, then f has a unique common fixed point.*

3. Application

Later, from the previous obtained results, we deduce some fixed point results for mapping satisfying a c-contraction of an integral type, as an application of Theorem 2.1.

Let $Y = \{\chi : [0, 1[\to [0, 1[, \chi$ is a Lebesgue integrable mapping which is summable, nonegative, and satisfies $\int_{1-\varepsilon}^{1} \chi(t)dt > 0$ for each $0 < \varepsilon < 1\}$.

Theorem 3.1. *Let $(X, M, *)$ be a complete weak non-archimedean fuzzy metric space and let ψ, φ, f and g four maps such that the forgoing conditions $(a), (b)$ and (c) are satisfied and if*

$$\begin{aligned} &\int_{1-\psi(M(fx,gy,t))}^{1} \chi(s)ds \\ &\leq \int_{1-\psi(m(x,y,t))}^{1} \chi(s)ds - \int_{1-\varphi(m(x,y,t))}^{1} \chi(s)ds \qquad \textit{for } \chi \in Y \end{aligned} \tag{3.1}$$

for all $x, y \in X$, with $x \neq y$ and $0 < M(x, y, t) < 1$ and if there exists $x_0 \in X$ such that $M(x_0, fx_0, t) > 0$ for all $t > 0$, then f and g have a unique common fixed point.

Proof. For $\chi \in Y$, we consider the function

$$\Lambda : [0, 1] \to [0, 1], \text{ by } \Lambda(\varepsilon) = \int_{1-\varepsilon}^{1} \chi(s)ds.$$

Λ is continuous, $\Lambda(0) = 0$, Λ is strictly increasing. Hence we can write the inequality (3.1) in the form

$$\Lambda(\psi(M(fx, gy, t))) \leq \Lambda(\psi(m(x, y, t))) - \Lambda(\varphi(m(x, y, t))). \tag{3.2}$$

Setting, $\psi_1 = \Lambda \circ \psi$ and $\varphi_1 = \Lambda \circ \varphi$. ψ_1 is strictly decreasing, continuous and satisfies the foregoing properties (a) and (c) for any $t > 0$, and φ_1 is lower semi-continuous and satisfies the properties (c) for any $t > 0$, then by Theorem 2.1, f and g have a unique common fixed point. □

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