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GABOR FRAMES AND MULTILINEAR MULTIPLIERS ON MODULATION SPACES

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Abstract. In this paper, we study multilinear multipliers on modulation spaces via Gabor frames. Thanks for the oscillatory integrals and the well-known inequalities, we develop a method to find several classes of multilinear multipliers and estimate the norms, some of them generalize the previous work. Finally, the application is presented.

1. Introduction

In this section, modulation spaces and the multilinear operators defined on them will be introduced briefly. Then Banach frames as a principal tool will be redefined precisely in our own words.

1.1. **Modulation Spaces.** Define a sequence space

$$\ell^{s,q} = \{a \in \ell^\infty : \|a\|^q = \sum_k \langle k\rangle^{sq}|a_k|^q < \infty\}, \langle x\rangle = 1 + |x|.$$

The modulation space $M^s_{p,q}$ owns the following norm

$$\|f\| = \big\|\|\mathscr{F}^{-1}(\psi_k\mathscr{F}f)\|_{L^p}\big\|_{\ell^{s,q}} = \|\{\mathscr{F}^{-1}(\psi_k\mathscr{F}f)\}\|_{\ell^{s,q}(L^p)},$$

where $\{\psi_k(\xi) = \psi(\xi - k) \in \mathscr{S}\}$ (Schwartz class) is a suitable partition of unity named the frequency uniform decomposition. $M^0_{p,q}$ is simplified as $M_{p,q}$. See [16, 22, 31] for more details on modulation spaces.

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The modulation spaces can be regarded as (the Fourier transform of) the special Wiener amalgam spaces[3, 11, 15]. Now we fix the definition of the spaces.

Definition 1.1. Given $\psi \in \mathscr{S}$ as in the modulation spaces, a function Banach space X and a sequence Banach space Y, the space $W(X,Y)$ consists of all distributions $u \in \mathscr{S}'$ ($u \in X_{\text{loc}}$ more precisely) for which

$$\|u\|_{W(X,Y)} := \left\|\{\|\psi_k u\|_X\}\right\|_Y < \infty.$$

Another type of spaces mentioned in the paper is

$$\mathscr{F}X := \{\mathscr{F}f | f \in X\} \text{ whose norm is } \|f\|_{\mathscr{F}X} := \|\mathscr{F}f\|_X,$$

where X is L^p or any other known space. We have $M^s_{p,q} = \mathscr{F}W(\mathscr{F}L^p, \ell^{s,q})$.

1.2. **Multilinear Operators.** The research of multilinear operators (or multipliers) keeps active due to the close relation with pseudo-differential equations (PDEs) initiated by Calderón about 50 years ago. It has not been too long people started to study the operators on modulation spaces since the spaces were introduced. In this paper, we study the multilinear operators determined by the multipliers on the modulation spaces that is defined as follows.

Definition 1.2. ([10]) Let

$$F(x)(\xi_1, \cdots, \xi_m) = \prod_{j=1}^m \hat{f}_j(\xi_j) \mathrm{e}^{2\pi \mathrm{i} \xi_j \cdot x} \in \mathscr{S}(\mathbb{R}^n, \mathscr{S}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n)),$$

we define

$$\begin{aligned} T(f_1, \cdots, f_m)(x) &= \mu(F(x)) \\ &= \int_{\mathbb{R}^{mn}} \prod_{j=1}^m \hat{f}_j(\xi_j) \mathrm{e}^{2\pi \mathrm{i} \sum_j \xi_j \cdot x} \mu(\xi_1, \cdots, \xi_m) \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_m, \end{aligned}$$

where $\mu \in \mathscr{S}'(\mathbb{R}^n \times \cdots \times \mathbb{R}^n)$ is the multiplier called the symbol of T.

In fact, we always assume that μ the symbol of T is a locally integrable function in the cases. If μ has the tensor form, then it will be reduced to the linear multiplier. We refer [18, 19, 20, 26, 29, 32] for further information.

T is an operator mapping a product space $X_1 \times \cdots \times X_m$ into another space X. The space of the symbols whose corresponding operators satisfying

$$\|T(f_1, \cdots, f_m)\|_X \lesssim \prod_j \|f_j\|_{X_j}$$

denoted with $\mathcal{M}(X_1, \cdots, X_m, X)$ while if $m = 1$, it is degenerated to the multiplier space. Nevertheless, symbols and the corresponding operators are seldom distinct since

$$\mu \mapsto T(\mu) : \mathcal{M}(X_1, \cdots, X_m, X) \hookrightarrow L(X_1, \cdots, X_m, Y).$$

Though multilinear operators generalize the Fourier multipliers, it can be treated as a special type of Fourier multipliers in the sense that

$$T(f_1, \cdots, f_m)(x) \quad = \quad \mathscr{F}^{-1}(\mu \mathscr{F} f)(x, x, \cdots, x), f = f_1 \otimes \cdots \otimes f_m, \tag{1.1}$$

where the variables take the values on a $n-$dimensional submanifold

$$S = \{(x, \cdots, x) \,|\, x \in \mathbb{R}^n\}$$

of $\mathbb{R}^{mn}$. It is straightforward to see that its Fourier transform is

$$\mathscr{F} T(f_1, \cdots, f_m)(\xi) \quad = \quad \int_{\xi = \sum_j \xi_j} \prod_{j=1}^{m} \hat{f}_j(\xi_j) \mu(\xi_1, \cdots, \xi_m) \mathrm{d}\nu, \tag{1.2}$$

where ν is the singular measure derived by $\xi = \sum_j \xi_j$. Notice that the Fourier transform is different from the one in (1.1). Moreover,

$$\langle T(f_1, \cdots, f_m), g\rangle = \int_{\mathbb{R}^{mn}} \mu(\xi_1, \cdots, \xi_m) \prod_{j=1}^{m} \hat{f}_j(\xi_j) \overline{\hat{g}(\sum_j \xi_j)} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_m. \tag{1.3}$$

1.3. **Banach Frames and Oscillatory Integrals.** The frames are thought as the remarkable tools for Hilbert spaces. On theory or practice, the frames perfumed better than the bases regarded as a kind of special frames. In 1991, Gröchenig successfully generalized the concepts of frames to Banach spaces [21] that makes the frames more prevailing and powerful. The definition is rephrased as follows(also see [25]).

Definition 1.3. (Banach Frames) Assume ℓ is a so-called BK (sequence) space (see Definition 1.5) and $x_k \in X, f_k \in X^*$ where X is a Banach space. If

(1) we have $Dx = \{f_k(x)\} : X \to \ell$ such that

$$\|\{f_k(x)\}\|_\ell \sim \|x\|,$$

(2) and $S\{a_k\} = \sum_k a_k x_k : \ell \to X$ such that

$$S(\{f_k(x)\}) = x, \ \ \forall x \in X, \tag{1.4}$$

then $(\{x_k \in X\}, \{f_k \in X^*\})$ is called the Banach frame of X (with respect to ℓ).

Remark 1.4. Note that it is unnecessary $S\{a_k\} = \sum_k a_k x_k$, but [9] said it is common in which ℓ is denoted as X_d. (1.4) means $SD = I$ where D, S are named decomposition operator and reconstruction operator respectively while $f_k(x)$ are named frame coefficients of x.

Definition 1.5. A BK space is a kind of sequence Banach spaces that any coordinator functional

$$f_n(\{a_k\}) = a_n$$

is continuous.

The definition of frames is contained in the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, shift right, bend right, "D_1"'] & Y \arrow[d, shift right, bend right, "D_2"'] \\ \ell_X \arrow[u, shift right, bend right, "S_1"'] \arrow[r, "G"] & \ell_Y \arrow[u, shift right, bend right, "S_2"'] \end{tikzcd} \tag{1.5}$$

that shows $T = S_2 G D_1$ and the boundedness of G yields the boundedness of T, precisely

$$\|T\| = \|S_2 G D_1\| \sim \|G\|,$$

where G can be represented with an infinite Gram-type matrix

$$\{g_j(Tx_k)\}_{jk}$$

where $(\{x_k \in X\}, \{f_k \in X^*\}), (\{y_k \in Y\}, \{g_k \in Y^*\})$ are the Banach frames of X and Y respectively. In the algebraic view, $\ell : X \mapsto \ell_X$ is a functor from the category of Banach spaces with frames to BK.

The argument can be extended to multilinear setting. Considering any multilinear operator $T : X_1 \times \cdots \times X_m \to Y$, we shall verify

$$G = \{g_j(T(x_{1k_1}, \cdots, x_{mk_m}))\}_{jk_1\cdots k_m} \tag{1.6}$$

is bounded on the corresponding BK spaces as a multilinear operator. The boundedness of T is reduced to the boundedness of G. Thus the remained task is to estimate $\|G\|$ under the matrix norm, namely

$$\| \sum_{k_1,\cdots,k_m} G_{jk_1\cdots k_m} a^{(1)}_{k_1} \cdots a^{(m)}_{k_2} \|_{\ell_j} \lesssim \|a^{(1)}\|_{\ell_1} \cdots \|a^{(m)}\|_{\ell_m}.$$

Notice that the elements in the frame matrix G can be expressed as forms of oscillatory integrals in common cases [1]. Thus the theory of oscillatory integrals should be applied into the estimate of the elements while most previous work ignored it. Besides the stationary phase method [28], we need the following fact. Integrating $\int_a^b \mathrm{e}^{\mathrm{i}\lambda\phi(x)}\psi(x)$ by parts where ϕ is smooth, we have

Lemma 1.6.

$$I(\lambda, \psi) = \frac{1}{\mathrm{i}\lambda}\left(\frac{\psi(a)}{\phi'(a)} - \frac{\psi(b)}{\phi'(b)}\right) + \frac{1}{\mathrm{i}\lambda} I\left(\lambda, \left(\frac{\psi}{\phi'}\right)'\right)$$
$$\lesssim \lambda^{-1},$$

where $\psi \in C^1$ unnecessarily vanishes at the ends. As the most interesting case, if $\psi(a) \neq 0, \psi(b) = 0$, then $I(\lambda) \sim \lambda^{-1}$ as $\lambda \to \infty$.

The lemma can be extended to high dimensional cases analogically.

We get the boundedness of the operators or the matrices as soon as we complete the estimate of the elements in G employing the famous inequalities. In words, the idea of the paper can be summarized as the following chain,

$$\text{multilinear operators} \xrightarrow{\text{Frames}} \text{frame matrices} \xrightarrow{\text{oscillatory integrals}} \text{estimate of the elements}$$

It is an original idea. Frames have been employed to estimate multipliers on modulation spaces [1] and multilinear operators even more general versions are dealt with in many papers [2, 5, 6, 10], but it is the first time to apply both frames and oscillatory integrals into multilinear operators. Furthermore, the method provided in the paper can be extended to pseudo-differential operators smoothly. In the paper we prove several theorems each of whom says if a multilinear multiplier μ is in some class, then μ is a bounded multilinear operators on the modulation spaces.

1.4. **Notations and Organization.** Through this paper, $A(x) \lesssim B(x)$ means that there exists a positive constant C such that $|A(x)| \leq C|B(x)|$ for all x in an abstract space where A, B are two functions with the same or opposite sign at any point on the space, while $A(x) \sim B(x)$ is used to denote $A(x) \lesssim B(x) \lesssim A(x)$. $\sum_{i\in\mathbb{Z}^n}$ ($\bigcup_{i\in\mathbb{Z}^n}, \int_{x\in X} f(x)\mathrm{d}x, \sup_{x\in X}$) will be simply written as $\sum_i$ ($\bigcup_i, \int f(x), \sup$), if there is no ambiguity. Even the sequence $\{x_k\}_{k\in I}$ is simplified as $\{x_k\}$ where I is always countable.

Hybrid norms appear frequently in the paper. $\|f(x,y)\|_{X_x}$ means we only take the X norm of the function (or the sequence) $f(\cdot) : x \mapsto f(x,y)$ unless it is very obvious to distinct the two variables. We would like to use the hybrid norm (space) such as

$$\|f(x,y)\|_{X_x Y_y} = \big\|\|f(x,y)\|_{X_x}\big\|_{Y_y}.$$

The order of the norms is always adapt to the order of variables, namely

$$\|f(x,y)\|_{XY} = \big\|\|f(x,y)\|_{X_x}\big\|_{Y_y},$$

if there is no index below the space symbols.

The main target of the paper is to study the boundedness of multilinear operators on modulation spaces through Gabor frames. This paper is organized as follows. In section 2, we introduce Gabor frames briefly. In section 3, we study the boundedness of the multilinear operators and estimate the Gabor matrix by oscillatory integrals. In section 4, some interesting application is presented.

2. Gabor Frames

In this section, we introduce a kind of frames, Gabor frames that have been applied in modern analysis frequently. Gabor frames turned out to be the appropriate tools for time-frequency such as representing an operator with so-called Gabor matrix(see, [1, 12, 13]). In this paper, we will estimate the Gabor matrix and the operator with the Gabor representation. The following theorem constructs a kind of Gabor frames for modulation spaces that will be used heavily(also see [17, 21, 27]).

Theorem 2.1. *Let $\phi \in M_{1,1}$ be such that $\{\phi_{kl} = \phi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x}, k, l \in \mathbb{Z}^n\}$ is a Gabor frame for L^2, and let $1 \le p, q < \infty$. Then there exists a (canonical) dual $\psi \in M_{1,1}$, $(\{\psi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x}, k, l \in \mathbb{Z}^n\}, \{\phi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x}, k, l \in \mathbb{Z}^n\})$ is a Banach frame for any modulation space $M^s_{p,q}$ with respect to $\ell^s_{p,q}$, namely*

$$f = \sum_{kl} \langle f, \psi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x}\rangle \phi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x} \tag{2.1}$$

unconditionally converges and

$$\|f\|_{M^s_{p,q}} \sim \|\{\langle f, \psi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x}\rangle\}\|_{\ell^s_{p,q}} \sim \|\{\langle f, \phi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x}\rangle\}\|_{\ell^s_{p,q}}. \tag{2.2}$$

Remark 2.2. If $p = \infty$ or $q = \infty$, then (2.1) is weakly convergent, but strongly in the space denoted $\tilde{M}^s_{p,q}$ as the completion of $\mathscr{S}$ under the $M^s_{p,q}$-norm. Fortunately, (2.2) always holds for the duality.

In the definition, the inner product with ψ is regarded as a linear functional. And it says that $\ell(M^s_{p,q}) = \ell^s_{p,q}$ actually and gives a concrete method to construct the frames for any modulation space. It is indeed easy to find such $\phi \in M^1$ the generator of a Gabor frame for L^2. When $\alpha\beta < 1$, if the Parseval formula holds for the frame, namely

$$\sum_{kl} |\langle f, \phi(x-\alpha k)\mathrm{e}^{2\pi \mathrm{i}\beta l\cdot x}\rangle|^2 = \|f\|_{L^2}, \tag{2.3}$$

equevilently

$$\sum_{k} |\hat{\phi}(x-\beta l)|^2 = \alpha^n, \quad \forall x \in \mathbb{R}^n, \tag{2.4}$$

where $\operatorname{supp}\hat{\phi} \subseteq [-\frac{1}{2\alpha}, \frac{1}{2\alpha}]^n$, then we take $\gamma = \phi$ and the frame is named the Parseval frame. We shall take $\alpha = \frac{1}{2}, \beta = 1, \phi \in \mathscr{S}$ for ease.

In linear case $m = 1$, we easily obtain the Gabor matrix

$$G_{klk'l'} \quad = \quad (-1)^C \int_{\mathbb{R}^n} \hat{\phi}(\eta)\overline{\hat{\phi}(\eta + l' - l)}\mu(\eta + l')\mathrm{e}^{\pi\mathrm{i}(k-k')\cdot\eta}\mathrm{d}\eta. \tag{2.5}$$

In fact, (2.5) has appeared in [1] implicatively of which the whole argument can be rebuilt on it with Lemma 1.6 while [12] studied the Fourier integral operators with the matrix. In next section, we will show the multilinear version of (2.5).

3. Boundedness of the multilinear operator

We begin with the Gabor matrix in the multilinear case. Notice

$$\hat{\phi}_{kl}(\xi) = \hat{\phi}(\xi - \beta l)\mathrm{e}^{-2\pi\mathrm{i}\alpha k\cdot(\xi-\beta l)}, \tag{3.1}$$

where ϕ is the frame generator. (3.1) is substituted into (1.3) to show the Gabor matrix of the multilinear multipliers with $2m + 2$ indexes taking $\alpha = \frac{1}{2}, \beta = 1$(also see (1.6)).

$$\begin{aligned} G_{klk_1l_1\cdots k_ml_m} \quad &= \quad C\int_{\mathbb{R}^{mn}} \prod_{j=1}^m \hat{\phi}(\xi_j - l_j)\overline{\hat{\phi}(\sum_j \xi_j - l)} \\ &\qquad \cdot\mu(\xi_1, \cdots, \xi_m)\mathrm{e}^{\pi\mathrm{i}\sum_j(k-k_j)\cdot\xi_j}\mathrm{d}\xi_1\cdots\mathrm{d}\xi_m \\ &= \quad C'\int_{\mathbb{R}^{mn}} \prod_{j=1}^m \hat{\phi}(\xi_j)\overline{\hat{\phi}(\sum_j \xi_j + \sum_j l_j - l)} \\ &\qquad \cdot\mu(\xi_1{+}l_1, \cdots, \xi_m{+}l_m)\mathrm{e}^{\pi\mathrm{i}\sum_j(k-k_j)\cdot\xi_j}\mathrm{d}\xi_1\cdots\mathrm{d}\xi_m, \end{aligned} \tag{3.2}$$

where the coefficients $C = (-1)^{\sum_j k_j\cdot l_j - k\cdot l}, C' = (-1)^{k\cdot(\sum_j l_j - l)}$ will be ignored. It is the multilinear version of (2.5). Notice $G_{klk_1l_1\cdots k_ml_m} = 0$ out of $|\sum_j l_j - l| < C$.

Actually (3.2) is an oscillatory integral

$$I(k, \Psi) = \int_{\mathbb{R}^{mn}} \Psi(\xi)\mathrm{e}^{\mathrm{i}\Phi(\xi,k)}\mathrm{d}\xi, \tag{3.3}$$

where $\xi = (\xi_1, \cdots, \xi_m)$ and Φ, Ψ are real valued functions with the indexes abstracted from (3.2).

3.1. **Smooth Case.** In the case when μ is smooth, the boundedness problem has been well studied, but it is the first time when we apply the theory of oscillatory integrals into it with Grame frames. Here $\mu \in C^\infty$ means $\|\partial^\gamma \mu\|_{L^\infty} < \infty$.

Theorem 3.1. *If $\mu \in C^\infty, 1 \le p_j, q \le \infty$, then*

$$\mu \in \mathcal{M}(M^s_{p_1,q_1}, \cdots, M^s_{p_m,q_m}, M^s_{p,q}),$$

namely T in Definition 1.2 is bounded on these modulation spaces where $\frac{1}{p} = \sum_j \frac{1}{p_j}, m-1+\frac{1}{q} = \sum_j \frac{1}{q_j}, s \ge 0$.

Proof. We follow the discussion above. If $\mu \in C^\infty$, it immediately follows from (3.2) that for large N

$$|G_{klk_1l_1\cdots k_ml_m}| \lesssim \langle k_1 - k\rangle^{-N} \cdots \langle k_m - k\rangle^{-N},$$

then

$$\begin{aligned} |(Ga)_{kl}| &\lesssim \sum_{|\sum_j l_j - l| < C} \sum_{k_1,\cdots,k_m} \langle k-k_1\rangle^{-N} \cdots \langle k-k_m\rangle^{-N} a^{(1)}_{k_1l_1} \cdots a^{(m)}_{k_ml_m} \\ &= \sum_{|\sum_j l_j - l| < C} \sum_{k_1,\cdots,k_m} \langle k-k_1\rangle^{-N} a^{(1)}_{k_1l_1} \cdots \langle k-k_m\rangle^{-N} a^{(m)}_{k_ml_m} \\ &= \sum_{|\sum_j l_j - l| < C} \langle k\rangle^{-N} * a^{(1)}_{\cdot,l_1} \cdots \langle k\rangle^{-N} * a^{(m)}_{\cdot,l_m}, \end{aligned}$$

where $a = a^{(1)} \otimes \cdots \otimes a^{(m)}$. Hence

$$\begin{aligned} \|(Ga)_{kl}\|_{\ell^p_k} &\lesssim \sum_{|\sum_j l_j - l| < C} \|\langle k\rangle^{-N} * a^{(1)}_{\cdot,l_1}\|_{\ell^{p_1}} \cdots \|\langle k\rangle^{-N} * a^{(m)}_{\cdot,l_m}\|_{\ell^{p_m}} \\ &\lesssim \sum_{|\sum_j l_j - l| < C} \|a^{(m)}_{\cdot,l_1}\|_{\ell^{p_1}} \cdots \|a^{(m)}_{\cdot,l_m}\|_{\ell^{p_m}}, \end{aligned}$$

where $\frac{1}{p} = \sum_j \frac{1}{p_j}$. Taking the $\ell^{s,q}$-norm $(s \ge 0)$ with respect to the variable l and with the Young's inequality we obtain

$$\|Ga\|_{\ell^s_{p,q}} \lesssim \|a^{(1)}\|_{\ell^s_{p_1,q_1}} \cdots \|a^{(m)}\|_{\ell^s_{p_m,q_m}},$$

where $m-1+\frac{1}{q} = \sum_j \frac{1}{q_j}$. It yields the boundedness of μ on the corresponding modulation spaces. $\square$

Remark 3.2. The theorem means $C^\infty \hookrightarrow \mathcal{M}(M^s_{p_1,q_1}, \cdots, M^s_{p_m,q_m}, M^s_{p,q})$ and it holds for $\mu \in C^N$ with large N.

In the proof, we have employed the following fact.

Lemma 3.3.

$$\Big\|\{\sum_{|\sum_j l_j - l|<C} a^{(1)}_{l_1}\cdots a^{(m)}_{l_m}\}\Big\|_{\ell^{s,q}_l} \quad\lesssim\quad \|a^{(1)}\|_{\ell^{s,q_1}}\cdots\|a^{(m)}\|_{\ell^{s,q_m}},$$

where $s\ge 0, m-1+\frac{1}{q}=\sum_j \frac{1}{q_j}, 1\le q, q_j, j=1,\cdots,m$. *It holds if* $q_j=q\le 1$.

Thus

$$C^\infty \hookrightarrow \mathcal{M}(M^s_{p_1,q},\cdots,M^s_{p_m,q},M^s_{p,q}), q\le 1.$$

A more general case is that $|\partial^\gamma\mu(\xi)|\lesssim\langle\xi\rangle^\delta$. The oscillatory integral estimate of the Gabor matrix now is that for large N

$$|G_{klk_1l_1\cdots k_ml_m}|\lesssim\sum_j\langle l_j\rangle^\delta\langle k_1-k\rangle^{-N}\cdots\langle k_m-k\rangle^{-N}.$$

Following the proof above, it holds that

$$\mu\in\mathcal{M}(M^{s+\delta}_{p_1,q_1},\cdots,M^{s+\delta}_{p_m,q_m},M^s_{p,q}).$$

One of ordinary examples is $D^\delta(\prod_j f_j)$ where $\delta\in\mathbb{Z}$ or is a large positive number.

3.2. **Non-smooth Case.** It is easy to see that (3.2) can be also rewritten as

$$\begin{aligned} G_{klk_1l_1\cdots k_ml_m} &= C\mathscr{F}^{-1}\Bigg(\prod_j\hat\phi(\xi_j-l_j)\overline{\hat\phi(\sum_j\xi_j-l)} \\ &\qquad\cdot\mu(\xi_1,\cdots,\xi_m)\Bigg)(k-k_1,\cdots,k-k_m) \\ &= C\mathscr{F}^{-1}\Bigg(\prod_j\hat\phi(\cdot-l_j)\mu\Bigg)*\psi(k-k_1,\cdots,k-k_m), \end{aligned} \tag{3.4}$$

where $\psi\in\mathscr{S}$ obtained by inserting a bump function.

With (3.4), we consider the case when μ is unnecessarily smooth.

Theorem 3.4. *If* $\mu\in W(\mathscr{F}L^1,\ell^r), 1\le p_j, q\le\infty$, *then*

$$\mu\in\mathcal{M}(M^s_{p_1,q_1},\cdots,M^s_{p_m,q_m},M^s_{p,q}),$$

where $\frac{1}{p}=\sum_j\frac{1}{p_j}, (m-1)(1-\frac{1}{r})+\frac{1}{q}=\sum_j\frac{1}{q_j}, s\ge 0.$

Proof. For (3.4), $(Ga)_{kl}$ is expressed as

$$\begin{aligned} \sum_{|\sum_j l_j-l|<C}\sum_{k_1,\cdots,k_m} &\mathscr{F}^{-1}\Bigg(\prod_j\hat\phi(\xi_j-l_j)\mu(\xi_1,\cdots,\xi_m)\Bigg) \\ &*\psi(k-k_1,\cdots,k-k_m)a_{k_1l_1}\cdots a_{k_ml_m}. \end{aligned}\tag{3.5}$$

Checking the fact with Minkowski's inequality and Hölder's inequality,

$$\begin{aligned}\Big\| \sum_{k_1,\cdots,k_m} F_{k-k_1,\cdots,k-k_m} \prod_j a^{(j)}_{k_j} \Big\|_{\ell^p_k} &\le \sum_{k_1,\cdots,k_m} |F_{k_1\cdots k_m}| \Big\| \prod_j a^{(j)}_{k-k_j} \Big\|_{\ell^p_k} \\ &\le \|F\|_{\ell^1} \prod_j \|a^{(j)}\|_{\ell^{p_j}}, \qquad (3.6)\end{aligned}$$

where $\frac{1}{p} = \sum_j \frac{1}{p_j}$. Taking ℓ^p-norm with respect to k on (3.5), we have

$$\|(Ga)_{kl}\|_{\ell^p_k} \lesssim \sum_{|\sum_j l_j - l| < C} \Big\|\mathscr{F}^{-1}\Big(\prod_j \hat{\phi}(\cdot - l_j)\mu\Big)\Big\|_{\ell^1} \prod_j \|a^{(j)}_{\cdot,l_j}\|_{\ell^{p_j}}.$$

Applying Proposition 1.3.3 in [30] and Lemma 3.3, we get

$$\begin{aligned}\|(Ga)\|_{\ell^s_{p,q}} &\lesssim \|\mu\|_{W(FL^1,\ell^r)} \Big(\sum_{|\sum_j l_j - l|<C} \prod_j \|a^{(j)}_{\cdot,l_j}\|^{r'}_{\ell^{p_j}} \Big)^{\frac{1}{r'}} \\ &\lesssim \|\mu\|_{W(\mathscr{F}L^1,\ell^r)} \prod_j \|a^{(j)}\|_{\ell^s_{p_j,q_j}},\end{aligned}$$

where $(m-1)(1-\frac{1}{r}) + \frac{1}{q} = \sum_j \frac{1}{q_j}$. It completes the proof. □

Since

$$M^{\infty,1} \hookrightarrow W(\mathscr{F}L^1, \ell^\infty) \hookrightarrow \mathcal{M}(M^s_{p_1,q_1}, \cdots, M^s_{p_m,q_m}, M^s_{p,q}), \text{ taking } r = \infty,$$

we get the multilinear multiplier version of the results in [2, 4, 6]. However, it is not sure

$$W(\mathcal{M}^p, \ell^\infty) \hookrightarrow \mathcal{M}(M^s_{p_1,q_1}, \cdots, M^s_{p_m,q_m}, M^s_{p,q})$$

as multipliers where $\mathcal{M}^p = \mathcal{M}(L^p, L^p)$ is the well-known Hörmander space, since (3.6) does not qualify to be the convolution.

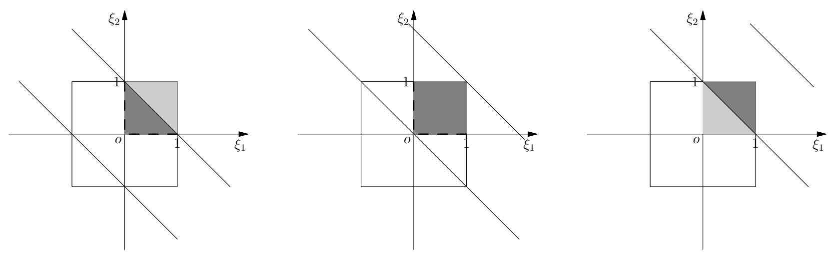
Now consider a new class of multipliers as a multilinear version of the class in [1], $\mu = \sum_{h_1\cdots h_m} c_{h_1\cdots h_m} \chi_{h_1\cdots h_m}$ where $\chi_{h_1\cdots h_m}$ are the characteristic functions on the high dimension squares

$$\prod_j [0,1]^n + h_j, h_j \in \mathbb{Z}.$$

Generally, $\mu \notin W(\mathscr{F}L^1, \ell^\infty)$. Let us consider $T(\chi_{h_1\cdots h_m})$ since $T(\mu)$ is a linear mapping. In (3.3), for $|\sum_j l_j - l| < C$

$$\Phi_{h_1\cdots h_m} = \begin{cases} \prod_{j=1}^m \hat{\phi}(\xi_j) \overline{\hat{\phi}(\sum_j \xi_j + \sum_j l_j - l)} \chi, & \max_j |h_j - l_j| \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

where χ is a characteristic function on a rectangle. The discontinuous edges of $\Phi_{h_1\cdots h_m}$ that must be parallel to the axes will appear when the rectangles do not contain the support of the bump function $\prod_{j=1}^m \hat{\phi}(\xi_j)\overline{\hat{\phi}(\sum_j \xi_j + \sum_j l_j - l)}$. We turn to estimate the corresponding oscillatory integral $I_{h_1\cdots h_m}$. For this purpose, it is better to assume that ϕ is radical and separated that means it is the product of the even functions. The additional assumption reduces the estimate to one dimensional case. With the following figures, we have a closer look at what the integrated functions look like.



(a) $\sum_j l_j = l$ (b) $\sum_j l_j - l = -1$ (c) $\sum_j l_j - l = -2$

Figure 1: The large squre, the domain between tow oblique lines, the grey area and the dack area represent (the supports of) $\prod_j \hat{\phi}(\xi_j)$, $\hat{\phi}(\sum_j \xi_j + \sum_j l_j - l)$, the characteristic function and their common domain respectively in each subfigure.

In Figure 1(c), the integrated function is continuous while Figure 1(a) and Figure 1(b) have discontinuous edges drawn by dashed lines where Lemma 1.6 should be applied to show the estimate. (Note that Figure 1(b) is continuous at origin.)

Theorem 3.5. *Suppose that the multiplier is the linear combination of the characteristic functions, precisely*

$$\mu = \sum_{h_1\cdots h_m} c_{h_1\cdots h_m}\chi_{h_1\cdots h_m}, \sup|c_{h_1\cdots h_m}| < \infty,$$

then $\mu \in \mathcal{M}(M^s_{p_1,q_1}, \cdots, M^s_{p_m,q_m}, M^s_{p,q})$ *where* $\frac{1}{p} = \sum_j \frac{1}{p_j}, 1 < p_j < \infty, m-1+\frac{1}{q} = \sum_j \frac{1}{q_j}, s \geq 0.$

Proof. Following the argument above, the analysis of oscillatory integrals gives the estimate of (3.2) or (3.3),

$$\begin{cases} \sim \prod_{j=1}^m K(k-k_j) + O(\prod_{j=1}^m J_j(k-k_j)), & \text{in cases as Figure 1(a),} \\ \lesssim \prod_{j=1}^m J_j(k-k_j), & \text{other discontinuous cases,} \\ \text{decays fast,} & \text{with out discontinuous edges,} \\ = 0, & \text{otherwise}(\max_j |h_j - l_j| > 1), \end{cases}$$

where $K(x) = \prod_i \frac{1}{x_i}$ the majorant at infinity and each $J_j(x)$ is the product of $\frac{1}{x_i}$s and $\frac{1}{\langle x_i\rangle^N}$s, $N \geq 2$.

Now corresponding Gabor matrix $(G_{h_1\cdots h_m}a)_{kl}$ can be controlled by a product of convolutions with kernels $K_j = K + J_j$, namely for $\max_j |h_j - l_j| \leq 1$

$$(G_{h_1\cdots h_m}a)_{kl} \quad \lesssim \quad c_{h_1\cdots h_m}\Bigg(\sum_{|\sum l_j - l| < C}\prod_j K_j * a_{\cdot,l_j}(k)\Bigg),$$

then

$$\begin{aligned}(Ga)_{kl} \quad &\lesssim \quad \Bigg|\sum_{|\sum l_j - l|<C}\ \sum_{\max_j |h_j - l_j| \leq 1} c_{h_1\cdots h_m}\prod_j K_j * a_{\cdot,l_j}\Bigg| \\ &\lesssim \quad \sup_{h_1\cdots h_m}|c_{h_1\cdots h_m}|\sum_{|\sum l_j - l|<C}\prod_j |K_j * a_{\cdot,l_j}(k)|.\end{aligned}$$

Taking ℓ^p-norm on two sides with respect to k and with the Hilbert operator [14, 28] that also implies a weak type inequality, we get

$$\begin{aligned}\|(Ga)_{\cdot,l}\|_{\ell^p} \quad &\lesssim \quad \sup_{h_1\cdots h_m}|c_{h_1\cdots h_m}|\sum_{|\sum l_j - l|<C}\prod_j \big\|K_j * a_{\cdot,l_j}\big\|_{\ell^{p_j}} \\ &\lesssim \quad \sup_{h_1\cdots h_m}|c_{h_1\cdots h_m}|\sum_{|\sum l_j - l|<C}\prod_j \|a_{\cdot,l_j}\|_{\ell^{p_j}}.\end{aligned}$$

Then take $\ell^{s,q}$-norm to complete the proof with Lemma 3.3. □

Of cause, it fails in the cases $p = 1$. Moreover, it holds when χ is the characteristic function of any bounded square even μ is piecewise smooth on the squares.

4. Comment and Application

We have obtained at least three kinds of multilinear multipliers bounded on modulation spaces and it is different from the linear case in some manner. Some of them generalize the previous work. More important contribution of the paper is providing the general method to estimate the multilinear multipliers that can be extended to pseudo-differential operators in parallel.

As is mentioned, the study of multilinear operators are motivated by the research of PDEs with a long history. As an example, the operator $D^\delta(\prod_j f_j)$ often acts as the nonlinearity in PDEs. If we deal with the equations on modulation spaces, then we need estimate the term. Now one can estimate it with the multiplier $\mu(\xi) = |\sum_j \xi_j|^\delta$. We refer to [7] for the application.

Another example is the paraproduct defined by [8]

$$\Pi(f,g) := \sum_{l\in\mathbb{Z}} (S_{l-1}f)(\triangle_l g),$$

where $\triangle_l$ constitute the homogeneous Littlewood-Paley decomposition and $S_l = \sum_{j\le l-1} \triangle_j$. The multiplier is discontinuous at origin while it is smooth in the inhomogeneous case. Its symbol is the sum of the singular part denoted

$$T(\mu)(f,g) := \sum_{l<0} (S_{l-1}f)(\triangle_l g), \mu(\xi,\eta) = \sum_{l<0} \hat{\psi}_{l-1}(\xi)\hat{\phi}_l(\eta)$$

discontinuous at origin and the regular part that is indeed in C^∞ class. Following (3.2), we present a rough estimate of the singular part applying Lemma 1.6 into the singular part and considering the boundedness and the support.

$$|G_{klk_1l_1k_2l_2}| \begin{cases} \sim K(k-k_1), & |l_1|,|l_2|,|l_1+l_2-l| < C, \\ = 0, & \text{otherwise,} \end{cases}$$

where K is the Hilbert kernel. It follows that

$$\mu \in \mathcal{M}(M^s_{p,\infty}, M^s_{1,\infty}, M^s_{p,1}), \;\; 1<p<\infty, \;\; s\ge 0.$$

In the similar way, we define a C^∞ class multilinear multiplier with the uniform decomposition.

$$\Pi(f,g) := \sum_{k\in\mathbb{Z}^n} (S_{k-1}f)(\Box_k g),$$

where $\Box_k g = \mathscr{F}^{-1}(\psi_k \mathscr{F} g), S_k = \sum_{j\le k-1} \Box_j$.

As a well-known example, the bilinear Hilbert transform proposed by Caldeón is defined as [23, 24]

$$H(f,g) = \int f(x+y)g(x-y)\frac{\mathrm{d}y}{y}$$

with the symbol $C\mathrm{sgn}(\xi-\eta)$ where C is a constant. Indeed it can be treated as the class in Theorem 3.5.

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