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ON G-ASYMPTOTIC QUASI-CONTRACTION IN METRIC SPACES WITH APPLICATION TO SOLOW GROWTH MODEL

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Abstract. In this paper, we introduce a new class of mappings called G-asymptotic quasi-contraction mappings which is a generalization of some existing contraction mappings in the literature. The fixed point theorem for this newly introduced maps is proved in a metric space equipped with a graph structure. The solution of the nonlinear differential equation of Solow growth model in Economics is established via fixed point theorem of this map. We equally provide examples to validate our results. Our results are generalization and extension of some related works in the literature.

1. Introduction

The fixed point theorem for asymptotic contraction mapping in metric spaces was established by Kirk [19] in 2003. Thereafter Jachymski and Jozwik [16] proved that the continuity of the operator in Kirk [19]'s result may not be necessary. However, the authors in [16] established the existence and uniqueness of fixed point for uniformly continuous asymptotic ψ-contractions in a metric space.

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In 1922, Banach [4] introduced a contraction mapping and proved that the fixed point of this operator in a metric space is unique. Many researchers extend and generalize this operator (see [2], [8], [10], [12], [17], [18]) for instance. One of the prominent generalized contraction map was introduced by Cirić [6] in 1971 and is called quasi-contraction mappings. In order to prove the existence and uniqueness of quasi contraction mappings in a metric space, Cirić [7] stated that the quasi contraction operator in a metric space must be orbitally complete.

In 2008, Jachymski [15] introduced graph structure to metric spaces and proved the fixed point theorem for Banach contraction mappings formulated in a graph language. Many researchers have carried out interesting works in this area (see [1], [3], [14]) and the references therein. In 2017, Fallahi [13] introduced G- asymptotic contraction mappings and established the existence and uniqueness of the fixed point of this operator in a complete metric space endowed with a graph.

In this paper, we generalize the result of [13] with quasi-contraction mapping of Cirić [6] and obtain G-asymptotic quasi-contraction mapping. The fixed point theorem for this newly introduced operator in a metric space endowed with a graph is proved.

The dominant discussion in contemporary economies may be why there has been limits to growth economics and what model could help create wealth and achieve higher sustainable growth rate, that are needed to meet national objectives. The popular Solow growth model in [22] attempts to provide a competitive factor market that reveals that output in a given period is determined by the available supplies of capital and labour, while total savings and investment assumes exogenous fraction of total income. To ensure Solows success in contemporary time, Eke et al. [9] found that a modern economy would need to improve the capital issuing capacity for higher capital formation, productivity and long-term growth. Moreover, Eke et al. [11] tested the prior savings theory towards achieving Solows long-run growth targets, that sustainable pension savings should be critical. In the model, labour is assumed to grow at a given population rate; capital grows via capital accumulation process. By extension, the model implies evolution of total income per worker, which may produce significance in the determination of how the long-run prosperity and wealth of a nation can be improved or otherwise the long-run poverty of nation, and conditions for transition of economies.

The purpose of this paper is to introduce a new class of mapping called G-asymptotic quasi-contraction mapping and apply the fixed point of this operator in a metric space endowed with a graph structure to obtain the solution of deterministic Solow growth model.

2. Preliminaries

In this section, we review some definitions and motivations that will be needed to prove our results. The following is brief description of graph theory. Details about this theory can also be found in [5] for interested readers.

Let (X, d) be a metric space, $\Delta = \Delta(X)$ is the diagonal of X. Let V be a set and $E \subset V \times V$ be a binary relation on V, the ordered pair (V, E) is called a graph G. The elements of E are called edges and are denoted by $E(G)$ while the elements of V are called vertices and it is denoted by $V(G)$. If the edges are directed then we have a directed graph. Suppose G has no parallel edges then the graph can be represented by the ordered pair $(V(G), E(G))$ and the metric space is equipped with G.

If the direction of the edges is reserved then we have graph G^{-1}. Also we have undirected graph $\bar{G}$, if the direction of the edges is ignored. In other words, we have $V(G^{-1}) = V(\bar{G}) = X$, $E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}$ and $E(\bar{G}) = E(G) \cup E(G^{-1})$.

If $x, y \in X$, then a finite sequence $\{x_i\}_{i=0}^{N}$ consisting of $N + 1$ vertices is called a path in G from x to y, whenever $x_0 = x$, $x_N = y$ and (x_{i-1}, x_i) is an edge of G for $i = 1, \cdots, N$. The graph G is called connected if there exists a path in G between each two vertices of G.

According to [20], Picard operators in metric spaces can be formulated as follows:

Definition 2.1. ([20]) Let (X, d) be a metric space. A self-map T on X is called a Picard operator if T has a unique fixed point x^* in X and $T^n x \to x^*$ for all $x \in X$.

Definition 2.2. ([15]) A mapping $T : X \to X$ is called G-contraction if T preserves edges of G that is, for all $x, y \in X, (x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ and T decreases weight of edges of G in the following way; there exists $\alpha \in [0, 1)$ and for all $x, y \in X, (x, y) \in E(G)$ implies $d(Tx, Ty) \leq \alpha d(x, y)$.

Definition 2.3. ([7]) Let $T : X \to X$ be a self-map on a metric space. For each $x \in X$ and for any positive whole number n,

$$O_T(x, n) = \{x, Tx, T^2x, T^3x, \cdots, T^n x\}$$

and

$$O_T(x, \infty) = \{x, Tx, T^2x, T^3x, \cdots\}.$$

The set $O_T(x, \infty)$ is called the orbit of T at x and the metric space X is called T-orbitally complete if every Cauchy sequence in $O_T(x, \infty)$ is convergent in X.

Definition 2.4. ([13]) Let (X, d) be a metric space endowed with a graph G. A self-map T on X is called a G-asymptotic contraction if

- (A_1) T preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;
- (A_2) there exists a sequence $\psi_n : [0, +\infty) \to [0, +\infty)$ converging uniformly to a $\psi \in \Psi$ on the range of d such that $d(T^n x, T^n y) \leq \psi_n(d(x, y))$ for all $n \geq 1$ and all $x, y \in X$ with $(x, y) \in E(G)$.

Definition 2.5. ([6]) Let (X, d) be a metric space and $T : X \to X$ be a self-map. The map T is called a quasi-contraction if there exists $0 \leq k < 1$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Recall that $\psi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and upper semi-continuous. As a consequence, we also have $\psi(t) < t$ for each $t > 0$, $\psi(0) = 0$. For example, $\psi(t) = at$ (where $a \in [0, 1)$), $\psi(t) = \frac{t}{1+t}$ and $\psi(t) = \ln(1 + t)$, $t \in R^+$. Likewise, example of the sequence $\psi_n : [0, \infty) \to [0, \infty)$ is $\psi_n(t) = \frac{t^n}{1+t^n}$ for $t > 0$ and $n \in \mathbb{N}$.

In the year 2000, Schenk- Hoppe and Schmalfub [21] applied the Banach fixed point theorem to analyze the random difference equations of stochastic Solow growth model. The stochastic Solow growth model considered by authors in [21] is presented as follows;

$$K_{t+1} = \frac{(1 - \delta(\eta^t \omega))k_t + \zeta(\eta^t \omega) f(k_t)}{1 + n(\eta^t \omega)} = h(\eta^t \omega, k_t), \tag{2.1}$$

where k_t is the capital per worker in period t, $f : R_+ \to R_+$ is a neoclassical production function, $\delta(\eta^t \omega)$, $\zeta(\eta^t \omega)$ and $n(\eta^t \omega)$ are ergodic processes that model stationary functions of the rate of depreciation, invested share of output and population growth rate, respectively.

The authors in [21] proved the following result.

Theorem 2.6. ([21]) *Assume the stochastic processes representing the rates of depreciation and population growth, and the product of saving rate and production shocks, respectively, take values* $\delta(\omega) \in [\delta_{min}, \delta_{max}] \subset [0, 1]$, $\eta(\omega) \in [\eta_{min}, \eta_{max}] \subset (-1, \infty)$ *and* $\zeta(\omega) \in [\zeta_{min}, \infty) \subset (0, \infty)$ *with the expected saving rate and production stocks,* $E\zeta < \infty$. *Assume further that* f *is non-negative, increasing, strictly concave, and continuously differentiable. Suppose,*

- (i) $\delta_{max} + \eta_{max} > 0$;
- (ii) $0 \leq \lim_{k\to\infty} f'(k) < \frac{\delta_{max}+\eta_{max}}{\zeta_{min}} < \lim_{k\to\infty} f'(k) \leq \infty$;

(iii) $E\log\frac{(1-\delta(\omega))+\zeta(\omega)f'(\bar{k})}{1+\eta(\omega)} < 0$, *where* $\bar{k} := \bar{k}(\delta_{max}, \eta_{max}, \zeta_{min})$ *is the non-trivial steady state of the deterministic Solow growth model with respective parameters and* $\bar{k}$ *is well- defined.*

Then there exists a unique non-trivial random fixed point K^* *for the random dynamical system* φ *generated by the stochastic Solow growth model* (2.1).

3. Main results

In this segment, we assume that (X, d) is a metric space endowed with a graph G. We denote by $Fix(T)$ the set of all fixed points of a self-map T on X, and we use C_T to denote the set of all points $x \in X$ such that$(T^m x, T^n x)$ is an edge of $\bar{G}$ for all $m, n \in \mathbb{N} \cup 0$. In other words,

$$C_T = \{x \in X : (T^m x, T^n x) \in E(\bar{G}),\ m, n = 0, 1 \cdots\}.$$

Now, we present the definition of G-asymptotic quasi-contraction in a metric space endowed with a graph.

Definition 3.1. Let (X, d) be a metric space endowed with a graph G. A self-map T on X is called G-asymptotic quasi-contraction if

(i) T preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;
(ii) there exists a sequence $\psi_n : [0, +\infty) \to [0, +\infty)$ such that $\limsup \psi_n(\epsilon) \leq \epsilon$ for all $\epsilon > 0$;
(iii) $d(T^n x, T^n y) \leq \psi_n(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\})$ for all $n \in N$ and $x, y \in X$ with $(x, y) \in E(G)$.

Here we state how G-asymptotic quasi-contraction mappings generalizes G-asymptotic contraction mappings in [13] and quasi-contraction mappings in [6].

Remark 3.2. (1) If $max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} = d(x, y)$ and the sequence $\psi_n : [0, +\infty) \to [0, +\infty)$ converges uniformly to ψ on the range of d then we obtain the result of Fallahi [13] for G-asymptotic contraction mapping.
(2) If $\psi(t) = kt$, $V(G) = X$, $E(G) = X \times X$ and $k \in [0, 1)$ then we obtain the definition of quasi-contraction mappings in [6].

Next, we give an example of G- asymptotic quasi-contraction mappings in a metric space endowed with a graph.

Example 3.3. Let $X = [1, 2]$ be equippied with the usual metric. Define the graph $G = (V(G), E(G))$ with $V(G) = X$ and $E(G) = \{i, j \in X \times X : i \leq j\}$. Let $T : X \to X$ be defined by $Tx = \frac{x}{1+x}$ for $x \in [1, 2]$ and $\psi_n(t) = \frac{t}{1+nt}$ for all $t \geq 0$. Then T is G-asymptotic quasi-contraction mapping.

To prove our fixed point theorem, we need the following lemma.

Lemma 3.4. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-asymptotic quasi-contraction mapping such that the functions ψ_n in Definition 3.1 are continuous on $[0, \infty)$ for sufficiently large indices n. Then $\{T^n x\}$ is a Cauchy sequence for all $x \in C_T$.*

Proof. Let $x_0 \in C_T$. Then $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \geq 0$. Suppose $\alpha = \limsup_n d(T^n x_0, T^{n+1} x_0) = 0$. Then clearly, T is G-asymptotic quasi-contraction. Using (ii) in Definition 3.1 we have,

$$\begin{aligned}\limsup_{n\to\infty} d(T^n x_0, T^{n+1} x_0) &\leq \limsup_{n\to\infty} \psi_n(d(x_0, Tx_0))\\ &\leq d(x_0, Tx_0)\\ &< \infty.\end{aligned}$$

On the contrary, we can assume $\alpha = \limsup_{n\to\infty} d(T^n x_0, T^{n+1} x_0) > 0$. Then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $d(T^{n_k} x_0, T^{n_k+1} x_0) \to \alpha$, and so by the continuity of ψ_n we obtain

$$\psi_n(\delta(O_T(x_0, n); n \in N)) \to \psi(\alpha) \leq \alpha.$$

Hence, there is a positive integer k_0 with $\psi(d(T^{n_{k_0}} x_0, T^{n_{k_0}+1} x_0)) < \alpha$ and so by (iii) of Definition 3.1 we get,

$$\begin{aligned}\alpha &= \limsup_{n\to\infty} d(T^n x_0, T^{n+1} x_0)\\ &= \limsup_{n\to\infty} d(T^n (T^{n_{k_0}} x_0, T^{n_{k_0}+1} x_0))\\ &\leq \limsup_{n\to\infty} \psi_n(d(T^{n_{k_0}} x_0, T^{n_{k_0}+1} x_0))\\ &\leq (d(T^{n_{k_0}} x_0, T^{n_{k_0}+1} x_0))\\ &\leq \psi_n(max\{d(T^{n_{k_0}-1} x_0, T^{n_{k_0}} x_0), d(T^{n_{k_0}-1} x_0, T^{n_{k_0}} x_0),\\ &\quad d(T^{n_{k_0}} x_0, T^{n_{k_0}+1} x_0), d(T^{n_{k_0}-1} x_0, T^{n_{k_0}+1} x_0), d(T^{n_{k_0}} x_0, T^{n_{k_0}} x_0))\\ &= \psi_n(\delta(O_T(x_0, n); n \in N) \to \alpha.\end{aligned}$$

This is a contradiction. Hence,

$$\limsup_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = 0.$$

Consequently, we have

$$0 \leq \liminf_{n\to\infty} d(T^n x_0, T^{n+1} x_0) \leq \limsup_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = 0.$$

Therefore,

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = 0. \tag{3.1}$$

Next we show that $\{T^n x_0\}$ is a Cauchy sequence. Assuming $\{T^n x_0\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ and decreasing sequence $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k \geq m_k \geq n$,

$$d(T^{m_k}x_0, T^{n_k}x_0) \geq \epsilon, \quad k = 1, 2, \cdots .$$

Keeping the integers n_k fixed for sufficiently large k, say $k \geq k_0$, we can assume without loss of generality that m_k is the smallest integer greater than n_k with $d(T^{m_k}x_0, T^{n_k}x_0) \geq \epsilon$ and $d(T^{m_k}x_0, T^{n_k-1}x_0) \leq \epsilon, (k \geq k_0)$.

By triangle inequality,

$$d(T^{m_k}x_0, T^{n_k}x_0) \leq d(T^{m_k}x_0, T^{n_k-1}x_0) + d(T^{n_k-1}x_0, T^{n_k}x_0).$$

Letting $k \to \infty$, then we obtain

$$d(T^{m_k}x_0, T^{n_k}x_0) \leq \epsilon.$$

Thus $d(T^{m_k}x_0, T^{n_k}x_0) \to \epsilon$ as $k \to \infty$. Also, we have

$$d(T^{m_k}x_0, T^{n_k}x_0) \leq d(T^{m_k}x_0, T^{m_k+1}x_0) + d(T^{m_k+1}x_0, T^{n_k}x_0)$$

and

$$d(T^{m_k+1}x_0, T^{n_k}x_0) \leq d(T^{m_k+1}x_0, T^{m_k}x_0) + d(T^{m_k}x_0, T^{n_k}x_0).$$

Letting $k \to \infty$, then we obtain

$$d(T^{m_k+1}x_0, T^{n_k}x_0) \leq \epsilon.$$

Thus $d(T^{m_k+1}x_0, T^{n_k}x_0) \to \epsilon$ as $k \to \infty$.

Similarly, we have $d(T^{m_k}x_0, T^{n_k+1}x_0) \to \epsilon$ as $k \to \infty$.

By (iii) of Definition 3.1 and (3.1) yields

$$\begin{aligned} d(T^{m_k+n}x_0, T^{n_k+n}x_0) &< \limsup_{n\to\infty} \psi_n(d(T^{m_k}x_0, T^{n_k}x_0)) \\ &\leq M(T^{m_k}x_0, T^{n_k}x_0) \\ &= \max\{d(T^{m_k}x_0, T^{n_k}x_0), d(T^{m_k}x_0, T^{m_k+1}x_0), \\ &\qquad d(T^{n_k}x_0, T^{n_k+1}x_0), d(T^{m_k}x_0, T^{n_k+1}x_0), \\ &\qquad d(T^{n_k}x_0, T^{m_k+1}x_0)\}. \end{aligned}$$

Taking $k \to \infty$, we obtain that $d(T^{m_k+n}x_0, T^{n_k+n}x_0) \to 0$. Thus $\{T^n x\}$ is a Cauchy sequence. □

Now we prove the main theorem for the fixed point of G-asymptotic quasi-contraction in metric space endowed with a graph G.

Theorem 3.5. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a G-asymptotic quasi-contraction mapping such that the function ψ_n in Definition 3.1 is continuous on $[0, +\infty)$ for sufficiently large indices n. If $T(X)$ is an orbitally complete subspace of X, then T has a fixed point in*

X if and only if $C_T \neq \emptyset$. Moreover, if the subgraph of G with the vertex set $Fix(T)$ is connected, then the restriction of T to C_T is a Picard operator.

Proof. Since $Fix(T) \subseteq G$, it implies that T has a fixed point, thus G is nonempty. Now let $x_0 \in G$, by Lemma 3.4, the sequence $(T^n x_0)$ is Cauchy. Since $T(X)$ is orbitally complete subspace of (X, d), then it implies that (X, d) is orbitally complete. Therefore, there exists $x \in X$ such that $\{T^n x\}$ converges to x. We need to prove that x is the fixed point of T. To prove this, recall that $x \in G$ gives $(T^n x, T^{n+1}x) \in E(G)$ for all $n \geq 0$.

Using (ii) of Definition 3.1

$$\begin{aligned} d(Tx, x) &= \lim_{n\to\infty} d(Tx, T^n x) \\ &= \limsup_{n\to\infty} d(Tx, T^n x) \\ &= \limsup_{n\to\infty} d(T^n(Tx), T^n(T^n x)) \\ &\leq \limsup_{n\to\infty} \psi_n(d(Tx, T^n x)) \\ &\leq d(Tx, T^n x). \end{aligned}$$

Applying the convergence of $\{T^n x\}$ we have $d(Tx, x) < d(Tx, x)$ which is a contradiction, unless $x = Tx$. Assume that the subgraph of G with the vertex set $Fix(T)$ is connected and $x^* \in X$ is a fixed point of T. Then there exists a path $\{x_i\}_{i=0}^N$ in G from x to x^* such that $x_1, \cdots, x_{N-1} \in Fix(T)$, that is $x_0 = x$, $x_N = x^*$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \cdots, N$. Since T is a G-asymptotic quasi-contraction for each $i = 1, \cdots, N$, it follows that

$$\begin{aligned} d(x_{i-1}, x_i) &= \limsup_{n\to\infty} d(T^n x_{i-1}, T^n x_i) \\ &= \limsup_{n\to\infty} d(T^n(T^n x_{i-1}), T^n(T^n x_i)) \\ &\leq \limsup_{n\to\infty} \psi_n(d(T^n x_{i-1}, T^n x_i)) \\ &\leq M(T^n x_{i-1}, T^n x_i) \\ &= \max\{(d(T^n x_{i-1}, T^n x_i), (d(T^n x_{i-1}, T^{n+1} x_{i-1}), \\ &\quad (d(T^n x_i, T^{n+1} x_i), (d(T^n x_{i-1}, T^{n+1} x_i), \\ &\quad (d(T^n x_i, T^{n+1} x_{i-1})\}. \end{aligned}$$

Applying the convergence of $\{T^n x_i\}$ we have $d(x_{i-1}, x_i) < d(x_{i-1}, x_i)$, a contradiction, unless $x_{i-1} = x_i$. Thus, $x = x_0 = x_1 = \cdots = x_{N-1} = x_N = x*$.

Consequently, the fixed point of T is unique and the restriction of T to G is a Picard operator. □

Remark 3.6. (1) If (X, d) is a metric space and G is the graph structure with the vertex set X, that is, $V(G) = X$ and $E(G) = X \times X$. Then condition (i) in Definition 3.1 holds trivially. Also if

$$\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} = d(x,y),$$

and $\psi_n : [0,+\infty) \to [0,+\infty)$ converges uniformly to a $\psi \in \Psi$ on the range of d. Then Theorem 3.5 generalizes the results of Kirk [19] .

(2) The result of Jachymski [15] is generalized by Theorem 3.5 because the result in [15] is in terms of G-contraction mappings in metric space endowed with a graph structure.

(3) Theorem 3.5 is more general than the result of Fallahi [13], Theorem 3.7 in the sense that our space is orbitally complete. Also, we prove our result using G-asymptotic quasi-contraction instead of G-asymptotic contraction used by Fallahi [13].

We give the consequence of Theorem 3.5. If the quasi-contraction mapping is replace with a generalized contraction mapping and the orbitally complete metric space in Theorem 3.5 is replace with a complete metric space then we have the next result.

Corollary 3.7. *Let (X,d) be a complete metric space endowed with a graph G and $T : X \to X$ be a G-asymptotic generalized contraction mapping satisfying the following conditions;*

(A_1) *T preserves the edges of G, that is, $(x,y) \in E(G)$ implies $(Tx,Ty) \in E(G)$ for all $x, y \in X$;*

(A_2) *there exists a sequence $\psi_n : [0,+\infty) \to [0,+\infty)$ such that $\limsup \psi_n(\epsilon) \le \epsilon$ for all $\epsilon > 0$;*

(A_3) *$d(T^n x, T^n y) \le \psi_n(\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\})$, for all $n \in N$ and $x, y \in X$ with $(x,y) \in E(G)$ such that the function ψ_n is continuous on $[0,+\infty)$ for sufficiently large indices n.*

Then T has a fixed point in X if and only if $C_T \neq \emptyset$. Moreover, if the subgraph of G with the vertex set Fix (T) is connected then the restriction of T to C_T is a Picard operator.

Example 3.8. Let $X = l_2$ be equipped with the usual metric $d(x,y) = [\sum_{i=1}^{\infty} |x_i - y_i|^2]^{\frac{1}{2}}$ for $x, y \in l_2$, and let B_H be the closed unit ball in l_2, that is, $B_H = \{x \in l_2 : \sum_{i=1}^{\infty} |x_i|^2 \le 1\}$. Define $T : l_2 \to l_2$ by $T(x) = (0, x_1^2, \alpha x_2, \alpha x_3, \alpha x_4, \dots)$ for $x = (x_1, x_2, x_3, \dots) \in l_2$ with α a real number in $(0,1)$. Consider the graph $G = (V(G), E(G))$ with $V(G) = B_H$ and $E(G) = B_H \times B_H$.

(1) For all $x \in B_H$, $d(Tx, 0) = |x_1|^4 + \alpha^2 \sum_{i=1}^{\infty} |x_i|^2 \le \sum_{i=1}^{\infty} |x_i|^2 \le 1$, hence $Tx \in B_H$. Thus, given $(x,y) \in E(G)$, $(Tx,Ty) \in E(G)$. Condition (i) of Definition 3.1 is satisfied.

(2) One can easily check that for all $n \in \mathbb{N}$, for all $x \in l_2$, and for all $y \in l_2$,

$$T^n x = (\underbrace{0, 0, 0, \ldots, 0}_{n}, \alpha^{n-1} x_1^2, \alpha^n x_2, \alpha^n x_3, \alpha^n x_4, \ldots)$$

and

$$\begin{aligned} d(T^n x, T^n y) &= [\alpha^{2(n-1)}|x_1^2 - y_1^2|^2 + \alpha^{2n}|x_2 - y_2|^2 + \alpha^{2n}|x_3 - y_3|^2 + \ldots]^{\frac{1}{2}} \\ &= \alpha^{n-1}[|x_1 - y_1|^2|x_1 + y_1|^2 + \alpha^2|x_2 - y_2|^2 + \alpha^2|x_3 - y_3|^2 + \ldots]^{\frac{1}{2}} \\ &\leq \alpha^{n-1}[4|x_1 - y_1|^2 + \alpha^2|x_2 - y_2|^2 + \alpha^2|x_3 - y_3|^2 + \ldots]^{\frac{1}{2}} \\ &\leq 2\alpha^{n-1}[|x_1 - y_1|^2 + |x_2 - y_2|^2 + |x_3 - y_3|^2 + \ldots]^{\frac{1}{2}} \\ &= 2\alpha^{n-1} d(x, y). \end{aligned}$$

Hence condition (ii) and (iii) of Definition 3.1 are satisfied, with $\psi_n : [0, \infty) \to [0, \infty)$ defined by $\psi_n(t) = 2\alpha^{n-1}t$ for all $n \in \mathbb{N}$ and $t \in [0, \infty)$.

One can also check that $0 = (0, 0, \cdots) \in l_2$ is the unique fixed point of T. Thus the conditions of Theorem 3.5 are satisfied.

4. Analysis of Solow growth model

In this section, rather than focus on short-run business cycle phenomena, the long-run output and development challenge is the focus of this study. This work reviews Solows productivity model by incorporating the dynamics of time in savings and investment and the stock of capital, labour and augmenting technology. The work also establish the solution of the differential equation of Solow growth model for continuous time frame. To improve national productivity overtime in the interval of two successive periods, the stock of capital will have to increase by an amount that equals to gross investment minus depreciation on the initial capital stock. The Solows model is however silent on the role of government and trade, excerpt for the unspecified sourcing of technology, it implicitly assumes a closed economy.

The long-run behavioral growth model of production is in the domain of neoclassical analysis. The model of long-run growth is

$$Y = F(K, L), \tag{4.1}$$

where Y is the production rate, K is the stock of capital and L is the labour. The technology possibilities are represented by a production function $F : R_+ \to R_+$.

Harrod-Domar model of economic growth studies long-run problems with the usual short-run tools. The model can be expressed in the form of;

$$Y = F(K, L) = \min(\frac{K}{a}, \frac{L}{b}), \tag{4.2}$$

where it takes a units of capital to produce a units of output; and b units of labour. The major parameters of the model are the saving ratio, capital - output ratio and the rate of increase of the labour. The consequence of the model is to grow unemployment or prolonged inflation.

The Cobb-Douglas' version of neoclassical growth model expresses a situation where the marginal productivity of capital rises indefinitely as the capital-labour ratio decreases. Since $a < 1$ in the equation below, the asymptotic behavior of the system is always a balanced growth at the natural rate.

$$Y = K^a L^{1-a} = F(K, L), \tag{4.3}$$

where $0 < a < 1$. The general solution of (4.3) is

$$K(t) = \left[K_0^b - \frac{s}{n}L_0^b + \frac{s}{n}L_0^b e^{nbt}\right]^{\frac{1}{b}},$$

where $b = 1 - a$ and K_0 is the initial capital stock.

The framework of Solow growth model is basically dynamic evolution of capital accumulation, labour, and technological progress. The model incorporates the dynamic link between savings and investment in stock of capital. That is, in two successive periods, the stock of capital would increase by a net investment, that is, an amount equal to gross investment minus depreciation on the initial capital stock. More formally, in continuity, the standard exposition of Solow neoclassical growth model is an extension of Cobb-Douglas production functional form by introducing technology change in continuous time $A(t)$ as a factor which multiples the production function by an increasing scale factor. So equation (4.3) becomes:

$$Y = A(t)F(K, L) = A(t)K^a L^{1-a}, \tag{4.4}$$

where $A(t)$ is the technological progression for a continuous time t. If we substitute $A(t) = e^{at}$, we obtain the nonlinear differential equation $\frac{dK}{dt} = K'$ as;

$$K' = se^{at}L_0^{1-\alpha}e^{(n(1-\alpha)+a)t}, \tag{4.5}$$

where $0 < a < 1$. The general solution of the model is

$$K(t) = \left[K_0^b - \frac{bs}{nb+g}L_0^b + \frac{bs}{nb+g}L_0^b e^{(nb+a)t}\right]^{\frac{1}{b}},$$

where $b = 1-\alpha$, $s > 0$ is the fraction of output being saved. In the long-run the capital stock increases at the relative rate $n + \frac{g}{b}$. The increase of real output is $n + \frac{ag}{b}$. This implies that real output gives more saving and investment and this compounds the rate of growth more.

5. Fixed point theorem of Solow growth model

Motivated by the result in [21], we apply the fixed point theorem of G-asymptotic quasi-contraction to find the solution of nonlinear differential equation of Solow growth model.

Consider (4.5), the nonlinear differential equation of Solow growth model for a continuous time t. The integral equation gives

$$K(t) = \int_0^t sk^a(u)L_0^{1-a}e^{(nb+g)u}du.$$

Theorem 5.1. *Let $X = C[0,T)$ and that $T : X \to X$ be an operator which is defined by*

$$(T^n x)(t) = \int_0^t sx^a(u)L_0^b e^{(nb+g)u}du \tag{5.1}$$

for all $x \in X$ and $t \in [0,T]$. The existence of a solution of the differential equation (4.5) *is equivalent to the existence of a fixed point of T in the integral equation* (5.1)*.*

Proof. Given a metric $d(x,y) = \sup_{t\in[0,T]} |x(u) - y(u)|$, we have that (X,d) is a complete metric space for all $x, y \in X$. If a graph G is defined by $G = (V(G), E(G))$ with $V(G) = X$ and $E(G) = X \times X$ then the metric is equipped with G. We prove that equation (4.4) satisfies the G-asymptotic quasi-contraction with the condition that $sL_0^b(\int_0^T e^{(nb+g)u}du) < 1$.

$$\begin{aligned}
|T^n x(t) - T^n y(t)| &= |(\int_0^T sx^a(u)L_0^{1-a}e^{(nb+g)u} - \int_0^T sy^a(u)L_0^{1-a}e^{(nb+g)u})du| \\
&= \int_0^T |sx^a(u)L_0^{1-a}e^{(nb+g)u} - sy^a(u)L_0^{1-a}e^{(nb+g)u}|du \\
&= sL_0^b \int_0^T |x^a(u)e^{(nb+g)u} - y^a(u)e^{(nb+g)u}|du \\
&= sL_0^b \int_0^T e^{(nb+g)u}|x^a(u) - y^a(u)|du \\
&= sL_0^b(\int_0^T e^{(nb+g)u})du\ d(x,y) \\
&\le sL_0^b(\int_0^T e^{(nb+g)u})du\ M(x,y) \qquad (5.2) \\
&\le sL_0^b(\int_0^T e^{(nb+g)u}du)M(x,y) < M(x,y) \\
&= \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}
\end{aligned}$$

for all $x, y \in X$.

Since operator (5.1) satisfies the G-asymptotic quasi-contraction conditions defined on a metric space endowed with a graph then the fixed point of G-asymptotic quasi-contraction mappings proved in Theorem 3.4 gives the solution of integral equation (4.5) which is also the solution to the differential equation of Solow growth model. □

Conclusion: In this research, we introduced a new class of G-asymptotic quasi-contraction mappings and proved the existence of the unique fixed point of the operator in a metric space endowed with a graph structure. The fixed point of this operator proves the applicability of our results by obtaining the solution of nonlinear differential equation of Solow growth model in economics. This result can also be established in other abstract spaces by interested researchers.

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