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COUPLED FIXED POINT THEOREMS FOR WEAKLY φ-CONTRACTIVE MIXED MONOTONE MAPPINGS IN ORDERED b-METRIC SPACES

Jamal Rezaei Roshan[1], Vahid Parvaneh[2] and Zoran Kadelburg[3]

[1]Department of Mathematics, Qaemshahr Branch
Islamic Azad University, Qaemshahr, Iran
e-mail: Jmlroshan@gmail.com

[2]Young Researchers and Elite Club, Kermanshah Branch
Islamic Azad University, Kermanshah, Iran
e-mail: vahid.parvaneh@kiau.ac.ir

[3]Faculty of Mathematics
University of Belgrade, Beograd, Serbia
e-mail: kadelbur@matf.bg.ac.rs

Abstract. In some recent papers, a method was developed of reducing coupled fixed point problems in (ordered) metric and various generalized metric spaces to the respective results for mappings with one variable. In this paper, we apply the mentioned method and obtain some coupled fixed point results for mappings satisfying φ-weak contractive conditions in ordered b-metric spaces. Examples show how these results can be used. Finally, an application to nonlinear Fredholm integral equations is presented, illustrating the effectiveness of our generalizations.

1. Introduction

Guo and Lakshmikantham introduced in [11] the notion of a coupled fixed point for a mapping in two variables. In subsequent papers several authors proved various coupled and common coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces (see, e.g., [7, 12, 22, 23,

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28]). These results were applied for investigation of solutions of differential and integral equations.

Czerwik introduced in [9] the concept of a b-metric space. Since then, several papers have dealt with fixed point theory for single-valued and multivalued operators in b-metric spaces (see, e.g., [1, 3, 8, 13, 14, 16, 20, 21, 24, 25, 26, 27]). We state a typical result of this kind.

Theorem 1.1. ([16]) *Let $(\mathcal{X}, d)$ be a complete b-metric space and let $f : \mathcal{X} \to \mathcal{X}$ satisfy the condition*

$$d(fx, fy) \leq \lambda \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2b}\right\}$$

for some $\lambda \in [0, 1/b)$ and all $x, y \in \mathcal{X}$. Then f has a unique fixed point in $\mathcal{X}$.

In some recent papers [2, 4, 5, 6, 10, 17, 18, 19], a method was developed of reducing coupled fixed point problems in (ordered) metric and various generalized metric spaces (such as partial metric spaces, G-metric spaces, spaces with the w- and c-distance, etc.) to the respective results for mappings with one variable. In this paper, we apply the mentioned method and obtain some coupled fixed point results for mappings satisfying φ-weak contractive conditions in ordered b-metric spaces. Examples show that these results can sometimes be used when the respective results in standard metric spaces cannot. Finally, an application to nonlinear Fredholm integral equations is presented, illustrating the effectiveness of our generalizations.

2. Preliminaries

Consistent with [9], the following definition will be needed in the sequel.

Definition 2.1. ([9]) Let $\mathcal{X}$ be a (nonempty) set and $b \geq 1$ be a given real number. A function $d : \mathcal{X} \times \mathcal{X} \to R^+$ is a b-metric if, for all $x, y, z \in \mathcal{X}$, the following conditions are satisfied:

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq b[d(x,y) + d(y,z)]$.

The pair $(\mathcal{X}, d)$ is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric if (and only if) $b = 1$. The following is a standard example illustrating this fact.

Example 2.2. Let $(\mathcal{X}, d)$ be a metric space, and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $b = 2^{p-1}$.

However, $(\mathcal{X}, \rho)$ is not necessarily a metric space. For example, if $\mathcal{X} = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $b = 2$, but it is not a metric on $\mathbb{R}$.

Notions as b-convergent and b-Cauchy sequences, b-continuous mappings and complete b-metric spaces are introduced in the standard way (see, e.g., [8]). Recently, Hussain et al. have presented an example of a b-metric which is not continuous(see [13, Example 2]).

We will also need the following definitions.

Definition 2.3. ([7, 11]) Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $f : \mathcal{X}^2 \to \mathcal{X}$.

(1) f is said to have the mixed monotone property if the following two conditions are satisfied:

$$(\forall x_1, x_2, y \in \mathcal{X})\ x_1 \preceq x_2 \implies f(x_1, y) \preceq f(x_2, y),$$
$$(\forall x, y_1, y_2 \in \mathcal{X})\ y_1 \preceq y_2 \implies f(x, y_1) \succeq f(x, y_2).$$

(2) A point $(x, y) \in \mathcal{X} \times \mathcal{X}$ is said to be a coupled fixed point of f if $f(x, y) = x$ and $f(y, x) = y$.

Definition 2.4. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called a partially ordered b-metric space if d is a b-metric on a partially ordered set $(\mathcal{X}, \preceq)$. The space $(\mathcal{X}, d, \preceq)$ is called regular if the following conditions hold:

(1) if a nondecreasing sequence $\{x_n\}$ b-converges to $x \in \mathcal{X}$, then $x_n \preceq x$ for all n;
(2) if a nonincreasing sequence $\{x_n\}$ b-converges to $x \in \mathcal{X}$, then $x_n \succeq x$ for all n.

The following simple lemma will be used in proving our main results.

Lemma 2.5. *Let $(\mathcal{X}, d, \preceq)$ be an ordered b-metric space.*

(a) *If a relation $\sqsubseteq$ is defined on $\mathcal{X}^2$ by*

$$X \sqsubseteq U \iff x \preceq u \wedge y \succeq v, \quad X = (x, y),\ U = (u, v) \in \mathcal{X}^2,$$

and a mapping $D : \mathcal{X}^2 \times \mathcal{X}^2 \to \mathbb{R}^+$ is given by

$$D(X, U) = d(x, u) + d(y, v), \quad X = (x, y),\ U = (u, v) \in \mathcal{X}^2,$$

then $(\mathcal{X}^2, D, \sqsubseteq)$ is an ordered b-metric space (with the same parameter b). The space $(\mathcal{X}^2, D)$ is complete iff $(\mathcal{X}, d)$ is complete.

(b) *If a mapping* $f : \mathcal{X}^2 \to \mathcal{X}$ *has the mixed monotone property, then the mapping* $F : \mathcal{X}^2 \to \mathcal{X}^2$ *given by*

$$FX = (f(x,y), f(y,x)), \quad X = (x,y) \in \mathcal{X}^2$$

is nondecreasing w.r.t. $\sqsubseteq$, *i.e.*

$$X \sqsubseteq U \implies FX \sqsubseteq FU.$$

(c) *If* f *is continuous from* $(\mathcal{X}^2, D)$ *to* $(\mathcal{X}, d)$ *then* F *is continuous in* $(\mathcal{X}^2, D)$.

3. Main results

Our first result will use a (ψ, φ)-weak contractive condition, with the parameter b entering the left-hand side of this condition. We will use the following sets of control functions:

Ψ is the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying:

(i_ψ) ψ is continuous and strictly increasing;
(ii_ψ) $\psi(0) = 0$.

Φ is the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying:

(i_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$;
(ii_φ) $\liminf_{n\to\infty} \varphi(t_n) > 0$ for any sequence $\{t_n\} \subset (0, +\infty)$ with $\liminf_{n\to\infty} t_n > 0$.

Remark 3.1. Note that it was shown by Jachymski [15] that the use of function $\psi \in \Psi$ is actually redundant, since the respective condition can always be substituted by the one involving just one function $\varphi' \in \Phi$. But, for practical purposes, this additional function ψ is still often used, and we will do so in our first result.

Theorem 3.2. *Let* $(\mathcal{X}, d, \preceq)$ *be a complete ordered b-metric space (with parameter* $b > 1$*) and let* $f : \mathcal{X}^2 \to \mathcal{X}$ *be a mixed monotone mapping for which there exist* $\psi \in \Psi$, $\varphi \in \Phi$ *and* $\varepsilon > 1$ *such that for all* $x, y, u, v \in \mathcal{X}$ *with* $x \preceq u$ *and* $y \succeq v$ *(or vice versa),*

$$\begin{aligned} &\psi(b^\varepsilon(d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)))) \\ &\le \psi(d(x,u) + d(y,v)) - \varphi(d(x,u) + d(y,v)). \end{aligned} \tag{3.1}$$

Suppose that

(a) f *is b-continuous, or*
(b) $\mathcal{X}$ *is regular.*

If there exist $x_0, y_0 \in \mathcal{X}$ *such that* $x_0 \preceq f(x_0, y_0)$ *and* $y_0 \succeq f(y_0, x_0)$ *(or vice versa), then* f *has a coupled fixed point* $(\bar{x}, \bar{y}) \in \mathcal{X}^2$.

Proof. Let D be the b-metric and $\sqsubseteq$ be the partial order on $\mathcal{X}^2$ defined in Lemma 2.5. Also, define a mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ by $F(X) = (f(x, y), f(y, x))$, $X = (x, y)$ as in Lemma 2.5. Then, $(\mathcal{X}^2, D, \sqsubseteq)$ is a complete ordered b-metric space (with the same parameter b as $\mathcal{X}$) and F is a nondecreasing mapping on it. Moreover, the contractive condition (3.1) implies that

$$\psi(b^\varepsilon D(FX, FU)) \leq \psi(D(X, U)) - \varphi(D(X, U)) \tag{3.2}$$

holds for all comparable (w.r.t. $\sqsubseteq$) $X, U \in \mathcal{X}^2$. Since φ has non-negative values and ψ is strictly increasing, (3.2) implies that

$$D(FX, FU) \leq \frac{1}{b^\varepsilon} D(X, U),$$

where $0 < 1/b^\varepsilon < 1/b$, for all comparable $X, U \in \mathcal{X}^2$. We will prove in the next lemma that under these circumstances, it follows that F has a fixed point $\overline{X} = (\bar{x}, \bar{y}) \in \mathcal{X}^2$ which is then obviously a coupled fixed point of f. □

The following lemma is an "ordered variant" of the basic result of Czerwik [9].

Lemma 3.3. *Let $(\mathcal{X}, d, \preceq)$ be a partially ordered b-metric space and let f be a self-mapping on $\mathcal{X}$. Assume that there exists $\lambda \in [0, \frac{1}{b})$ such that*

$$d(fx, fy) \leq \lambda d(x, y) \tag{3.3}$$

for all comparable $x, y \in \mathcal{X}$. If the following conditions hold:

(i) *f is nondecreasing;*
(ii) *there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq fx_0$;*
(iii) *f is continuous and $(\mathcal{X}, d)$ is complete, or*
(iii') *$(\mathcal{X}, d, \preceq)$ is regular and $f(\mathcal{X})$ is complete.*

Then f has a fixed point in $\mathcal{X}$.

Proof. Starting with the given x_0, define an iterative sequence by

$$x_{n+1} = fx_n \text{ for } n = 0, 1, 2, \cdots .$$

It can be proved by induction that $x_n \preceq x_{n+1}$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f. Hence, we will suppose that $x_n \neq x_{n+1}$ for all n. It can be proved in a standard way (see, e.g., [16, Lemma 3.1]) that $\{x_n\}$ is a Cauchy sequence.

Suppose first that (iii) holds. Then there exists

$$\lim_{n\to\infty} x_n = z \in \mathcal{X}.$$

Further, since f is continuous we get that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fz.$$

In the case (iii'), it follows that

$$\lim_{n\to\infty} fx_n = fu$$

for some $u \in \mathcal{X}$. Because of regularity, we have $x_n \preceq u$. Applying (3.3) with $x = x_n$, $y = u$, we have

$$d(fx_n, fu) \leq \lambda d(x_n, u) \to 0 \quad (n \to \infty).$$

It follows that $d(fx_n, fu) \to 0$ when $n \to \infty$, that is, $fx_n \to fu$. Hence, f has a fixed point $u \in X$. □

The following example supports our result.

Example 3.4. Let $\mathcal{X} = \mathbb{R}$ be endowed with the usual ordering $\leq$ and the b-metric $d(x,y) = (x-y)^2$ $(b=2)$. Let $f : \mathcal{X}^2 \to \mathcal{X}$ be defined by

$$f(x,y) = \frac{x-2y}{9}, \quad (x,y) \in \mathcal{X}^2.$$

We define $\psi, \varphi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \ln(t+1) \text{ and } \varphi(t) = \ln\left(\frac{t+1}{ct+1}\right),$$

where $c = \frac{40}{81}$ and take $\varepsilon = 2$. It is easy to check that $\psi \in \Psi$ and $\varphi \in \Phi$. Also, f is mixed monotone and satisfies the condition (3.1) with $\varepsilon = 1$. Indeed, for all $x, y, u, v \in \mathcal{X}$ with $x \leq u$ and $y \geq v$ (or vice versa), we have

$$\begin{aligned}
&\psi(2^2(d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)))) \\
&= \ln\left(4\left(\frac{x-2y}{9} - \frac{u-2v}{9}\right)^2 + 4\left(\frac{y-2x}{9} - \frac{v-2u}{9}\right)^2 + 1\right) \\
&= \ln\left(\frac{4}{81}((x-u) + 2(v-y))^2 + \frac{4}{81}((y-v) + 2(u-x))^2 + 1\right) \\
&\leq \ln\left(\frac{8}{81}((x-u)^2 + 4(v-y)^2) + \frac{8}{81}((y-v)^2 + 4(u-x)^2) + 1\right) \\
&= \ln\left(\frac{40}{81}((x-u)^2 + (y-v)^2) + 1\right) \\
&= \ln(c(d(x,u) + d(y,v)) + 1) \\
&= \ln(d(x,u) + d(y,v) + 1) - \ln\left(\frac{d(x,u) + d(y,v) + 1}{c(d(x,u) + d(y,v)) + 1}\right) \\
&= \psi(d(x,u) + d(y,v)) - \varphi(d(x,u) + d(y,v)).
\end{aligned}$$

Hence, by Theorem 3.2 we obtain that f has a coupled fixed point (which is $(0,0)$).

Corollary 3.5. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered b-metric space (with parameter $b > 1$) and let $f : \mathcal{X}^2 \to \mathcal{X}$ be a mixed monotone mapping for which there exist $\varphi \in \Phi$ and $\varepsilon > 1$ such that for all $x, y, u, v \in \mathcal{X}$ with $x \preceq u$ and $y \succeq v$ (or vice versa),*

$$d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)) \leq \frac{d(x,u) + d(y,v)}{b^\varepsilon} - \frac{\varphi(d(x,u) + d(y,v))}{b^\varepsilon}.$$

Suppose that

(a) *f is b-continuous, or*
(b) *$\mathcal{X}$ is regular.*

If there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$ (or vice versa), then f has a coupled fixed point $(\bar{x}, \bar{y}) \in \mathcal{X}^2$.

We provide now an additional condition to ensure that the coupled fixed point in Theorem 3.2 is unique, as was done in several earlier papers.

Theorem 3.6. *In addition to the hypotheses of Theorem 3.2, suppose that for all (x, y) and $(x^*, y^*) \in \mathcal{X}^2$, there exists $(u, v) \in \mathcal{X}^2$, such that $(f(u,v), f(v,u))$ is $\sqsubseteq$-comparable with both $(f(x,y), f(y,x))$ and $(f(x^*,y^*), f(y^*,x^*),)$. Then, f has a unique coupled fixed point. In particular, if $(\bar{x}, \bar{y})$ is the coupled fixed point of f, then $\bar{x} = \bar{y}$.*

Proof. We shall use the notation as in the proof of Theorem 3.2 (i.e., Lemma 2.5). It was proved in Theorem 3.2 that the set of coupled fixed points of f, i.e., the set of fixed points of F in $\mathcal{X}^2$ is nonempty. We shall show that if X and X^* are coupled fixed points of F, that is, $X = FX$ and $X^* = FX^*$, then $X = X^*$.

Choose an element $U = (u, v) \in \mathcal{X}^2$ such that $FU = (f(u,v), f(v,u))$ is comparable with FX and FX^*. Let $U_0 = U$ and inductively define a sequence $\{U_n\}$ such that $U_{n+1} = FU_n$. Since $X = FX$ and $U_1 = FU_0$ are $\sqsubseteq$-comparable, we may assume that $X \sqsubseteq U_1$. Using the mathematical induction, it is easy to prove that $X \sqsubseteq U_n$, for all $n \geq 0$.

Applying (3.1), one obtains that

$$\begin{aligned}\psi(s^\varepsilon D(X, U_{n+1})) = \psi(s^\varepsilon D(FX, FU_n)) &\leq \psi(D(X, U_n)) - \varphi(D(X, U_n)) \\ &\leq \psi(D(X, U_n)).\end{aligned}$$

From the properties of ψ, we deduce that the sequence $\{D(X, U_n)\}$ is nonincreasing. Hence, if we proceed as in Theorem 3.2, we can show that

$$\lim_{n \to \infty} D(X, U_n) = 0,$$

that is, $\{U_n\}$ is b-convergent to X. Similarly, we can show that $\{U_n\}$ is b-convergent to X^*. Since the limit is unique, it follows that $X = X^*$. $\square$

In order to formulate our next result, we will denote by Φ_b the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ such that

(i_{φ_b}) $\varphi(0) = 0$ and $\varphi(t) > (1 - \frac{1}{b^{1+\varepsilon}})t$ for $t > 0$, where $\varepsilon > 0$ is fixed.

Note that now the parameter b enters the conditions for the control function φ. Taking into account Remark 3.1, we will not use the control function ψ.

Theorem 3.7. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered b-metric space with the parameter $b > 1$ and let $f : \mathcal{X}^2 \to \mathcal{X}$ be a mixed monotone mapping such that*

$$d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)) \le m_b(x,y,u,v) - \varphi(m_b(x,y,u,v)) \quad (3.4)$$

holds for some $\varphi \in \Phi_b$ and all $x, y, u, v \in \mathcal{X}$ such that $x \preceq u$ and $y \succeq v$ (or vice versa), where

$$m_b(x,y,u,v) = \max\Big\{ d(x,u) + d(y,v), d(x, f(x,y)) + d(y, f(y,x)), d(u, f(u,v)) + d(v, f(v,u)), \frac{d(x, f(u,v)) + d(y, f(v,u)) + d(u, f(x,y)) + d(v, f(y,x))}{2b} \Big\}.$$

If there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$ (or vice versa), then f has a coupled fixed point $(\bar{x}, \bar{y}) \in \mathcal{X}^2$.

Proof. Let D be the b-metric and $\sqsubseteq$ be the partial order on $\mathcal{X}^2$ defined in Lemma 2.5. Also, define a mapping $F : \mathcal{X}^2 \to \mathcal{X}^2$ by $F(X) = (f(x,y), f(y,x))$, $X = (x,y)$ as in Lemma 2.5. Then, $(\mathcal{X}^2, D, \sqsubseteq)$ is a complete ordered b-metric space (with the same parameter b as $\mathcal{X}$) and F is a nondecreasing mapping on it. Moreover, the condition (3.4) implies that

$$D(FX, FU) \le M_b(X,U) - \varphi(M_b(X,U)) \quad (3.5)$$

holds for all comparable $X, U \in \mathcal{X}^2$, where

$$M_b(X,U) = \max\left\{ D(X,U), D(X,FX), D(U,FU), \frac{D(X,FU) + D(U,FX)}{2b} \right\}.$$

Finally, there exists $X_0 = (x_0, y_0)$, comparable with FX_0.

Now (3.5) and (i_{φ_b}) imply that

$$D(FX, FU) \le M_b(X,U) - \left(1 - \frac{1}{b^{1+\varepsilon}}\right) M_b(X,U) = \frac{1}{b^{1+\varepsilon}} M_b(X,U),$$

where $\lambda = 1/b^{1+\varepsilon} < 1/b$. Using Theorem 1.1 (adapted to the "ordered" situation in a standard way), it follows that F has a fixed point. □

Remark 3.8. In the case $\varepsilon = 1$ (i.e., if $\varphi(t) > (1 - \frac{1}{b^2})t$ holds for $t > 0$), the same conclusion can be obtained using the results from the paper [3].

The uniqueness of a coupled fixed point can be obtained using additional assumptions, similarly as in Theorem 3.6.

By a simple modification of Theorem 3.7 (see also the previous remark), the following result can be deduced.

Corollary 3.9. *Let $(\mathcal{X}, d, \preceq)$ be a complete ordered b-metric space with the parameter $b > 1$ and let $f : \mathcal{X}^2 \to \mathcal{X}$ be a mixed monotone mapping such that*

$$\begin{aligned} &d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)) \\ &\le d(x,u) + d(y,v) - \varphi(d(x,u) + d(y,v)) \end{aligned} \tag{3.6}$$

holds for some $\varphi \in \Phi_b$ and all $x, y, u, v \in \mathcal{X}$ such that $x \preceq u$ and $y \succeq v$ (or vice versa). If there exist $x_0, y_0 \in \mathcal{X}$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$ (or vice versa), then f has a coupled fixed point $(\bar{x}, \bar{y}) \in \mathcal{X}^2$.

Example 3.10. Let $\mathcal{X} = \mathbb{R}$ be equipped with standard order and the b-metric given by $d(x,y) = (x-y)^2$ $(b = 2)$. Consider the mapping $f : \mathcal{X}^2 \to \mathcal{X}$ and the function $\varphi \in \Phi_b$ defined by

$$f(x,y) = \frac{x - 4y}{12}, \qquad \varphi(t) = \frac{55}{72}\,t$$

(note that $\frac{55}{72} > \frac{3}{4} = 1 - \frac{1}{b^2}$). Then f obviously has the mixed monotone property. Let us check that the condition (3.6) holds true for all $x, y, u, v \in \mathcal{X}$. Indeed,

$$\begin{aligned} &d(f(x,y), f(u,v)) + d(f(y,x), f(v,u)) \\ &= \left(\frac{x-4y}{12} - \frac{u-4v}{12}\right)^2 + \left(\frac{y-4x}{12} - \frac{v-4u}{12}\right)^2 \\ &= \frac{1}{144}[((x-u) + 4(v-y))^2 + ((y-v) + 4(u-x)^2)] \\ &\le \frac{1}{72}[(x-u)^2 + 16(v-y)^2 + (y-v)^2 + 16(u-x)^2] \\ &= \frac{17}{72}[(x-u)^2 + (y-v)^2] \\ &= d(x,u) + d(y,v) - \varphi(d(x,u) + d(y,v)). \end{aligned}$$

Finally, there are obviously $x_0, y_0 \in \mathcal{X}$ such that $x_0 \le f(x_0, y_0)$ and $y_0 \ge f(y_0, x_0)$. Thus, all the assumptions of Corollary 3.9 are fulfilled and we conclude that the mapping f has a coupled fixed point (which is $(0,0)$).

Consider now the same example, but with the standard metric $\rho(x,y) = |x-y|$ on $\mathcal{X} = \mathbb{R}$. The respective contractive condition

$$\rho(f(x,y), f(u,v)) + \rho(f(y,x), f(v,u)) \leq \rho(x,u) + \rho(y,v) - \varphi(\rho(x,u) + \rho(y,v))$$

does not hold for all $x, y, u, v \in \mathcal{X}$ such that $x \geq u$ and $y \leq v$. Indeed, for $x = 1$, $y = u = v = 0$ it reduces to

$$\begin{aligned}\rho(f(1,0), f(0,0)) + \rho(f(0,1), f(0,0)) &= \rho(\tfrac{1}{12}, 0) + \rho(-\tfrac{4}{12}, 0) = \tfrac{5}{12} \\ &\leq \tfrac{17}{72} = \tfrac{17}{72}(\rho(1,0) + \rho(0,0)),\end{aligned}$$

which is not true. We conclude that using a b-metric instead of the standard one, one has more possibilities for choosing a control function in order to get a fixed point result.

4. An application to integral equations

As an application of the coupled fixed point theorems established in the previous section, we study the existence and uniqueness of solutions to a Fredholm nonlinear integral equation.

In order to compare our results to the ones in [5, 23], we shall consider the same integral equation, that is,

$$x(t) = \int_a^b (K_1(t,s) + K_2(t,s))(f(s, x(s)) + g(s, x(s)))\, ds + h(t), \tag{4.1}$$

where, $t \in I = [a, b]$.

Let Θ denote the set of all functions $\theta : [0, \infty) \to [0, \infty)$ satisfying

(i_θ) θ is non-decreasing and $(\theta(r))^p \leq \theta(r^p)$, for all $p \geq 1$ and $r \in [0, +\infty)$.
(ii_θ) There exists $\varphi \in \Phi$ such that $\theta(r) = \frac{r}{2} - \varphi\left(\frac{r}{2}\right)$, for all $r \in [0, \infty)$.

As in [23], Θ is nonempty, as $\theta_1(r) = kr$ with $0 \leq 2k < 1$ and $\theta_2(r) = \frac{r^2}{2(r+1)}$ are elements of Θ.

As in [23], we assume that the functions K_1, K_2, f and g fulfill the following conditions:

(i) $K_1(t,s) \geq 0$ and $K_2(t,s) \leq 0$, for all $t, s \in I$;
(ii) There exist positive numbers λ, μ and $\theta \in \Theta$, such that for all $x, y \in \mathbb{R}$, with $x \geq y$, the following conditions hold:

$$0 \leq f(t,x) - f(t,y) \leq \lambda\theta(x-y)$$

and

$$-\mu\theta(x-y) \leq g(t,x) - g(t,y) \leq 0;$$

(iii)

$$(\lambda^p+\mu^p)\cdot\sup_{t\in I}\left[\left(\int_a^b K_1(t,s)\,ds\right)^p+\left(\int_a^b -K_2(t,s)\,ds\right)^p\right] \\ \leq \frac{1}{2^{4p-4}} \quad \text{for some} \quad p>1. \tag{4.2}$$

We will consider on $\mathcal{X}=C(I,\mathbb{R})$ the natural partial order relation, that is, for all $x,y\in C(I,\mathbb{R})$,

$$x\preceq y \;\Leftrightarrow\; x(t)\leq y(t), \quad \forall t\in I.$$

Obviously, for any $(x,y)\in\mathcal{X}^2$, the functions $\max\{x,y\}$ and $\min\{x,y\}$ are the upper and lower bounds of x,y, respectively. Therefore, for every $(x,y),(u,v)\in\mathcal{X}^2$, there exists the element $(\max\{x,u\},\min\{y,v\})$ which is comparable to (x,y) and (u,v).

Definition 4.1. ([23]) A pair $(\alpha,\beta)\in\mathcal{X}^2$ with $\mathcal{X}=C(I,R)$ is called a coupled lower-upper solution of the equation (4.1) if, for all $t\in I$,

$$\alpha(t)\leq\int_a^b K_1(t,s)[f(s,\alpha(s))+g(s,\beta(s))]\,ds \\ +\int_a^b K_2(t,s)[f(s,\beta(s))+g(s,\alpha(s))\,ds+h(t)$$

and

$$\beta(t)\geq\int_a^b K_1(t,s)[f(s,\beta(s))+g(s,\alpha(s))]\,ds \\ +\int_a^b K_2(t,s)[f(s,\alpha(s))+g(s,\beta(s))]\,ds+h(t).$$

Theorem 4.2. *Let $K_1,K_2\in C(I\times I,\mathbb{R})$ and $h\in C(I,\mathbb{R})$. Suppose that there exists a coupled lower-upper solution (α,β) of* (4.1) *and that conditions* (i)–(iii) *are fulfilled. Then the integral equation* (4.1) *has a unique solution in* $C(I,\mathbb{R})$.

Proof. It is well known that $\mathcal{X}$ is a complete metric space with respect to the max metric

$$\rho(x,y)=\max_{t\in I}|x(t)-y(t)|\,,\;\; x,y\in C(I,\mathbb{R}).$$

Now, for $p>1$ define

$$d(x,y)=\rho(x,y)^p=\left(\max_{t\in I}|x(t)-y(t)|\right)^p=\max_{t\in I}|x(t)-y(t)|^p\,,\;\; x,y\in C(I,\mathbb{R}).$$

It is easy to see that $(\mathcal{X}, d)$ is a complete b-metric space with $b = 2^{p-1} > 1$ (see Example 2.2).

Define now the mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned} F(x,y)(t) = &\int_a^b K_1(t,s)\,[f(s,x(s)) + g(s,y(s))]\;ds \\ &+ \int_a^b K_2(t,s)\,[f(s,y(s)) + g(s,x(s))]\;ds + h(t), \text{ for all } t \in I. \end{aligned}$$

It is not difficult to prove, as in [23], that F has the mixed monotone property. Now for all $x, y, u, v \in \mathcal{X}$ with $x \succeq u$ and $y \preceq v$, we have

$$\rho(F(x,y), F(u,v)) = \max_{t\in I} |F(x,y)(t) - F(u,v)(t)|^p .$$

Let us first evaluate the expression in the right-hand side:

$$\begin{aligned} &F(x,y)(t) - F(u,v)(t) \\ &= \int_a^b K_1(t,s)\,[f(s,x(s)) + g(s,y(s))]\;ds \\ &\quad + \int_a^b K_2(t,s)\,[f(s,y(s)) + g(s,x(s))]\;ds \\ &\quad - \int_a^b K_1(t,s)\,[f(s,u(s)) + g(s,v(s))]\;ds \\ &\quad - \int_a^b K_2(t,s)\,[f(s,v(s)) + g(s,u(s))]\;ds \\ &= \int_a^b K_1(t,s)\,[f(s,x(s)) - f(s,u(s)) + g(s,y(s)) - g(s,v(s))]\;ds \\ &\quad + \int_a^b K_2(t,s)\,[f(s,y(s)) - f(s,v(s)) + g(s,x(s)) - g(s,u(s))]\;ds \\ &= \int_a^b K_1(t,s)\,[(f(s,x(s)) - f(s,u(s))) - (g(s,v(s)) - g(s,y(s)))]\;ds \\ &\quad - \int_a^b K_2(t,s)\,[(f(s,v(s)) - f(s,y(s))) - (g(s,x(s)) - g(s,u(s)))]\;ds \\ &\le \int_a^b K_1(t,s)\,[\lambda\theta(x(s) - u(s)) + \mu\theta(v(s) - y(s))]\;ds \\ &\quad - \int_a^b K_2(t,s)\,[\lambda\theta(v(s) - y(s)) + \mu\theta(x(s) - u(s))]\;ds. \end{aligned} \tag{4.3}$$

Since the function θ is non-decreasing and $x \succeq u$ and $y \preceq v$, we have

$$\theta(x(s) - u(s)) \le \theta(\max_{t\in I} |x(t) - u(t)|) = \theta(\rho(x,u))$$

and

$$\theta(v(s) - y(s)) \leq \theta(\max_{t\in I} |v(t) - y(t)|) = \theta(\rho(v, y)).$$

Hence, by (4.3), in view of the fact that $K_2(t, s) \leq 0$, we obtain

$$\begin{aligned} |F(x,y)(t) - F(u,v)(t)| \leq &\int_a^b K_1(t,s)\,[\lambda\theta(d(x,u)) + \mu\theta(\rho(v,y))]\; ds \\ &- \int_a^b K_2(t,s)\,[\lambda\theta(d(v,y) + \mu\theta(\rho(x,u))]\; ds, \end{aligned} \tag{4.4}$$

as all quantities in the right-hand side of (4.4) are non-negative.

Now, from (4.3) we have

$$\begin{aligned} &|F(x,y)(t) - F(u,v)(t)|^p \\ &\leq \begin{pmatrix} \int_a^b K_1(t,s)\,[\lambda\theta(\rho(x,u)) + \mu\theta(\rho(v,y))]\; ds \\ -\int_a^b K_2(t,s)\,[\lambda\theta(\rho(v,y)) + \mu\theta(\rho(x,u))]\; ds \end{pmatrix}^p \\ &\leq 2^{p-1}\begin{pmatrix} \left(\int_a^b K_1(t,s)\,ds\right)^p (\lambda\theta(\rho(x,u)) + \mu\theta(\rho(v,y)))^p \\ + \left(-\int_a^b K_2(t,s)\,ds\right)^p (\lambda\theta(\rho(v,y) + \mu\theta(\rho(x,u)))^p \end{pmatrix} \\ &\leq 2^{p-1}\begin{pmatrix} \left(\int_a^b K_1(t,s)\,ds\right)^p 2^{p-1}\,(\lambda^p\theta(\rho(x,u))^p + \mu^p\theta(\rho(v,y))^p) \\ + \left(-\int_a^b K_2(t,s)\,ds\right)^p 2^{p-1}\,(\lambda^p\theta(\rho(v,y))^p + \mu^p\theta(\rho(x,u))^p) \end{pmatrix} \\ &\leq 2^{p-1}\begin{pmatrix} \left(\int_a^b K_1(t,s)\,ds\right)^p 2^{p-1}\,(\lambda^p\theta(d(x,u)) + \mu^p\theta(d(v,y))) \\ + \left(-\int_a^b K_2(t,s)\,ds\right)^p 2^{p-1}\,(\lambda^p\theta(d(v,y)) + \mu^p\theta(d(x,u)) \end{pmatrix} \\ &\leq 2^{2p-2}\begin{bmatrix} \left(\lambda^p\left(\int_a^b K_1(t,s)\,ds\right)^p + \mu^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta(d(x,u)) \\ + \left(\mu^p\left(\int_a^b K_1(t,s)\,ds\right)^p + \lambda^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\right)\theta(d(v,y)) \end{bmatrix}. \end{aligned}$$

Hence, we have

$$|F(x,y)(t) - F(u,v)(t)|^p \leq 2^{2p-2}\begin{bmatrix} \begin{pmatrix} \lambda^p\left(\int_a^b K_1(t,s)\,ds\right)^p \\ +\mu^p\left(-\int_a^b K_2(t,s)\,ds\right)^p \end{pmatrix}\theta(d(x,u)) \\ + \begin{pmatrix} \mu^p\left(\int_a^b K_1(t,s)\,ds\right)^p \\ +\lambda^p\left(-\int_a^b K_2(t,s)\,ds\right)^p \end{pmatrix}\theta(d(v,y)) \end{bmatrix}.$$

Similarly, one can obtain

$$|F(y,x)(t)-F(v,u)(t)|^p \le 2^{2p-2}\left[\begin{array}{c}\left(\begin{array}{c}\lambda^p\left(\int_a^b K_1(t,s)\,ds\right)^p\\ +\mu^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\end{array}\right)\theta(d(v,y))\\ +\left(\begin{array}{c}\mu^p\left(\int_a^b K_1(t,s)\,ds\right)^p\\ +\lambda^p\left(-\int_a^b K_2(t,s)\,ds\right)^p\end{array}\right)\theta(d(x,u))\end{array}\right].$$

By summing up the above two inequalities and then taking the maximum with respect to t and using (4.2), we get,

$$\begin{aligned}
&d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\\
&\le 2^{2p-2}(\lambda^p+\mu^p)\max_{t\in I}\left[\left(\int_a^b K_1(t,s)\,ds\right)^p+\left(\int_a^b -K_2(t,s)\,ds\right)^p\right]\\
&\qquad\times(\theta(d(v,y))+\theta(d(x,u)))\\
&\le \frac{2^{2p-2}}{2^{4p-4}}(\theta(d(v,y))+\theta(d(x,u)))\\
&=\frac{1}{2^{2p-2}}(\theta(d(v,y))+\theta(d(x,u))).
\end{aligned}$$

Now, since θ is non-decreasing, we have

$$\theta(d(x,u)) \le \theta(d(x,u)+\rho(v,y)) \text{ and } \theta(d(v,y)) \le \theta(d(x,u)+d(v,y))$$

and so,

$$\theta(d(v,y))+\theta(d(x,u)) \le d(v,y)+d(x,u)-\varphi\left(d(v,y)+d(x,u)\right),$$

by the definition of θ. Thus we finally get

$$\begin{aligned}
&d(F(x,y),F(u,v))+d(F(y,x),F(v,u))\\
&\le \frac{1}{2^{2p-2}}(d(v,y)+d(x,u))-\frac{1}{2^{2p-2}}\varphi\left(d(v,y)+d(x,u)\right),
\end{aligned}$$

which is just the contractive condition of Corollary 3.5 with $\varepsilon=2$.

Now, let $(\alpha,\beta)\in\mathcal{X}^2$ be a coupled lower-upper solution of (4.1). Then we have

$$\alpha(t)\preceq F(\alpha,\beta)(t) \text{ and } \beta(t)\succeq F(\beta,\alpha)(t),$$

for all $t\in I$, which show that the hypotheses of Theorem 3.6 are satisfied. This proves that F has a unique coupled fixed point $(\overline{x},\overline{x})$ in $\mathcal{X}^2$. This means that $\overline{x}=F(\overline{x},\overline{x})$, and therefore $\overline{x}\in C(I,\mathbb{R})$ is a unique solution of the integral equation (4.1). □

Remark 4.3. Since a b-metric is a metric, when $b = 1$, our results can be viewed as generalizations and extensions of the corresponding results in the literature.

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