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EXISTENCE RESULTS FOR KIRCHHOFF TYPE PROBLEMS WITH CRITICAL POTENTIAL AND CRITICAL CAFFARELI-KOHN-NIREMBERG EXPONENT

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Abstract. In this paper, we establish the existence of at least two distinct solutions to a p-Laplacian problem involving critical exponents and singular cylindrical potential, by using the Nehari manifold.

1. Introduction

The original one-dimensional Kirchhoff equation was introduced by Kirchhoff [8] in 1883. His model takes into account the changes in length of the strings produced by transverse vibrations.

In recent years, the existence and multiplicity of solutions to the nonlocal problem

$$\begin{cases} -\left(m + n \int_\Omega |\nabla u|^2 \, dx\right) \Delta u = & g(x;u) \ \text{ in } \Omega, \\ u = 0, & \text{on } \ \partial\Omega \end{cases} \tag{1.1}$$

has been studied by various researchers and many interesting and important results can be found. For instance, positive solutions could be obtained in [1, 4, 9, 10]. Especially, Chen et al. [3] discussed a Kirchhoff type problem when

$$g(x;u) = f(x) u^{p-2} u + \lambda g(x) |u|^{q-2} u,$$

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where $1 < q < 2 < p < 2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$, $f(x)$ and $g(x)$ with some proper conditions are sign-changing weight functions. And they have obtained the existence of two positive solutions if $p > 4$, $0 < \lambda < \lambda_0(m)$.

Researchers, such as Mao and Zhang [12], Mao and Luan [11], found sign-changing solutions. As for infinitely many solutions, we refer readers to [7, 14]. He and Zou [6] considered the class of Kirchhoff type problem when $g(x;u) = \lambda f(x;u)$ with some conditions and proved a sequence of almost everywhere positive weak solutions tending to zero in $L^\infty(\Omega)$.

In the case of a bounded domain of $\mathbb{R}^N$ with $N \geq 3$, Tarantello [7] proved, under a suitable condition on f, the existence of at least two solutions to (1.1) for $m = 0$, $n = 1$ and $g(x;u) = |u|^{\frac{4}{N-2}} u + f$.

In this paper, we consider the multiplicity results of positive solutions of the following Kirchhoff problem

$$\begin{cases} Lu = |x|^{-2_* b} f(x) |u|^{2_*-2} u + \mu g(x) |u|^{q-2} u \text{ in } \mathbb{R}^3, \quad x \neq 0, \\ u \in H_0^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

where $L := -m + n \left(\int_{\mathbb{R}^3} |x|^{-2a} |\nabla u|^2 dx \right) div \left(|x|^{-2a} \nabla u \right)$, $m > 0$, $n > 0$, $\mu \neq 0$ is a real parameter, $1 < q < 2$, $-\infty < a < \frac{1}{2}$, $a \leq b < a + (1/4)$, $2_* = \frac{6}{1+2(b-a)}$ is the critical Caffareli-Kohn-Niremberg exponent and f, g are continuous and sign-changing functions which we will specify later.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorems 3.1 and 3.2.

2. Preliminaries

Before formulating our results, we give some definitions and notation.

The space $\mathcal{H} = \mathcal{H}_0^1(\mathbb{R}^3)$ is equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |x|^{-2a} |\nabla u|^2 dx \right)^{1/2}.$$

Let S_μ be the best Sobolev constant. Then,

$$S_a = \inf_{u \in \mathcal{H}_0^1(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |x|^{-2a} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |x|^{-2_* b} f |u|^{2_*} dx \right)^{\frac{2}{2_*}}}. \tag{2.1}$$

Since our approach is variational, we define the functional J on $\mathcal{H}_0^1\left(\mathbb{R}^3\right)$ by

$$\begin{aligned} J\left(u\right) &= \left(1/2\right) m \left\|u\right\|^2 + \left(1/4\right) n \left\|u\right\|^4 \\ &\quad - \left(1/2_*\right) \int_{\mathbb{R}^3} \left|x\right|^{-2_* b} f \left|u\right|^{2_*} dx - \left(\mu/q\right) \int_{\mathbb{R}^3} g \left|u\right|^q dx. \end{aligned} \tag{2.2}$$

A point $u \in \mathcal{H}_0^1\left(\mathbb{R}^3\right)$ is a weak solution of the equation (1.1) if it is the critical point of the functional J. Generally speaking, a function u is called a solution of (1.1) if $u \in \mathcal{H}_0^1\left(\mathbb{R}^3\right)$ and for all $v \in \mathcal{H}_0^1\left(\mathbb{R}^3\right)$ it holds

$$\begin{aligned} &\left(m + n \left\|u\right\|^2\right) \int_{\mathbb{R}^3} \left(\left|x\right|^{-2a} \nabla u \nabla v\right) dx \\ &\quad - \int_{\mathbb{R}^3} \left|x\right|^{-2_* b} f \left|u\right|^{2_*-2} uvdx - \mu \int_{\mathbb{R}^3} g \left|u\right|^{q-2} uvdx = 0. \end{aligned}$$

Throughout this work, we consider the following assumptions:

(F) f is a continuous function satisfies $f(0) = \max\limits_{x\in\mathbb{R}^3} f(x) > 0$,
$f(x) = f(0) + o(x^\beta)$ with $\beta > \frac{3(1-2a)}{1+2(b-a)}$,

(G) h is a continuous function and there exists g_0 and ϱ_0 positive such that $g(x) \geq g_0$ for all $x \in B(0, 2\varrho_0)$, where $B(a, r)$ denotes the ball centered at a with radius r.

In our work, we research the critical points as the minimizers of the energy functional associated to the problem (1.1) on the constraint defined by the Nehari manifold, which are solutions of our problem.

Let μ_0 be A real number such that

$$\mu_0 = \frac{2n\left(A+B\right)}{\left(2_* - q\right)\left|g^+\right|_\infty} X_0^{\frac{2-q}{2_*-q}} - \frac{2nA'}{\left(2_* - q\right)\left|g^+\right|_\infty} X_0^{\frac{2_*-q}{2_*-q}},$$

where

$$A = \frac{2m}{n},\ B = \frac{\left(q-2\right)m}{\left(4-q\right)n},\ A' = \left(\frac{2_* - q}{4-q}\right)\frac{\left(S_a\right)^{\frac{2}{2_*}}}{n},\ X_0 = \left[\frac{\left(2_* - q\right)A}{\left(2-q\right)\left(A'+B\right)}\right]^{\frac{1}{2-2_*}}.$$

Definition 2.1. ([15]) Let $c \in \mathbb{R}$, E a Banach space and $I \in C^1\left(E, \mathbb{R}\right)$.

(1) $\{u_n\}_n$ is a Palais-Smale sequence at level c (in short $\left(PS\right)_c$) in E for I if

$$I\left(u_n\right) = c + o_n\left(1\right) \text{ and } I'\left(u_n\right) = o_n\left(1\right),$$

where $o_n\left(1\right)$ tends to 0 as n goes at infinity.

(2) We say that I satisfies the $\left(PS\right)_c$ condition if any $\left(PS\right)_c$ sequence in E for I has a convergent subsequence.

Lemma 2.2. ([16]) *Let X be a Banach space and $J \in C^1(X, \mathbb{R})$ verifying the Palais-Smale condition. Suppose that $J(0) = 0$ and that*

(*i*) *there exist $R > 0$, $r > 0$ such that if $\|u\| = R$, then $J(u) \geq r$;*
(*ii*) *there exist $(u_0) \in X$ such that $\|u_0\| > R$ and $J(u_0) \leq 0$.*

Let $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} (J(\gamma(t)))$, where

$$\Gamma = \{\gamma \in C([0,1]; X) \text{ such that } \gamma(0) = 0, \ \gamma(1) = u_0\}.$$

Then c is critical value of J such that $c \geq r$.

It is well known that the functional J is of class C^1 in $\mathcal{H}_0^1(\mathbb{R}^3)$ and the solutions of (1.1) are the critical points of J which is not bounded below on $\mathcal{H}_0^1(\mathbb{R}^3)$.

Consider the following Nehari manifold

$$\mathcal{N} = \left\{u \in \mathcal{H}_0^1(\mathbb{R}^3) \setminus \{0\} : \ \left\langle J'(u), u \right\rangle = 0\right\}.$$

Thus, $u \in \mathcal{N}$ if and only if

$$\left(m + n\|u\|^2\right) \int_{\mathbb{R}^3} \left(|x|^{-2a} |\nabla u|^2\right) dx - \int_{\mathbb{R}^3} |x|^{-2_* b} f |u|^{2_*} dx - \mu \int_{\mathbb{R}^3} g |u|^q dx = 0. \tag{2.3}$$

Define

$$\varphi(u) = \left\langle J'(u), u \right\rangle.$$

Then, for $u \in \mathcal{N}$,

$$\begin{aligned} \left\langle \varphi'(u), u \right\rangle &= \left(2m + 4n\|u\|^2\right)\|u\|^2 - 2_* \int_{\mathbb{R}^3} |x|^{-2_* b} f |u|^{2_*} dx - \mu q \int_{\Omega} g |u|^q dx \\ &= \left[(2-q)m + (4-q)n\|u\|^2\right]\|u\|^2 - (2_* - q)\int_{\mathbb{R}^3} |x|^{-2_* b} f |u|^{2_*} dx \\ &= (2_* - q)\mu \int_{\mathbb{R}^3} g|u|^q dx - \left(4m + 2n\|u\|^2\right)\|u\|^2. \end{aligned} \tag{2.4}$$

Now, we split $\mathcal{N}$ in three parts:

$$\begin{aligned} \mathcal{N}^+ &= \left\{u \in \mathcal{N} : \ \left\langle \varphi'(u), u \right\rangle > 0\right\}, \\ \mathcal{N}^0 &= \left\{u \in \mathcal{N} : \ \left\langle \varphi'(u), u \right\rangle = 0\right\}, \\ \mathcal{N}^- &= \left\{u \in \mathcal{N} : \ \left\langle \varphi'(u), u \right\rangle < 0\right\}. \end{aligned}$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem (1.1). Moreover, we have the following results.

Lemma 2.3. *J is coercive and bounded from below on $\mathcal{N}$.*

Proof. If $u \in \mathcal{N}$, then by (2.4) and the Hölder inequality, we deduce that

$$\begin{aligned} J(u) &= (1/2)\, m \|u\|^2 + (1/4)\, n \|u\|^4 \\ &\quad - (1/2_*) \int_{\mathbb{R}^3} |x|^{-2_* b} f |u|^{2_*} dx - (\mu/q) \int_{\mathbb{R}^3} g |u|^q dx \\ &\geq m \left(\frac{1}{2} - \frac{1}{2_*}\right) \|u\|^2 + n \left(\frac{1}{4} - \frac{1}{2_*}\right) \|u\|^4 - \mu \left(\frac{1}{q} - \frac{1}{2_*}\right) \left|g^+\right|_\infty \|u\|^q . \end{aligned}$$

Thus, J is coercive and bounded from below on $\mathcal{N}$. □

We have the following results.

Lemma 2.4. *Suppose that u_0 is a local minimizer for J on $\mathcal{N}$. Then, if $u_0 \notin \mathcal{N}^0$, u_0 is a critical point of J.*

Proof. If u_0 is a local minimizer for J on $\mathcal{N}$, then u_0 is a solution of the optimization problem

$$\min_{\{u/\ \varphi(u)=0\}} J(u) .$$

Hence, there exists a Lagrange multipliers $\theta \in \mathbb{R}$ such that

$$J'(u_0) = \theta \varphi'(u_0) \text{ in } \mathcal{H}^{-1}.$$

Thus,

$$\left\langle J'(u_0), u_0 \right\rangle = \theta \left\langle \varphi'(u_0), u_0 \right\rangle .$$

But, $\left\langle \varphi'(u_0), u_0 \right\rangle \neq 0$, since $u_0 \notin \mathcal{N}^0$. Hence $\theta = 0$. This completes the proof. □

Lemma 2.5. *There exists a positive number μ_0 such that, for all $\mu \in (0, \mu_0)$, we have $\mathcal{N}^0 = \emptyset$.*

Proof. Let us reason by contradiction. Suppose $\mathcal{N}^0 \neq \emptyset$ such that $0 < \mu < \mu_0$. Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|u\|^4 \geq A' \|u\|^2 - B' \|u\|^q \tag{2.5}$$

and

$$\|u\|_\mu^4 \leq A \|u\|^{2_*} - B \|u\|^2 \tag{2.6}$$

with

$$A = \left(\frac{2_* - q}{4 - q}\right) \frac{(S_a)^{\frac{2}{2_*}}}{n},\ B = \left(\frac{2 - q}{6 - q}\right)\left(\frac{m}{n}\right),\ A' = \frac{2m}{n},\ B' = \left(\frac{2_* - q}{2n}\right) \mu \left|g^+\right|_\infty .$$

From (2.5) and (2.6), we obtain $\mu \geq \mu_0$, which contradicts an hypothesis. □

Since, $\mathcal{N} = \mathcal{N}^{+} \cup \mathcal{N}^{-}$. Define

$$\delta := \inf_{u \in \mathcal{N}} J(u), \ \delta^{+} := \inf_{u \in \mathcal{N}^{+}} J(u) \text{ and } \delta^{-} := \inf_{u \in \mathcal{N}^{-}} J(u).$$

For the sequel, we need the following lemmas.

Lemma 2.6. (1) *For all μ such that $0 < \mu < \mu_0$, one has $\delta \leq \delta^{+} < 0$.*
(2) *There exists $\mu_1 > 0$ such that for all $0 < \mu < \mu_1$, one has*

$$\delta^{-} > C_0 = C_0\left(m, \ n, \ q, \ \left|g^{+}\right|_{\infty}\right).$$

Proof. (1) Let $u \in \mathcal{N}^{+}$. By (2.4), we have

$$\left[\left((2-q)\, m + (4-q)\, n \left\|u\right\|^2\right) / \left(2_* - q\right)\right] \left\|u\right\|^2 > \int_{\mathbb{R}^3} |x|^{-2_* b} f \left|u\right|^{2_*} dx$$

and so

$$\begin{aligned} J(u) &= \left(\frac{1}{4} - \frac{1}{q}\right) n \left\|u\right\|^4 + \left(\frac{1}{2} - \frac{1}{q}\right) m \left\|u\right\|^2 + \left(\frac{2_*}{q} - \frac{1}{2_*}\right) \int_{\mathbb{R}^3} |x|^{-2_* b} f \left|u\right|^{2_*} dx \\ &< -\left[\left(\frac{1}{4} - \frac{1}{q}\right) - \left(\frac{2_*}{q} - \frac{1}{2_*}\right)\left(\frac{4-q}{2_* - q}\right)\right] n \left\|u\right\|^4 \\ &\quad -\left[\left(\frac{1}{q} - \frac{1}{2}\right) - \left(\frac{2_*}{q} - \frac{1}{2_*}\right)\left(\frac{2-q}{2_* - q}\right)\right] m \left\|u\right\|^2. \end{aligned}$$

We conclude that $\delta \leq \delta^{+} < 0$.

(2) Let $u \in \mathcal{N}^{-}$. By (2.4) and the Hölder inequality, we get

$$\begin{aligned} J(u) &= m\left(\frac{1}{2} - \frac{1}{2_*}\right) \left\|u\right\|^2 + n\left(\frac{1}{4} - \frac{1}{2_*}\right) \left\|u\right\|_{\mu}^4 - \mu \int_{\mathbb{R}^3} g \left|u\right|^q dx. \\ &\geq m\left(\frac{1}{2} - \frac{1}{2_*}\right) \left\|u\right\|^2 - \mu\left(\frac{1}{q} - \frac{1}{2_*}\right) \left|g^{+}\right|_{\infty} \left\|u\right\|^q. \end{aligned}$$

Thus, for all μ such that $0 < \mu < \mu_1 = \frac{m\left(\frac{1}{2} - \frac{1}{2_*}\right)}{\left(\frac{1}{q} - \frac{1}{2_*}\right)\left|g^{+}\right|_{\infty}}$, we have $J(u) \geq C_0$. $\square$

We define:

$$\begin{aligned} F^{+} &:= \left\{u \in \mathcal{N} / \textstyle\int_{\mathbb{R}^3} |x|^{-2_* b} f \left|u\right|^{2_*} dx > 0\right\}, \\ F_0^{-} &:= \left\{u \in \mathcal{N} / \textstyle\int_{\mathbb{R}^3} |x|^{-2_* b} f \left|u\right|^{2_*} dx \leq 0\right\}, \\ G^{+} &:= \left\{u \in \mathcal{N} / \textstyle\int_{\mathbb{R}^3} g \left|u\right|^q dx > 0\right\}, \\ G_0^{-} &:= \left\{u \in \mathcal{N} / \textstyle\int_{\mathbb{R}^3} g \left|u\right|^q dx \leq 0\right\} \end{aligned}$$

and for each $u \in \mathcal{H}$ with $u \in F^{+}$, we write

$$t_m := t_{\max}(u) = \left[\frac{8n\left(\frac{1}{q} - \frac{1}{4}\right)}{2_*\left(2_* - 2\right)\left(\frac{2_*}{q} - \frac{1}{2_*}\right)\int_{\mathbb{R}^3} |x|^{-2_* b} f\, |u|^{2_*}\, dx} \right]^{\frac{1}{2_*-4}} > 0.$$

Lemma 2.7. *Let μ real parameters such that $0 < \mu < \mu_0$. For each $u \in \mathcal{H}$ we have*

(1) *if $u \in F^{+} \cap G_0^{-}$, then there exists unique $t^{+} > t_M$ such that $t^{+}u \in \mathcal{N}^{-}$ and*

$$J\left(t^{+}u\right) \geq J\left(tu\right) \text{ for } t > t_M,$$

(2) *if $u \in F^{+} \cap G^{+}$, then there exist unique t^{+} and t^{-} such that $0 < t^{-} < t_M < t^{+}$, $(t^{+}u) \in \mathcal{N}^{-}$, $t^{-}u \in \mathcal{N}^{+}$ and*

$$J\left(t^{+}u\right) \geq J\left(tu\right) \text{ for } t \geq t^{-} \text{ and } J\left(t^{-}u\right) \leq J\left(tu\right) \text{ for } t \in \left[0, t^{+}\right],$$

(3) *if $u \in F^{-} \cap G^{-}$, then does not exist $t > 0$ such that $(tu) \in \mathcal{N}$,*

(4) *if $u \in F_0^{-} \cap G^{+}$, then there exists unique $0 < t^{-} < +\infty$ such that $(t^{-}u) \in \mathcal{N}^{+}$ and*

$$J\left(t^{-}u\right) = \inf_{t \geq 0} J\left(tu\right).$$

Proof. With minor modifications, we refer to [2]. □

Proposition 2.8. ([2])

(1) *For all μ such that $0 < \mu < \mu_0$, there exists a $(PS)_{\delta^{+}}$ sequence in $\mathcal{N}^{+}$.*

(2) *For all μ such that $0 < \mu < \mu_1$, there exists a $(PS)_{\delta^{-}}$ sequence in $\mathcal{N}^{-}$.*

3. Main results

Now, taking as a starting point the work of Tarantello [13], we establish the existence of a local minimum for J on $\mathcal{N}^{+}$.

Theorem 3.1. *Assume that $1 < q < 2$, $-\infty < a < \frac{1}{2}$, $a \leq b < a + (1/4)$, and (F) satisfied and μ verifying $\mu < \mu_0$. Then the problem (1.1) has at least one positive solution.*

Proof. If $0 < \mu < \mu_0$, then from Proposition 2.8, there exists a $\{u_n\}_n$ $(PS)_{\delta^{+}}$ sequence in $\mathcal{N}^{+}$, thus it bounded by Lemma 2.3. Then, there exists $u_0^{+} \in \mathcal{H}$ and we can extract a subsequence which will denoted by $\{u_n\}_n$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^{+} \text{ weakly in } \mathcal{H}_0^{1}\left(\mathbb{R}^3\right), \qquad (3.1)\\ u_n &\rightharpoonup u_0^{+} \text{ weakly in } L^{2_*}\left(\mathbb{R}^3\right),\\ u_n &\rightarrow u_0^{+} \text{ strongly in } L^{q}\left(\mathbb{R}^3\right),\\ u_n &\rightarrow u_0^{+} \text{ a.e in } \mathbb{R}^3. \end{aligned}$$

Thus, by (3.1), u_0^+ is a weak nontrivial solution of (1.1).

Now, we show that $\{u_n\}$ converges to u_0^+ strongly in $\mathcal{H}_0^1(\mathbb{R}^3)$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\|u_0^+\| < \liminf_{n\longrightarrow\infty} \|u_n\|$ and we obtain

$$\begin{aligned}\delta \leq J(u_0^+) &= m\left(\frac{1}{2}-\frac{1}{2_*}\right)\|u_0^+\|^2 + n\left(\frac{1}{4}-\frac{1}{2_*}\right)\|u_0^+\|^4 - \mu\int_{\mathbb{R}^3} g\,|u_0^+|^q\,dx \\ &< \liminf_{n\longrightarrow\infty} J(u_n) = \delta,\end{aligned}$$

which is a contradiction. Therefore, $\{u_n\}$ converge to u_0^+ strongly in $\mathcal{H}_0^1(\mathbb{R}^3)$. Moreover, we have $u_0^+ \in \mathcal{N}^+$. If not, then by Lemma 2.7, there are two numbers t_0^+ and t_0^-, uniquely defined so that $(t_0^+ u_0^+) \in \mathcal{N}^+$ and $(t^- u_0^+) \in \mathcal{N}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J(tu_0^+)_{\downarrow t=t_0^+} = 0 \quad \text{and} \quad \frac{d^2}{dt^2}J(tu_0^+)_{\downarrow t=t_0^+} > 0,$$

there exists $t_0^+ < t^- \leq t_0^-$ such that $J_\lambda(t_0^+ u_0^+) < J_\lambda(t^- u_0^+)$. By Lemma 2.7, we get

$$J_\lambda(t_0^+ u_0^+) < J_\lambda(t^- u_0^+) < J_\lambda(t_0^- u_0^+) = J_\lambda(u_0^+),$$

which contradicts the fact that $J(u_0^+) = c^+$. Since $J(u_0^+) = J(|u_0^+|)$ and $|u_0^+| \in \mathcal{N}^+$, then by Lemma 2.4, we may assume that u_0^+ is a nontrivial nonnegative solution of (1.1). By the Harnack inequality, we conclude that $u_0^+ > 0$, see for example [5]. □

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$.

Theorem 3.2. *In addition to the assumptions of the Theorem* 3.1*, if the condition* (G) *hold, then there exists* $\mu_1 > 0$ *such that for all* μ *verifying* $0 < \mu < \min(\mu_0, \mu_1)$ *the problem* (1.1) *has at least two positive solutions.*

Proof. If $0 < \delta < \delta_1$, then from Proposition 2.8, there exists a $\{u_n\}_n$, $(PS)_{\delta^-}$-sequence in $\mathcal{N}^-$, thus it bounded by Lemma 2.3. Then, there exists $u_0^- \in \mathcal{H}_0^1(\mathbb{R}^3)$ and we can extract a subsequence which will denoted by $\{u_n\}_n$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^- \text{ weakly in } \mathcal{H}_0^1(\mathbb{R}^3), \\ u_n &\rightharpoonup u_0^- \text{ weakly in } L^{2_*}(\mathbb{R}^3), \\ u_n &\to u_0^- \text{ strongly in } L^q(\mathbb{R}^3), \\ u_n &\to u_0^- \text{ a.e in } \mathbb{R}^3. \end{aligned}$$

This implies that

$$\int_{\mathbb{R}^3} |x|^{-2_*b} f |u_n|^{2_*} dx \quad \to \quad \int_{\mathbb{R}^3} |x|^{-2_*b} f \left|u_0^-\right|^{2_*} dx, \text{ as } n \text{ goes to } \infty.$$

Moreover, by (G) and (2.4) we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |x|^{-2_*b} f |u_n|^{2_*} dx &> \left[\frac{(4-q)}{(2_*-q)} n \|u_n\|^4 + \frac{(2-q)}{(2_*-q)} m \|u_n\|^2\right] \\ &> \left[\frac{(4-q)}{(2_*-q)} n \|u_n\|^4 + \frac{(2-q)}{(2_*-q)} m \|u_n\|^2\right] - \|u_n\|^2 \\ &> C_1 \\ &= \left[\frac{(2-q)}{(2_*-q)} m\right]^2 \left[\frac{(4-q)}{(2_*-q)} n - 3\right] \left[\frac{2(4-q)}{(2_*-q)} n - 2\right]^{-2}.\end{aligned}$$

If $n > \frac{3(6-q)}{(4-q)}$, we get

$$\int_{\mathbb{R}^3} |x|^{-2_*b} f |u_n|^{2_*} dx > C_1 > 0. \tag{3.2}$$

This implies that

$$\int_{\mathbb{R}^3} |x|^{-2_*b} f \left|u_0^-\right|^{2_*} dx \geq C_1.$$

Now, we prove that $\{u_n\}_n$ converges to u_0^- strongly in $\mathcal{H}_0^1(\mathbb{R}^3)$. Suppose otherwise. Then, either $\left\|u_0^-\right\| < \liminf\limits_{n \longrightarrow \infty} \|u_n\|$. By Lemma 2.7, there is a unique t_0^- such that $\left(t_0^- u_0^-\right) \in \mathcal{N}^-$. Since

$$u_n \in \mathcal{N}^-, \ J(u_n) \geq J(tu_n) \ \text{ for all } \ t \geq 0,$$

we have

$$J\left(t_0^- u_0^-\right) < \lim_{n \longrightarrow \infty} J\left(t_0^- u_n\right) \leq \lim_{n \longrightarrow \infty} J(u_n) = \delta^-,$$

and this is a contradiction. Hence,

$$(u_n)_n \quad \to \quad u_0^- \text{ strongly in } \mathcal{H}_0^1(\mathbb{R}^3).$$

Thus,

$$J(u_n) \text{ converges to } J\left(u_0^-\right) = \delta^- \quad \text{as } n \text{ tends to } +\infty.$$

Since $J\left(u_0^-\right) = J\left(\left|u_0^-\right|\right)$ and $u_0^- \in \mathcal{N}^-$, then by (3.2) and Lemma 2.4, we may assume that u_0^- is a nontrivial nonnegative solution of (1.1). By the maximum principle, we conclude that $u_0^- > 0$.

Now, we obtain that (1.1) has two positive solutions $u_0^+ \in \mathcal{N}^+$ and $u_0^- \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that u_0^+ and u_0^- are distinct. □

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