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N-BIPOLAR SOFT CONNECTEDNESS, DISCONNECTEDNESS AND COMPACTNESS SPACES WITH APPLICATION IN INFINITE GAME

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Abstract. In this paper, we introduce the notion of N-bipolar soft disjoint sets, N-bipolar soft separate, and explore the attributes and characterizations of N-bipolar soft connectedness and N-bipolar soft disconnectedness and study the relation between them. By providing a detailed picture of N-bipolar soft connected and disconnected spaces. We study N-bipolar soft compact spaces and obtain outcomes associated with this idea. Also, we introduce a new game using N-bipolar soft open covering and study some of the characterizations of this game.

1. Introduction

Molodtsov [14] used acceptable parametrization. He initiated the introductory notion of soft-set propositions in 1999 and presented the first result of the proposition. He has attracted numerous research to this proposition.

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Topology is prominent in colorful branches of mathematics. Therefore, the idea of soft topological spaces was introduced by Shabir and Naz [22]. Akdag and Ozkan ([1], [2]) presented the concepts of soft α-open, the soft b-open, and their respective continuous functions. The soft b^*-closed and soft b^*-continuous functions are studied by Hameed et al. [5], [6]. Hussain [9], defined a note on soft connectedness. Peygh et al. [16] studied soft locally connected and soft connected spaces rely on soft disjoint open set. The bipolar soft set was proposed by Shabir and Naz [23]. Shabir and Bakhtawar introduced the bipolar soft connected and also studied bipolar soft compact [20]. In [4], Fatimah et al. introduced the concept of N soft sets, which serves as an expanded model of the soft set. Mustafa [15] presented and studied the N-bipolar soft set and N-bipolar soft topological spaces. Shabir and Naz [21], defined the concept of N-bipolar soft and elucidated its utilization in the context of decision-making. Also, some basic operations on the bipolar N-soft sets was described by Kamac et al. in [10].

The study of selection rules has a long history in mathematics, dating back to the influential work of Menger et al. [13] and Hurawicz ([7], [8]). The systematic investigation of selection principles was initiated with the research of Scheepers [19]. This theory has significant connections to various branches of mathematics, including set theory, general topology, game theory, Ramsey theory, uniform spaces, hyperspaces, and topological groups. For the theory of the selection principle and game theory, see Kočinac [12], LjDR [11], Babinkostova et al. [3], Radwan et al. [17], Sakai et al. [18].

In the present work, we introduce N-bipolar soft disjoint sets, N-bipolar soft separated, N-bipolar soft connectedness and N-bipolar soft disconnectedness. Also, we define the N-bipolar soft compact. Moreover, we study a new game by using N-bipolar soft open covering and some of the characterizations of this game.

2. Preliminaries

In this study, let $\hat{X}$ be an initial universe and $2^{\hat{X}}$ be the power set of $\hat{X}$. Moreover, $\hat{H} \neq \emptyset$ is the collection of parameters that are being considered.

Definition 2.1. ([14]) (Ψ, ϑ) is referred to be a soft set over $\hat{X}$ if Ψ is a map from ϑ to $P(\hat{X})$.

Definition 2.2. ([4]) Let $\hat{X}$ be an N-soft set on $\hat{X}$ if $\Psi : \vartheta \to 2^{\hat{X}\times\hat{R}}$ for each $\nabla \in \vartheta$ and $\wp \in \hat{X}$, there exists a unique $(\wp, r_\nabla) \in \hat{X} \times \hat{R}$ such that $(\wp, r_\nabla) \in \Psi(\nabla), r_\nabla \in \hat{R}$, $\hat{R} = \{0, 1, ..., N-1\}$ with $N = \{2, 3, ...\}$.

Definition 2.3. ([23]) (Ψ, ζ, ϑ) is a bipolar soft set on $\hat{X}$ if $\Psi : \vartheta \to 2^{\hat{X}}$ and $\zeta : \neg\vartheta \to 2^{\hat{X}}$ with the property that for each $\nabla \in \vartheta$, $\Psi(\nabla) \cap \zeta(\neg\nabla) = \phi$.

Definition 2.4. ([15]) $(\Psi, \zeta, \vartheta, N)$ is an NBS-set on $\hat{X}$ if $\Psi : \vartheta \to 2^{\hat{X}\times\hat{R}}$ and $\zeta : \neg\vartheta \to 2^{\hat{X}\times\hat{R}}$ with the property that for each $\nabla \in \vartheta$ and $\wp \in \hat{X}$, there exists a unique $(\wp, r_\nabla), (\wp, r_{\neg\nabla}) \in \hat{X} \times \hat{R}$ such that $(\wp, r_\nabla) \in \Psi(\nabla)$, $(\wp, r_{\neg\nabla}) \in \zeta(\neg\nabla)$, $r_\nabla \neq r_{\neg\nabla}$ and $0 < r_\nabla + r_{\neg\nabla} \leq N-1$,$r_\nabla, r_{\neg\nabla} \in \hat{R}$, $\hat{R} = \{0, 1, ..., N-1\}$ with $N = \{2, 3, ...\}$.

Definition 2.5. ([15]) Let $(\Psi, \zeta, \vartheta, N)$ be an NBS-set on $\hat{X}$. The complement of $(\Psi, \zeta, \vartheta, N)$, denoted as $(\Psi, \zeta, \vartheta, N)^c$ can be as follows: $(\Psi, \zeta, \vartheta, N)^c = (\Psi^c, \zeta^c, \vartheta, N)$ with $\Psi^c(\nabla)(\wp) = \zeta(\neg\nabla)(\wp)$ and $\zeta^c(\neg\nabla)(\wp) = \Psi(\nabla)(\wp)$ for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$.

Definition 2.6. ([15]) An NBS-set $(\Psi, \zeta, \vartheta, N)$ on $\hat{X}$ is referred to as an empty NBS-set, denoted as $\phi^N_\vartheta = (\Psi_0, \zeta_{N-1}, \vartheta, N)$ satisfying the condition for each $\nabla \in \vartheta$, $\Psi_0(\nabla)(\wp) = 0$ and $\zeta_{N-1}(\neg\nabla)(\wp) = N-1$ for all $\wp \in \hat{X}$.

Definition 2.7. ([15]) An NBS-set $(\Psi, \zeta, \vartheta, N)$ on $\hat{X}$ is referred to as a universal NBS-set, denoted as $\hat{X}^N_\vartheta = (\Psi_{N-1}, \zeta_0, \vartheta, N)$ satisfying the condition for each $\nabla \in \vartheta, \Psi_{N-1}(\nabla)(\wp) = N-1$ and $\zeta_0(\neg\nabla)(\wp) = 0$ for all $\wp \in \hat{X}$.

Definition 2.8. ([15]) The NBS power whole set $\varphi\omega(\Psi, \zeta, \vartheta, N)$ of the NBS-set $(\Psi, \zeta, \vartheta, N)$ is defined by

$$\varphi\omega(\Psi, \zeta, \vartheta, N) = \{(\Psi, \zeta)_i : (\Psi, \zeta)_i \sqsubseteq (\Psi, \zeta), \vartheta, N, i \in N\}$$

such that $\Psi(\nabla)(\wp) = \Psi_i(\nabla)(\wp)$ and $\zeta(\neg\nabla)(\wp) = \zeta_i(\neg\nabla)(\wp)$; $\nabla \in \vartheta$ and $\wp \in \hat{X}$, where $(\Psi, \zeta)_i = (\Psi_i, \zeta_i, \vartheta, N)$ is NBS-subset of $(\Psi, \zeta, \vartheta, N)$.

Definition 2.9. ([15]) Let $(\Psi, \zeta, \vartheta, N)$ be an NBS-set on $\hat{X}$. A collection of NBS-subsets of $(\Psi, \zeta, \vartheta, N)$ is referred to as N-bipolar soft topology ($NBST_S$) on $(\Psi, \zeta, \vartheta, N)$ denoted as $\mathfrak{J}^N_\vartheta$, if the following conditions are satisfied.

(1) $\phi^N_\vartheta, (\Psi, \zeta, \vartheta, N) \in \mathfrak{J}^N_\vartheta$.
(2) Arbitrary unions of members $\mathfrak{J}^N_\vartheta$ of belong to $\mathfrak{J}^N_\vartheta$.
(3) Finite intersections of members $\mathfrak{J}^N_\vartheta$ of belong to $\mathfrak{J}^N_\vartheta$.

The pair $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ is said to be an N-bipolar soft topological spaces ($NBST_S$). Every member of $\mathfrak{J}^N_\vartheta$ is referred to as an N-bipolar soft open set (NBS-open set). In addition, the complement of an N-bipolar soft open set is called an N-bipolar soft closed set (NBS-closed set).

Definition 2.10. ([15]) Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ be an $NBST_S$ and $(\Psi, \zeta)_1 = (\Psi_1, \zeta_1, \vartheta, N) \sqsubseteq (\Psi, \zeta, \vartheta, N)$. Then the collection

$$\widetilde{\mathfrak{J}}_{(\Psi,\zeta)_1}^{N} = \{(\Psi, \zeta)_i \cap (\Psi_1, \zeta_1, \vartheta, N) : (\Psi, \zeta)_i \in \mathfrak{J}_{\vartheta}^{N}\}$$

is referred to as an N-bipolar sub-topology or N-bipolar relative topology on $(\Psi_1, \zeta_1, \vartheta, N)$. The pair $(\Psi_1, \zeta_1, \vartheta, N, \mathfrak{J}_{(\Psi,\zeta)_1}^{N})$ is referred to as an N-bipolar subspace of $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$.

Definition 2.11. ([15]) Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$, $(\Psi, \zeta, \vartheta, N, \xi_{\vartheta}^{N})$ be two $NBST_{SS}$ over $(\Psi, \zeta, \vartheta, N)$. If $\mathfrak{J}_{\vartheta}^{N} \subseteq \xi_{\vartheta}^{N}$ then ξ_{ϑ}^{N} is referred to as finer than $\mathfrak{J}_{\vartheta}^{N}$.

Definition 2.12. ([15]) Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ be an $NBST_S$. Every member of $\mathfrak{J}_{\vartheta}^{N}$ can be expressed as a union of some elements from subfamily Λ of $\mathfrak{J}_{\vartheta}^{N}$ is referred to as an N-bipolar basis for $\mathfrak{J}_{\vartheta}^{N}$.

For more details, one may see [10], [15] and [21].

3. N-bipolar soft connected spaces

In this section, we discuss and explore the properties of N-bipolar topological spaces called the N-bipolar connectedness and N-bipolar disconnectedness.

Definition 3.1. Two NBS-sets $(\Psi_1, \zeta_1, \vartheta, N)$, $(\Psi_2, \zeta_2, \vartheta, N)$ are referred to as N-bipolar disjoint if

$$\min(\Psi_1(\nabla)(\wp), \Psi_2(\nabla)(\wp)) = 0$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$.

Definition 3.2. Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ be an $NBST_S$. An N-bipolar separation of $(\Psi, \zeta, \vartheta, N)$ is a pair $(\Psi_1, \zeta_1, \vartheta, N)$, $(\Psi_2, \zeta_2, \vartheta, N)$ of NBS-open sets are disjoint non-null such that

$$\max(\Psi_1(\nabla)(\wp), \Psi_2(\nabla)(\wp)) = \Psi(\nabla)(\wp)$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$.

Definition 3.3. An $NBST_S$ $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ is referred to as an N-bipolar soft disconnected space (NBS-disconnected space) if there exists an N-bipolar separation of $(\Psi, \zeta, \vartheta, N)$. Moreover, $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ is referred to as an N-bipolar connected space (NBS-connected space) if and only if it is not an N-bipolar disconnected space.

Example 3.4. Consider a set of houses under consideration denoted as $\hat{X} = \{\wp_1, \wp_2\}$,

$$\vartheta = \{\nabla_1 = Morbled, \ \ \nabla_2 = Modern\}$$

and

$$\neg\vartheta = \{\neg\nabla_1 = Wooden, \ \ \neg\nabla_2 = Traditional\}.$$

Consider a $5BS$-set to describe the design of houses in the following manner:

$$(\Psi, \zeta, \vartheta, 5) = \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 2)\} >), \\ (< \nabla_2, \{(\wp_1, 3), (\wp_2, 1)\} >, < \neg\nabla_2, \{(\wp_1, 1), (\wp_2, 2)\} >)\}$$

then

$$\mathfrak{I}^5_\vartheta = \{(\Psi, \zeta, \vartheta, 5), \phi^5_\vartheta, (\Psi, \zeta)_1, (\Psi, \zeta)_2\},$$

where $(\Psi, \zeta)_1, (\Psi, \zeta)_2$ are $5BS$-subsets on $5BS$-set $(\Psi, \zeta, \vartheta, 5)$ defined as follows:

$$(\Psi, \zeta)_1 = \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 0)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 4)\} >), \\ (< \nabla_2, \{(\wp_1, 3), (\wp_2, 0)\} >, < \neg\nabla_2, \{(\wp_1, 1), (\wp_2, 4)\} >)\}$$

and

$$(\Psi, \zeta)_2 = \{(< \nabla_1, \{(\wp_1, 0), (\wp_2, 1)\} >, < \neg\nabla_1, \{(\wp_1, 4), (\wp_2, 2)\} >), \\ (< \nabla_2, \{(\wp_1, 0), (\wp_2, 1)\} >, < \neg\nabla_2, \{(\wp_1, 4), (\wp_2, 2)\} >)\}.$$

Then $(\Psi, \zeta, \vartheta, 5, \mathfrak{I}^5_\vartheta)$ is 5-bipolar soft disconnected space because $(\Psi, \zeta)_1$ and $(\Psi, \zeta)_2$ form 5-bipolar soft separation of $(\Psi, \zeta, \vartheta, 5)$.

Theorem 3.5. *An $NBST_S$ $(\Psi, \zeta, \vartheta, N, \mathfrak{I}^N_\vartheta)$ is $N-bipolar$ soft disconnected space if and only if there exist two NBS-closed sets $(\Psi_1, \zeta_1, \vartheta, N), (\Psi_2, \zeta_2, \vartheta, N)$ with $\zeta_1(\neg\nabla)(\wp) \neq 0$, $\zeta_2(\neg\nabla)(\wp) \neq 0$ for some $\nabla \in \vartheta$ and $\wp \in \hat{X}$ such that*

$$\max(\zeta_1(\neg\nabla)(\wp), \zeta_2(\neg\nabla)(\wp)) = \Psi(\nabla)(\wp)$$

for all $\neg\nabla \in \neg\vartheta$ and

$$\min(\zeta_1(\neg\nabla)(\wp), \zeta_2(\neg\nabla)(\wp)) = 0$$

for all $\neg\nabla \in \neg\vartheta$ and $\wp \in \hat{X}$.

Proof. Suppose that $(\Psi, \zeta, \vartheta, N, \mathfrak{I}^N_\vartheta)$ is an NBS-disconnected space. Then there exists an NBS-separation of $(\Psi, \zeta, \vartheta, N)$. Let $(\mathcal{Q}, \zeta, \vartheta, N)$ and $(\mathcal{H}, \{, \vartheta, N)$ form NBS-separation of $(\Psi, \zeta, \vartheta, N)$. Then

$$\max(\mathcal{Q}(\nabla)(\wp), \mathcal{H}(\nabla)(\wp)) = \Psi(\nabla)(\wp)$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$, $\mathcal{Q}(\nabla)(\wp) \neq 0$, $\mathcal{H}(\nabla)(\wp) \neq 0$ for some $\nabla \in \vartheta$ and

$$\min(\mathcal{Q}(\nabla)(\wp), \mathcal{H}(\nabla)(\wp)) = 0$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$.

As $\mathcal{Q}(\nabla)(\wp) = \zeta^c(\neg\nabla)(\wp)$ and $\mathcal{H}(\nabla)(\wp) = F^c(\neg\nabla)(\wp)$. Therefore, we have

$$\max(\zeta^c(\neg\nabla)(\wp), F^c(\neg\nabla)(\wp)) = \Psi(\nabla)(\wp)$$

and

$$\min(\zeta^c(\neg\nabla)(\wp), F^c(\neg\nabla)(\wp)) = 0$$

for all $\neg\nabla \in \neg\nabla$ and $\wp \in \hat{X}$, where $\zeta^c(\neg\nabla)(\wp) \neq 0, F^c(\neg\nabla)(\wp) \neq 0$ for some $\neg\nabla \in \neg\vartheta$. Moreover, $(\mathcal{Q}, \zeta, \vartheta, N)^c$ and $(\mathcal{H}, \mathcal{F}, \vartheta, N)^c$ are NBS closed sets, since $(\mathcal{Q}, \zeta, \vartheta, N)$, $(\mathcal{H}, \mathcal{F}, \vartheta, N) \in \mathfrak{J}^N_\vartheta$.

Conversely, suppose that there exist two NBS closed sets $(\Psi_1, \zeta_1, \vartheta, N)$, $(\Psi_2, \zeta_2, \vartheta, N)$ with $\zeta_1(\neg\nabla)(\wp) \neq 0$, $\zeta_2(\neg\nabla)(\wp) \neq 0$ for some $\nabla \in \vartheta$ such that

$$\max(\zeta_1(\neg\nabla)(\wp), \zeta_2(\neg\nabla)(\wp)) = \Psi(\nabla)(\wp)$$

for all $\neg\nabla \in \neg\vartheta$ and $\wp \in \hat{X}$, and

$$\min(\zeta_1(\neg\nabla)(\wp), \zeta_2(\neg\nabla)(\wp)) = 0$$

for all $\neg\nabla \in \neg\vartheta$ and $\wp \in \hat{X}$. Then $(\Psi_1, \zeta_1, \vartheta, N)^c$ and $(\Psi_2, \zeta_2, \vartheta, N)^c$ are NBS-open sets with $\Psi_1^c(\nabla)(\wp) = \zeta_1(\neg\nabla)(\wp) \neq 0$, $\Psi_2^c(\nabla)(\wp) = \zeta_2(\neg\nabla)(\wp) \neq 0$ for some $\nabla \in \vartheta$ such that

$$\max(\Psi_1^c(\nabla)(\wp), \Psi_2^c(\nabla)(\wp)) = \max(\zeta_1(\neg\nabla)(\wp), \zeta_2(\neg\nabla)(\wp)) = \Psi(\nabla)(\wp)$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$, and

$$\min(\Psi_1^c(\nabla)(\wp), \Psi_2^c(\nabla)(\wp)) = 0$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. Therefore, $(\Psi_1, \zeta_1, \vartheta, N)^c$ and $(\Psi_2, \zeta_2, \vartheta, N)^c$ form an NBS-separation of $(\Psi, \zeta, \vartheta, N)$. Thus, $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ is an NBS-connected space. □

Remark 3.6. The union of two NBS-connected spaces over the same NBS-set need not to be an NBS-connected space.

Example 3.7. Let $\hat{X} = \{\wp_1, \wp_2\}$ and $\vartheta = \{\nabla_1, \nabla_2\}$, $\neg\vartheta = \{\neg\nabla_1, \neg\nabla_2\}$. Consider a $5BS$ soft set in Example 3.4 as follows:

$$\begin{aligned}(\Psi, \zeta, \vartheta, 5) = \{(&< \nabla_1, \{(\wp_1, 2), (\wp_2, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 2)\} >), \\ &< \nabla_2, \{(\wp_1, 3), (\wp_2, 1)\} >, < \neg\nabla_2, \{(\wp_1, 1), (\wp_2, 2)\} >)\}\end{aligned}$$

and

$$\mathfrak{J}^5_\nabla = \{(\Psi, \zeta, \nabla, 5), \phi^5_\nabla, (\Psi, \zeta)_1\},\ \ \xi^5_\nabla = \{(\Psi, \zeta, \vartheta, 5), \phi^5_\vartheta, (\Psi, \zeta)_2\},$$

where $(\Psi, \zeta)_1, (\Psi, \zeta)_2$ are $5BS$-subsets on $5BS$-set $(\Psi, \zeta, \vartheta, 5)$ defined as follows:

$$\begin{aligned}(\Psi, \zeta)_1 = \{(&< \nabla_1, \{(\wp_1, 2), (\wp_2, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 2)\} >), \\ (&< \nabla_2, \{(\wp_1, 3)\} >, < \neg\nabla_2, \{(\wp_1, 1), (\wp_2, 2)\} >)\}\end{aligned}$$

and

$$(\Psi,\zeta)_2 = \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 2)\} >),$$
$$(< \nabla_2, \{(\wp_1, 3)\} >, < \neg\nabla_2, \{(\wp_1, 1)\} >)\}.$$

Then $(\Psi,\zeta,\vartheta,5,\mathfrak{J}^5_\vartheta)$ and $(\Psi,\zeta,\vartheta,5,\xi^5_\vartheta)$ are $5BS$-connected spaces. But we note that $\mathfrak{J}^5_\vartheta \cup \xi^5_\vartheta$ is not $5BS$-connected space because $(\Psi,\zeta)_1$ and $(\Psi,\zeta)_2$ form $5BS$-separation of $(\Psi,\zeta,\vartheta,5)$.

Theorem 3.8. *The intersection of two NBS-connected spaces over the same NBS-set is also a NBS-connected space.*

Proof. Let $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta)$ and $(\Psi,\zeta,\vartheta,N,\mathfrak{L}^N_\vartheta)$ be two NBS-connected spaces. Suppose that $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta \cap \mathfrak{L}^N_\vartheta)$ is not an NBS-connected space. Then there exist NBS-sets $(\Psi_1,\zeta_1,\vartheta,N)$, $(\Psi_2,\zeta_2,\vartheta,N)$ belonging to $\mathfrak{J}^N_\vartheta \cap \mathfrak{L}^N_\vartheta$, which forms an NBS-separation of (Ψ,ζ,ϑ,N) in $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta \cap \mathfrak{L}^N_\vartheta)$.

Since $(\Psi_1,\zeta_1,\vartheta,N)$, $(\Psi_2,\zeta_2,\vartheta,N) \in \mathfrak{J}^N_\vartheta \cap \mathfrak{L}^N_\vartheta$, $(\Psi_1,\zeta_1,\vartheta,N)$, $(\Psi_2,\zeta_2,\vartheta,N) \in \mathfrak{J}^N_\vartheta$ and $(\Psi_1,\zeta_1,\vartheta,N)$, $(\Psi_2,\zeta_2,\vartheta,N) \in \mathfrak{L}^N_\vartheta$.

This implies that $(\Psi_1,\zeta_1,\vartheta,N)$, $(\Psi_2,\zeta_2,\vartheta,N)$ form an NBS-separation of (Ψ,ζ,ϑ,N) in $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta)$ and $(\Psi_1,\zeta_1,\vartheta,N)$, $(\Psi_2,\zeta_2,\vartheta,N)$ form an NBS-separation of (Ψ,ζ,ϑ,N) in $(\Psi,\zeta,\vartheta,N,\mathfrak{L}^N_\vartheta)$, which is a contradiction to the given hypothesis. Thus, $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta \cap \mathfrak{L}^N_\vartheta)$ is an NBS-connected space. □

Remark 3.9. The intersection of two NBS-disconnected spaces over the same NBS-set needs to not be a NBS-disconnected space.

Example 3.10. Let $\hat{X} = \{\wp_1, \wp_2, \wp_3\}$ and $\vartheta = \{\nabla_1, \nabla_2\}$, $\neg\nabla = \{\neg\nabla_1, \neg\nabla_2\}$. Consider a $5BS$-set as follows:

$$(\Psi,\zeta,\vartheta,5)$$
$$= \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 3), (\wp_3, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 1), (\wp_3, 3)\} >),$$
$$(< \nabla_2, \{(\wp_1, 1), (\wp_2, 2), (\wp_3, 3)\} >, < \neg\nabla_2, \{(\wp_1, 2), (\wp_2, 1), (\wp_3, 1)\} >)\}$$

and

$$\mathfrak{J}^5_\vartheta = \{(\Psi,\zeta,\vartheta,5), \phi^5_\vartheta, (\Psi,\zeta)_1, (\Psi,\zeta)_2\},\ \ \xi^5_\vartheta = \{(\Psi,\zeta,\vartheta,5), \phi^5_\vartheta, (\Psi,\zeta)_3, (\Psi,\zeta)_4\},$$

where $(\Psi,\zeta)_1, (\Psi,\zeta)_2, (\Psi,\zeta)_3, (\Psi,\zeta)_4$ are $5BS$ soft subset on $5BS$' soft set $(\Psi,\zeta,\vartheta,5)$ defined as follows:

$$(\Psi,\zeta)_1$$
$$= \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 0), (\wp_3, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 4), (\wp_3, 3)\} >),$$
$$(< \nabla_2, \{(\wp_1, 1), (\wp_2, 0), (\wp_3, 3)\} >, < \neg\nabla_2, \{(\wp_1, 2), (\wp_2, 4), (\wp_3, 1)\} >)\},$$

$$(\Psi,\zeta)_2 = \{(<\nabla_1,\{(\wp_1,0),(\wp_2,3),(\wp_3,0)\}>,<\neg\nabla_1,\{(\wp_1,4),(\wp_2,1),(\wp_3,4)\}>),\\ (<\nabla_2,\{(\wp_1,0),(\wp_2,2),(\wp_3,0)\}>,<\neg\nabla_2,\{(\wp_1,4),(\wp_2,1),(\wp_3,4)\}>)\},$$

$$(\Psi,\zeta)_3 = \{(<\nabla_1,\{(\wp_1,2),(\wp_2,3),(\wp_3,0)\}>,<\neg\nabla_1,\{(\wp_1,1),(\wp_2,1),(\wp_3,4)\}>),\\ (<\nabla_2,\{(\wp_1,0),(\wp_2,0),(\wp_3,3)\}>,<\neg\nabla_2,\{(\wp_1,4),(\wp_2,4),(\wp_3,1)\}>)\}$$

and

$$(\Psi,\zeta)_4 = \{(<\nabla_1,\{(\wp_1,0),(\wp_2,0),(\wp_3,1)\}>,<\neg\nabla_1,\{(\wp_1,4),(\wp_2,4),(\wp_3,3)\}>),\\ (<\nabla_2,\{(\wp_1,1),(\wp_2,2),(\wp_3,0)\}>,<\neg\nabla_2,\{(\wp_1,2),(\wp_2,1),(\wp_3,0)\}>)\}.$$

Then $(\Psi,\zeta,\vartheta,5,\mathfrak{J}^5_\vartheta)$ and $(\Psi,\zeta,\vartheta,5,\xi^5_\vartheta)$ are $5BS$-disconnected spaces. But we note that $\mathfrak{J}^5_\vartheta\cap\xi^5_\vartheta$ is not $5BS$-disconnected space.

Theorem 3.11. *The union of two NBS disconnected spaces over the same NBS-set is a NBS disconnected.*

Proof. The proof is obvious. □

Theorem 3.12. *Let $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta)$ be an $NBST_S$ over (Ψ,ζ,ϑ,N). And let the NBS-sets $(\Psi_2,\zeta_2,\vartheta,N)$ and $(\Psi_3,\zeta_3,\vartheta,N)$ form an NBS-separation of (Ψ,ζ,ϑ,N). If $(\Psi_1,\zeta_1,\vartheta,N,\mathfrak{J}^N_{(\Psi,\zeta)_1})$ is an NBS-connected subspace of $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta)$, then $\Psi_1(\nabla)(\wp)\subseteq\Psi_2(\nabla)(\wp)$ for all $\nabla\in\vartheta$ and $\wp\in\hat{X}$ or $\Psi_1(\nabla)(\wp)\subseteq\Psi_3(\nabla)(\wp)$ for all $\nabla\in\vartheta$ and $\wp\in\hat{X}$.*

Proof. Since $(\Psi_2,\zeta_2,\vartheta,N)$ and $(\Psi_3,\zeta_3,\vartheta,N)$ form an NBS-separation of (Ψ,ζ,ϑ,N), we have

$$\min(\Psi(\nabla)(\wp),\max(\Psi_2(\nabla)(\wp),\Psi_3(\nabla)(\wp)))=\Psi(\nabla)(\wp) \tag{3.1}$$

for all $\nabla\in\vartheta$ and $\wp\in\hat{X}$, $\Psi_2(\nabla)(\wp)\neq 0$ and $\Psi_3(\nabla)(\wp)\neq 0$ for some $\nabla\in\vartheta$ and

$$\min(\Psi_2(\nabla)(\wp),\Psi_3(\nabla)(\wp))=0 \tag{3.2}$$

for all $\nabla\in\vartheta$ and $\wp\in\hat{X}$.

As $(\Psi_1,\zeta_1,\vartheta,N)\subseteq(\Psi,\zeta,\vartheta,N)$, and $(\Psi_1,\zeta_1,\vartheta,N,\mathfrak{J}^N_{(\Psi,\zeta)_1})$ is an $NBST_{SS}$ of $(\Psi,\zeta,\vartheta,N,\mathfrak{J}^N_\vartheta)$, then we have $(\Psi_1,\zeta_1,\vartheta,N)\cap(\Psi_2,\zeta_2,\vartheta,N)\in\mathfrak{J}^N_{(\Psi,\zeta)_1}$ and $(\Psi_1,\zeta_1,\vartheta,N)\cap(\Psi_3,\zeta_2,\vartheta,N)\in\mathfrak{J}^N_{(\Psi,\zeta)_1}$. So,

$$\min(\Psi_1(\nabla)(\wp),\max(\Psi_2(\nabla)(\wp),\Psi_3(\nabla)(\wp)))=\Psi_1(\nabla)(\wp)$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. But

$$\begin{aligned}&\min(\Psi_1(\nabla)(\wp), \max(\Psi_2(\nabla)(\wp), \Psi_3(\nabla)(\wp)))\\ &\quad = \max(\min(\Psi_1\nabla(\wp), \Psi_2(\nabla)(\wp), \min(\Psi_1(\nabla)(\wp), \Psi 3(\nabla)(\wp))))\\ &\quad = \Psi_1(\nabla)(\wp). \end{aligned} \tag{3.3}$$

Since $\min(\Psi_2(\nabla)(\wp), \Psi_3(\nabla)(\wp)) = 0$, then

$$\begin{aligned}&(\min(\min(\Psi_1(\nabla)(\wp), \Psi_2(\nabla)(\wp), \min(\Psi_1(\nabla)(\wp), \Psi_3(\nabla)(\wp)))))\\ &\quad = \min(\Psi_1(\nabla)(\wp), \min(\Psi_2(\nabla)(\wp), \Psi_3(\nabla)(\wp)))\\ &\quad = \min(\Psi_1(\nabla)(\wp), 0) = 0,\end{aligned}$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. Since $(\Psi_1, \zeta_1, \vartheta, N)$ is a NBS connected subspace of $(\Psi, \zeta, \vartheta, N)$, we have either

$$\min(\Psi_1(\nabla)(\wp), \Psi_2(\nabla)(\wp)) = 0$$

or

$$\min(\Psi_1(\nabla)(\wp), \Psi_3(\nabla)(\wp)) = 0$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. If $\min(\Psi_1(\nabla)(\wp), \Psi_2(\nabla)(\wp)) = 0$, then from equation (3.3)

$$\min(\Psi_1(\nabla)(\wp), \Psi_3(\nabla)(\wp)) = \Psi_1(\nabla)(\wp)$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. So $\Psi_1(\nabla)(\wp) \subseteq \Psi_3(\nabla)(\wp)$ for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. So, $(\Psi_1, \zeta_1, \vartheta, N) \subseteq (\Psi_3, \zeta_3, \vartheta, N)$.

Similarly, if $\min(\Psi_1(\nabla)(\wp), \Psi_3(\nabla)(\wp)) = 0$, then from equation (3.3)

$$\min(\Psi_1(\nabla)(\wp), \Psi_2(\nabla)(\wp)) = \Psi_1(\nabla)(\wp)$$

for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. So, $\Psi_1(\nabla)(\wp) \subseteq \Psi_2(\nabla)(\wp)$ for all $\nabla \in \vartheta$ and $\wp \in \hat{X}$. □

4. N-bipolar soft compact spaces

We discuss and explore in this section a new compact space called the N-bipolar compact and investigate certain characteristics of them.

Definition 4.1. A family $\mho = (\Psi_\alpha, \zeta_\alpha, \vartheta, N)_{\alpha\in\zeta}$ of NBS-sets is called N-bipolar soft cover (NBS-cover) of an NBS-set $(\Psi, \zeta, \vartheta, N)$ if

$$(\Psi, \zeta, \vartheta, N) \subseteq \cup_{\alpha\in\zeta}(\Psi_\alpha, \zeta_\alpha, \vartheta, N).$$

Also, it is called the NBS-open cover of an NBS-set $(\Psi, \zeta, \vartheta, N)$ if every member of $\mho$ is an NBS-open set. An N-bipolar soft subcover of $\mho$ is a subfamily of $\mho$ which is also a N-bipolar soft cover.

Definition 4.2. A $NBST_S$ $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_\vartheta^N)$ is called an NBS compact space, if each NBS-open cover of $(\Psi, \zeta, \vartheta, N)$ has a finite N-bipolar soft subcover.

Example 4.3. Let $\hat{X} = \{\wp_1, \wp_2, \wp_3\}$ and $\vartheta = \{\nabla_1, \nabla_2\}$, $\neg\vartheta = \{\neg\nabla_1, \neg\nabla_2\}$. Consider a $5BS$-set as follows:

$$\begin{aligned}&(\Psi, \zeta, \vartheta, 5)\\ &= \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 3), (\wp_3, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 1), (\wp_3, 3)\} >),\\ &\quad (< \nabla_2, \{(\wp_1, 1), (\wp_2, 2), (\wp_3, 3)\} >, < \neg\nabla_2, \{(\wp_1, 2), (\wp_2, 1), (\wp_3, 1)\} >)\}\end{aligned}$$

and

$$\mathfrak{J}_\vartheta^5 = \{(\Psi, \zeta, \vartheta, 5), \phi_\vartheta^5, (\Psi, \zeta)_1, (\Psi, \zeta)_2, (\Psi, \zeta)_3\},$$

where $(\Psi, \zeta)_1, (\Psi, \zeta)_2, (\Psi, \zeta)_3$ are $5BS$-subsets on $5BS$-set $(\Psi, \zeta, \vartheta, 5)$ defined as follows:

$$\begin{aligned}&(\Psi, \zeta)_1\\ &= \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 0), (\wp_3, 1)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 4), (\wp_3, 3)\} >),\\ &\quad (< \nabla_2, \{(\wp_1, 1), (\wp_2, 0), (\wp_3, 3)\} >, < \neg\nabla_2, \{(\wp_1, 2), (\wp_2, 4), (\wp_3, 1)\} >)\},\end{aligned}$$

$$\begin{aligned}&(\Psi, \zeta)_2\\ &= \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 0), (\wp_3, 0)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 4), (\wp_3, 4)\} >),\\ &\quad (< \nabla_2, \{(\wp_1, 1), (\wp_2, 0), (\wp_3, 0)\} >, < \neg\nabla_2, \{(\wp_1, 2), (\wp_2, 4), (\wp_3, 4)\} >)\}\end{aligned}$$

and

$$\begin{aligned}&(\Psi, \zeta)_3\\ &= \{(< \nabla_1, \{(\wp_1, 2), (\wp_2, 3), (\wp_3, 0)\} >, < \neg\nabla_1, \{(\wp_1, 1), (\wp_2, 1), (\wp_3, 4)\} >),\\ &\quad (< \nabla_2, \{(\wp_1, 1), (\wp_2, 2), (\wp_3, 0)\} >, < \neg\nabla_2, \{(\wp_1, 2), (\wp_2, 1), (\wp_3, 4)\} >)\}.\end{aligned}$$

Then $(\Psi, \zeta, \vartheta, 5, \mathfrak{J}_\vartheta^5)$ is $5BS$ soft space over $(\Psi, \zeta, \vartheta, 5)$. Hence, it is easy to show that $(\Psi, \zeta, \vartheta, 5, \mathfrak{J}_\vartheta^5)$ is $5BS$-compact space, since every $5BS$-open cover of $(\Psi, \zeta, \vartheta, 5)$ has a finite $5BS$-subcover.

Definition 4.4. Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_\vartheta^N)$ and $(\Psi, \zeta, \vartheta, N, \xi_\vartheta^N)$ be two $NBST_S$ over $(\Psi, \zeta, \vartheta, N)$. If $\mathfrak{J}_\vartheta^N \subseteq \xi_\vartheta^N$ or $\xi_\vartheta^N \subseteq \mathfrak{J}_\vartheta^N$ then $\mathfrak{J}_\vartheta^N$ is said to be comparable with ξ_ϑ^N.

Theorem 4.5. *Let $(\Psi, \zeta, \vartheta, N, \xi_\vartheta^N)$ be an NBS-compact space and $\mathfrak{J}_\vartheta^N \subseteq \xi_\vartheta^N$. Then $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_\vartheta^N)$ is an NBS-compact.*

Proof. Let $(\Psi_\alpha, \zeta_\alpha, \vartheta, N)_{\alpha\in\zeta}$ be the NBS-open cover of $(\Psi, \zeta, \vartheta, N)$ in $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$. Since $\mathfrak{J}^N_\vartheta \subseteq \xi^N_\vartheta$, $(\Psi_\alpha, \zeta_\alpha, \vartheta, N)_{\alpha\in\zeta}$ is the NBS-open cover of $(\Psi, \zeta, \vartheta, N)$ by NBS-open sets of $(\Psi, \zeta, \vartheta, N, \xi^N_\vartheta)$. But $(\Psi, \zeta, \vartheta, N, \xi^N_\vartheta)$ is an NBS-compact space,

$$(\Psi, \zeta, \vartheta, N) \subseteq (\Psi_{\alpha_1}, \zeta_{\alpha_1}, \vartheta, N) \cup (\Psi_{\alpha_2}, \zeta_{\alpha_2}, \vartheta, N) \cup ... \cup (\Psi_{\alpha_n}, \zeta_{\alpha_n}, \vartheta, N)$$

for some $\alpha_1, \alpha_2, ..., \alpha_n \in \zeta$. Hence, $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ is an NBS-compact. □

Theorem 4.6. *Let $(\Psi_1, \zeta_1, \vartheta, N, \mathfrak{J}^N_{(\Psi,\zeta)_1})$ be an $NBST_{SS}$ of $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$. Then $(\Psi_1, \zeta_1, \vartheta, N, \mathfrak{J}^N_{(\Psi,\zeta)_1})$ is an NBS-compact space if and only if every NBS cover of $(\Psi_1, \zeta_1, \vartheta, N)$ by NBS-open sets in $(\Psi, \zeta, \vartheta, N)$ contains a finite subcover.*

Proof. Let $(\Psi_\alpha, \zeta_\alpha, \vartheta, N)_{\alpha\in\zeta}$ be a cover of $(\Psi_1, \zeta_1, \vartheta, N)$ by NBS-open sets in $(\Psi, \zeta, \vartheta, N)$. Now, $(\Psi_1, \zeta_1, \vartheta, N) \subseteq \cup_{\alpha\in\zeta}(\Psi_\alpha, \zeta_\alpha, \vartheta, N)$. Then there exists $\alpha_c \in \zeta$ such that

$$(\Psi_1, \zeta_1, \vartheta, N) \subseteq \cup_{\alpha\in\zeta}(\Psi_\alpha, \zeta_\alpha, \vartheta, N) \subseteq (\Psi_{\alpha_c}, \zeta_{\alpha_c}, \vartheta, N).$$

Since $(\Psi_{\alpha_c}, \zeta_{\alpha_c}, \vartheta, N) \in \mathfrak{J}^N_\vartheta$, we have

$$\begin{aligned}(\Psi_1, \zeta_1, \vartheta, N) &\subseteq (\Psi_1, \zeta_1, \vartheta, N) \cap (\Psi_{\alpha_c}, \zeta_{\alpha_c}, \vartheta, N)\\ &\subseteq \cup_{\alpha\in\zeta}(\Psi_1, \zeta_1, \vartheta, N) \cap (\Psi_\alpha, \zeta_\alpha, \vartheta, N).\end{aligned}$$

Then $(\Psi_{(\Psi,\zeta)_{1\alpha}}, \zeta_{(\Psi,\zeta)_{1\alpha}}, \vartheta, N)$ is an NBS open cover of $(\Psi_1, \zeta_1, \vartheta, N)$. Since $(\Psi_1, \zeta_1, \vartheta, N, \mathfrak{J}^N_{(\Psi,\zeta)_1})$ is an NBS-compact space, we have

$$\begin{aligned}(\Psi_1, \zeta_1, \vartheta, N) \subseteq\ & (\Psi_{(\Psi,\zeta)_{1\alpha_1}}, \zeta_{(\Psi,\zeta)_{1\alpha_1}}, \vartheta, N)\\ & \cup (\Psi_{(\Psi,\zeta)_{1\alpha_2}}, \zeta_{(\Psi,\zeta)_{1\alpha_2}}, \vartheta, N) \cup \cdots \cup (\Psi_{(\Psi,\zeta)_{1\alpha_n}}, \zeta_{(\Psi,\zeta)_{1\alpha_n}}, \vartheta, N)\end{aligned}$$

for some $\alpha_1, \alpha_2, ..., \alpha_n \in \zeta$. This implies that $\{(\Psi_{\alpha_i}, \zeta_{\alpha_i}, \vartheta, N)\}_{i=1}^n$ is an NBS subcover of $(\Psi_1, \zeta_1, \vartheta, N)$ by NBS open sets in $(\Psi, \zeta, \vartheta, N)$.

Conversely, suppose that $\{(\Psi_{(\Psi,\zeta)_{1\alpha}}, \zeta_{(\Psi,\zeta)_{1\alpha}}, \vartheta, N)\}_{\alpha\in\zeta}$ is an NBS-open cover of $(\Psi_1, \zeta_1, \vartheta, N)$. We see that $\{(\Psi_\alpha, \zeta_\alpha, \vartheta, N)\}_{\alpha\in\zeta}$ is an NBS open cover of $(\Psi_1, \zeta_1, \vartheta, N)$ by NBS-open sets in $(\Psi, \zeta, \vartheta, N)$. Therefore, by the given hypothesis we have

$$(\Psi_1, \zeta_1, \vartheta, N) \subseteq (\Psi_{\alpha_1}, \zeta_{\alpha_1}, \vartheta, N) \cup (\Psi_{\alpha_2}, \zeta_{\alpha_2}, \vartheta, N) \cup \cdots \cup (\Psi_{\alpha_n}, \zeta_{\alpha_n}, \vartheta, N)$$

for some $\alpha_1, \alpha_2, \cdots, \alpha_n \in \zeta$. Thus, $\{(\Psi_{(\Psi,\zeta)_{1\alpha_i}}, \zeta_{(\Psi,\zeta)_{1\alpha_i}}, \vartheta, N)\}_{i=1}^n$ is an NBS subcover of $(\Psi_1, \zeta_1, \vartheta, N)$. Hence, $(\Psi_1, \zeta_1, \vartheta, N, \mathfrak{J}^N_{(\Psi,\zeta)_1})$ is an NBS-compact space. □

Theorem 4.7. *An $NBST_S$ $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ is an NBS-compact space if and only if there is an N-bipolar basis Λ for $\mathfrak{J}^N_\vartheta$ such that every NBS-cover of $(\Psi, \zeta, \vartheta, N)$ by the elements of Λ has a finite NBS-subcover.*

Proof. Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ be an NBS-compact space. Then, $\mathfrak{J}_{\vartheta}^{N}$ is an NBS basis for $\mathfrak{J}_{\vartheta}^{N}$. Therefore, every NBS-cover of $(\Psi, \zeta, \vartheta, N)$ by elements of $\mathfrak{J}_{\vartheta}^{N}$ has a finite NBS-subcover.

Conversely, let $\{(L_\alpha, K_\alpha, \vartheta, N)\}_{\alpha \in \zeta}$ be anNBS open cover of $(\Psi, \zeta, \vartheta, N)$. We can write $(L_\alpha, K_\alpha, \vartheta, N)$ as a union of basis elements for each $\alpha \in \zeta$. These elements form an NBS-open cover of $(\Psi, \zeta, \vartheta, N)$ such that $\{(\Psi_\Lambda, \zeta_\Lambda, \vartheta, N)\}_{\Lambda \in I}$.

Now, by given hypothesis, for some $\Lambda_1, \Lambda_2, ..., \Lambda_n \in I$, we have

$$\Psi(\vartheta)(\wp) = \max(\Psi_{\Lambda_1}(\vartheta)(\wp), \Psi_{\Lambda_2}(\vartheta)(\wp), ..., \Psi_{\Lambda_n}(\vartheta)(\wp))$$

for each $\vartheta \in \vartheta$ and $\zeta(\neg\vartheta)(\wp) = \min(\zeta_{\Lambda 1}(\neg\vartheta)(\wp), \zeta_{\Lambda_2}(\neg\vartheta)(\wp), ..., \zeta_{\Lambda_n}(\neg\vartheta)(\wp)) = 0$ for each $\neg\vartheta \in \vartheta$ and $\wp \in \hat{X}$. That is,

$$(\Psi, \zeta, \vartheta, N) = (\Psi_{\Lambda_1}, \zeta_{\Lambda_1}, \vartheta, N) \cup (\Psi_{\Lambda_2}, \zeta_{\Lambda_2}, \vartheta, N) \cup ... \cup (\Psi_{\Lambda_n}, \zeta_{\Lambda_n}, \vartheta, N)$$

for some $\Lambda_1, \Lambda_2, ..., \Lambda_n \in I$.

Now, $(\Psi_{\Lambda_1}, \zeta_{\Lambda_1}, \vartheta, N) \subseteq (L_{\alpha_2}, K_{\alpha_2}, \vartheta, N)$ for each $1 \leq i \leq n$. This implies that $\{(L_{\alpha_i}, K_{\alpha_i}, \vartheta, N)\}_{i=1}^{n}$ is a finite NBS subcover of $(\Psi, \zeta, \vartheta, N)$. Hence, $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ is an NBS-compact space. □

5. N-Bipolar soft covers and infinitely long game

In this section, we define a new game by NBS open covering and study the characterizations and properties of this game. We denoted to the N-bipolar game by NBSĢ.

Definition 5.1. Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ be an $NBST_S$ and $\mho$ be a collection of all NBS-open covers of $(\Psi, \zeta, \vartheta, N)$. We define infinitely long game NBSĢ$(\mho)$ as follows: play an inning for every positive integer. In the $\upsilon - th$ inning, ONE chooses an NBS open cover $(\Psi, \zeta)_\upsilon$ of $\mho$ and TWO responds by selecting a finite subset $(\text{Ḿ}, \zeta)_\upsilon$ of $(\Psi, \zeta)_\upsilon$ such that

$$(\text{ß}, \zeta)_\upsilon = \cup\{(\text{Ş}, \zeta) : (\text{Ş}, \zeta) \in (\text{Ḿ}, \zeta)_\upsilon\}$$

is an NBS-open set. ONE wins the play $(\Psi, \zeta)_1, (\text{ß}, \zeta)_1, \cdots, (\Psi, \zeta)_\upsilon, (\text{ß}, \zeta)_\upsilon, \cdots$ of game if $\{(\text{ß}, \zeta)_\upsilon, \upsilon \in \mathbb{N}\}$ is an NBS-open cover of $(\Psi, \zeta, \vartheta, N)$. Otherwise, TWO wins.

Corollary 5.2. *If $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ is an NBS-compact topological space, then TWO has a losing strategy in the game NBSĢ.*

Proof. Obviously, from Definition 5.1. □

Definition 5.3. Let $(\Psi, \zeta, \vartheta, N, \mathfrak{J}_{\vartheta}^{N})$ be an $NBST_S$ and $\mho$ be a collection of all NBS-open covers of $(\Psi, \zeta, \vartheta, N)$. We define infinitely long game NBSĢ$_D(\mho)$ (resp. NBSĢ$_O(\mho)$) as follows: They play an inning for each positive integer.

In the υ-th inning ONE chooses an NBS open cover $(\Psi, \zeta)_\upsilon$ of $\mho$ and TWO responds by selecting a nonempty finite subset $(\c{M}, \zeta)_\upsilon$ of $(\Psi, \zeta)_\upsilon$ such that

$$(ß, \zeta)_\upsilon = \cup\{(\c{S}, \zeta) : (\c{S}, \zeta) \in (\c{M}, \zeta)_\upsilon\}$$

is an NBS-open set. ONE wins the play $(\Psi, \zeta)_1, (ß, \zeta)_1, \cdots, (\Psi, \zeta)_\upsilon, (ß, \zeta)_\upsilon, \cdots$ of this game $NBS\c{G}_D(\mho)$ (resp. $NBS\c{G}_O(\mho)$) if $cl(\cup\{(ß, \zeta)_\upsilon, \upsilon \in \mathbb{N}\}) \in (\Psi, \zeta, \vartheta, N)$ (resp. $\cup\{cl(ß, \zeta)_\upsilon, \upsilon \in \mathbb{N}\} \in (\Psi, \zeta, \vartheta, N)$). Otherwise, TWO wins.

Lemma 5.4. *If $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ is an $NBST_S$, then the following statements are held:*

(1) *ONE has a winning strategy in the game $NBS\c{G}(\mho)$ implies ONE has a winning strategy in the game $NBS\c{G}_O(\mho)$.*
(2) *ONE has a winning strategy in the game $NBS\c{G}_O(\mho)$ implies ONE has a winning strategy in the game $NBS\c{G}_D(\mho)$.*

Proof. This is derived from the fact that

$$\{(ß, \zeta)_\upsilon, \upsilon \in \mathbb{N}\} \subseteq \cup cl\{(ß, \zeta)_\upsilon, \upsilon \in \mathbb{N}\} \subseteq cl \cup \{(ß, \zeta)_\upsilon, \upsilon \in \mathbb{N}\}$$

and by the *hereditary* property of an $NBST_S$. □

Lemma 5.5. *If $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ is an $NBST_S$ such that for each NBS open subset of $(\Psi, \zeta, \vartheta, N)$ is an N-bipolar closed, then the following statements are held:*

(1) *ONE has a winning strategy in the game $NBS\c{G}(\mho)$.*
(2) *ONE has a winning strategy in the game $NBS\c{G}_D(\mho)$.*
(3) *ONE has a winning strategy in the game $NBS\c{G}_O(\mho)$.*

Lemma 5.6. *If $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ is an $NBST_S$ such that for every NBS open subset of $(\Psi, \zeta, \vartheta, N)$ is dense, then the following statements are held:*

(1) *TWO has no winning strategy in the game $NBS\c{G}_D(\mho)$.*
(2) *TWO has no winning strategy in the game $NBS\c{G}_O(\mho)$.*

Proof. Omitted. □

Theorem 5.7. *Let $(\Psi_1, \zeta_1, \vartheta, N)$ be an NBS open subset of an $NBST_S$ on $(\Psi, \zeta, \vartheta, N, \mathfrak{J}^N_\vartheta)$ and $\mho_{(\Psi,\zeta)_1}$ be a collection of all NBS-open covers for $(\Psi_1, \zeta_1, \vartheta, N)$ with respect to the $NBST_{SS}$. Then the following statements are held:*

(1) *ONE has a winning strategy in the game $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$ implies ONE has a winning strategy in the game $NBS\c{G}(\mho)$.*

(2) *ONE has a winning strategy in the game* $NBS\c{G}_D(\mho_{(\Psi,\zeta)_1})$ *on* $(\Psi_1, \zeta_1, \vartheta, N)$ *implies ONE has a winning strategy in the game* $NBS\c{G}_D(\mho)$.

(3) *ONE has a winning strategy in the game* $NBS\c{G}_O(\mho_{(\Psi,\zeta)_1})$ *on* $(\Psi_1, \zeta_1, \vartheta, N)$ *implies ONE has a winning strategy in the game* $NBS\c{G}_O(\mho)$.

Proof. (1) Let $\mathfrak{L}'$ be a winning strategy for ONE in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$. Define $\mathfrak{L}$ to be a winning strategy for ONE in $NBS\c{G}(\mho)$ on $(\Psi, \zeta, \vartheta, N)$. Player ONE aims to make the first step $(\Psi,\zeta)_1' = \mathfrak{L}'(\phi)$ of player ONE in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$ and take

$$(\Psi,\zeta)_1 = \mathfrak{L}(\phi) = (\Psi,\zeta)_1' \cup ((\Psi,\zeta,\vartheta,N) - (\Psi_1,\zeta_1,\vartheta,N))$$

to be the first step in $NBS\c{G}(\mho)$ on $(\Psi, \zeta, \vartheta, N)$. TWO responds by selecting a nonempty N-bipolar soft finite subset $(\c{M},\zeta)_1$ of $(\Psi,\zeta)_1$ such that

$$(\text{ß},\zeta)_1 = \cup\{(S,\zeta) : (S,\zeta) \in (\c{M},\zeta)_1\}.$$

Now,

$$(\c{M},\zeta)_1' = \{(S,\zeta) \cup (\Psi_1,\zeta_1,\vartheta,N) : (S,\zeta) \in (\c{M},\zeta)_1\} \subseteq (\Psi,\zeta)_1',$$

where $(\text{ß},\zeta)_1' = \cup\{(\text{Ł},\zeta) : (\text{Ł},\zeta) \in (\c{M},\zeta)_1'\}$ is a reasonable step for TWO in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$. Player ONE aims to make the second step

$$(\Psi,\zeta)_2' = \mathfrak{L}'((\Psi,\zeta)_1', (\text{ß},\zeta)_1')$$

of player ONE in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$ and take

$$(\Psi,\zeta)_2 = \mathfrak{L}((\Psi,\zeta)_1, (\text{ß},\zeta)_1) = (\Psi,\zeta)_2' \cup ((\Psi,\zeta,\vartheta,N) - (\Psi_1,\zeta_1,\vartheta,N))$$

to be the second step in $NBS\c{G}(\mho)$ on $(\Psi, \zeta, \vartheta, N)$. TWO responds by selecting a non-empty finite subset $(\c{M},\zeta)_2$ of $(\Psi,\zeta)_2$ such that

$$(\text{ß},\zeta)_2 = \cup\{(S,\zeta) : (S,\zeta) \in (\c{M},\zeta)_2\}.$$

Now,

$$(\c{M},\zeta)_2' = \{(S,\zeta) \cup (\Psi_1,\zeta_1,\vartheta,N) : (S,\zeta) \in (\c{M},\zeta)_2\} \subseteq (\Psi,\zeta)_2',$$

where $(\text{ß},\zeta)_2' = \cup\{(\text{Ł},\zeta) : (\text{Ł},\zeta) \in (\c{M},\zeta)_2'\}$ is a reasonable step for TWO in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$. In υ-th inning, Player ONE aims to make the $\upsilon - th$ step

$$(\Psi,\zeta)_\upsilon' = \mathfrak{L}'((\Psi,\zeta)_1', (\text{ß},\zeta)_1', \cdots, (\Psi,\zeta)_{\upsilon-1}', (\text{ß},\zeta)_{\upsilon-1}')$$

of Player ONE in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$ and takes

$$\begin{aligned}(\Psi,\zeta)_2 &= \mathfrak{L}((\Psi,\zeta)_1, (\text{ß},\zeta)_1, \cdots, (\Psi,\zeta)_{\upsilon-1}, (\text{ß},\zeta)_{\upsilon-1}) \\ &= (\Psi,\zeta)_\upsilon' \cup ((\Psi,\zeta,\vartheta,N) - (\Psi_1,\zeta_1,\vartheta,N))\end{aligned}$$

to be the υ-th step in $NBS\c{G}(\mho)$ on $(\Psi, \zeta, \vartheta, N)$. TWO responds by selecting a nonemptyNBS finite subset $(\c{M}, \zeta)_\upsilon$ of $(\Psi, \zeta)_\upsilon$ such that

$$(\ss, \zeta)_\upsilon = \cup\{(S, \zeta) : (S, \zeta) \in (\c{M}, \zeta)_\upsilon.$$

Now,

$$(\c{M}, \zeta)_\upsilon^{'} = \{(S, \zeta) \cup (\Psi_1, \zeta_1, \vartheta, N) : (S, \zeta) \in (\c{M}, \zeta)_\upsilon\} \subseteq (\Psi, \zeta)_\upsilon^{'},$$

where $(\ss, \zeta)_\upsilon^{'} = \cup\{(\text{Ł}, \zeta) : (\text{Ł}, \zeta)\} \in (\c{M}, \zeta)_\upsilon^{'}$ is a reasonable step for TWO in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$. Now, since $\mathfrak{L}^{'}$ is a winning strategy for ONE in $NBS\c{G}(\mho_{(\Psi,\zeta)_1})$ on $(\Psi_1, \zeta_1, \vartheta, N)$ implies

$$\cup\{(\ss, \zeta)_\upsilon^{'}, \upsilon \in \mathbb{N}\} \in (\Psi_1, \zeta_1, \vartheta, N).$$

So, $\cup\{(\ss, \zeta_\upsilon, \upsilon \in \mathbb{N}\} \in (\Psi, \zeta, \vartheta, N)$. Hence, $\mathfrak{L}$ is a winning strategy for ONE in $NBS\c{G}(\mho)$ on $\Psi, \zeta, \vartheta, N)$.

In a similar way, we can prove (2) and (3). □

Definition 5.8. Let $(\Psi, \zeta, \vartheta, N, \mathfrak{I}_\vartheta^N)$ be an$NBST_S$. We define the infinitely long game $NBS\c{G}((\Psi, \zeta, \vartheta, N))$ as follows: They play an inning for each positive integer. In the n-th inning, ONE chooses NBS-compact subset $(\Psi, \zeta)_\upsilon$ of $(\Psi, \zeta, \vartheta, N)$ and TWO responds by selecting and choosing NBS-compact subset $(\c{M}, \zeta)_\upsilon$ of $(\Psi, \zeta, \vartheta, N)$ such that

$$(\Psi, \zeta)_\upsilon \cap (\c{M}, \zeta)_\upsilon = 0.$$

ONE wins the play $(\Psi, \zeta)_1, (\ss, \zeta)_1, \cdots, (\Psi, \zeta)_\upsilon, (\ss, \zeta)_\upsilon, \cdots$ the game if $\{(\ss, \zeta)_\upsilon : \upsilon \in \mathbb{N}\}$ is locally finite; else, TWO wins.

Now, we can use the natural game $NBSG_1(\gamma, \lambda)$ to explain the status of the two Players' odds of winning in the game $NBS\c{G}(\Psi, \zeta, \vartheta, N)$, where γ and λ are defined as follows:

A family $\gamma = (\c{M}, \zeta)_\upsilon : \upsilon \in \mathbb{N}$ of nonempty NBS-compact subsets of $(\Psi, \zeta, \vartheta, N)$ satisfying that for every NBS-compact subsets $(\Psi, \zeta)_1$ of $(\Psi, \zeta, \vartheta, N)$, there exists $(\Psi, \zeta)_2 \in (\c{M}, \zeta)_\upsilon$ such that $(\Psi, \zeta)_1 \cap (\Psi, \zeta)_2 = 0$.

If $(\Psi, \zeta, \vartheta, N)$ is non-NBS-compact space, then for every NBS-compact subset $(\Psi, \zeta)_1$ of $(\Psi, \zeta, \vartheta, N)$, there exists $\{x\} \subseteq (\Psi, \zeta, \vartheta, N) - (\Psi, \zeta)_1$ such that $\{x\} \cap (\Psi, \zeta)_1 = 0$.

Let $\lambda = \{(S, \zeta) : \upsilon \in \mathbb{N}\}$ be an infinite NBS locally finite family of NBS-compact subsets of $(\Psi, \zeta, \vartheta, N)$.

Theorem 5.9. *For any non-NBS-compact of $NBST_S$ $(\Psi, \zeta, \vartheta, N)$, TWO has a winning strategy in $NBS\c{G}((\Psi, \zeta, \vartheta, N))$ if and only if ONE has a winning strategy in $NBSG_1(\gamma, \lambda)$.*

Proof. Let $\mathfrak{I}$ be a winning strategy for TWO in $NBS\c{G}(\Psi,\zeta,\vartheta,N)$. Define $\mathfrak{L}$ to be a winning strategy for ONE in $NBSG_1(\gamma,\lambda)$.

Let $(\Psi,\zeta)_1$ be the first step of ONE in $NBS\c{G}(\Psi,\zeta,\vartheta,N)$ and

$$(\ss,\zeta)_1 = \{(S,\zeta) : (S,\zeta) \text{ is NBS-compact subset of } (\Psi,\zeta,\vartheta,N) \text{ such that } \mathfrak{I}((\Psi,\zeta)_1) = (S,\zeta)\}.$$

I am playing that $(\ss,\zeta)_1 \in \gamma$ and $(\ss,\zeta)_1$ is a reasonable step to be the first selecting of ONE in $NBSG_1(\gamma,\lambda)$. TWO, in $NBSG_1(\gamma,\lambda)$ responds by selecting $(\c{M},\zeta)_1 = \mathfrak{I}((\ss,\zeta)_1)$ since $(\c{M},\zeta)_1 \in (\ss,\zeta)_1$, then

$$(\Psi,\zeta)_1 \cap (\c{M},\zeta)_1 = 0 \text{ and } \mathfrak{I}((\Psi,\zeta)_1) = (\c{M},\zeta)_1$$

is the first step for TWO in $NBS\c{G}(\Psi,\zeta,\vartheta,N)$.

Second step, let $(\Psi,\zeta)_2$ be the second step of ONE in $NBS\c{G}(\Psi,\zeta,\vartheta,N)$ and

$$(\ss,\zeta)_2 = \{(S,\zeta) : (S,\zeta)\text{is NBS-compact subset of}(\Psi,\zeta,\vartheta,N) \text{ such that } \mathfrak{I}((\Psi,\zeta)_1,(\c{M},\zeta)_1,(\Psi,\zeta)_2) = (S,\zeta)\}.$$

Implies, $(\ss,\zeta)_2 \in \gamma$ and $(\ss,\zeta)_2$ is a reasonable step to be the second selecting of ONE in $NBSG_1(\gamma,\lambda)$. TWO, in $NBSG_1(\gamma,\lambda)$, responds by selecting $(\c{M},\zeta)_2 = \mathfrak{L}((\ss,\zeta)_1,(\c{M},\zeta)_1,(\ss,\zeta)_2)$, since $(\c{M},\zeta)_2 \in (\ss,\zeta)_2$, then

$$\mathfrak{I}((\Psi,\zeta)_1,(\c{M},\zeta)_1,(\Psi,\zeta)_2) = (\c{M},\zeta)_2 \text{ and } (\Psi,\zeta)_2 \cap (\c{M},\zeta)_2 = 0.$$

Now, in the υ-th step of $NBS\c{G}(\Psi,\zeta,\vartheta,N)$, ONE chooses $(\Psi,\zeta)_n$.

Let

$$(\ss,\zeta)_2 = \{(S,\zeta) : (S,\zeta) \text{ is NBS compact subset of } (\Psi,\zeta,\vartheta,N) \text{ such that } \mathfrak{I}((\Psi,\zeta)_1,(\c{M},\zeta)_1,\cdots,(\Psi,\zeta)_{\upsilon-1},(\c{M},\zeta)_{\upsilon-1},(\Psi,\zeta)_\upsilon) = (S,\zeta)\}.$$

Implies, $(\ss,\zeta)_n \in \gamma$ and $(\ss,\zeta)_\upsilon$ is a reasonable step to be the υ-th selecting of ONE in $NBSG_1(\gamma,\lambda)$. TWO, in $NBSG_1(\gamma,\lambda)$, responds by selecting $(\c{M},\zeta)_\upsilon = \mathfrak{L}((\ss,\zeta)_1,(\c{M},\zeta)_1,\cdots,(\ss,\zeta)_{\upsilon-1},(\c{M},\zeta)_{\upsilon-1},(\ss,\zeta)_\upsilon)$ since $(\c{M},\zeta)_\upsilon \in (\ss,\zeta)_\upsilon$, then

$$\mathfrak{I}((\Psi,\zeta)_1,(\c{M},\zeta)_1,\cdots,(\Psi,\zeta)_{\upsilon-1},(\c{M},\zeta)_{\upsilon-1},(\Psi,\zeta)_\upsilon) = (\c{M},\zeta)_\upsilon$$

and

$$(\Psi,\zeta)_\upsilon \cap (\c{M},\zeta)_\upsilon = 0.$$

Since $\mathfrak{I}$ is a winning strategy for TWO in $NBS\c{G}(\Psi,\zeta,\vartheta,N)$, then $\{(\c{M},\zeta)_\upsilon : \upsilon \in \mathbb{N}\}$ is not a locally finite family. Therefore, $\{(\c{M},\zeta)_\upsilon : \upsilon \in \mathbb{N}\} \notin \lambda$. Hence, $\mathfrak{L}$ is a winning strategy for ONE in $NBSG_1(\gamma,\lambda)$.

Conversely, let $\mathfrak{L}$ be a winning strategy for ONE in $NBSG_1(\gamma,\lambda)$. Define $\mathfrak{I}$ to be a winning strategy for TWO in $NBS\c{G}(\Psi,\zeta,\vartheta,N)$.

Player ONE in $NBS\c{G}(\Psi,\zeta,\vartheta,N)$ choose an NBS compact subset $(\Psi,\zeta)_1$ of (Ψ,ζ,ϑ,N). TWO in $(NBS\c{G}(\Psi,\zeta,\vartheta,N)$ aims to make the first step for player ONE in $NBSG_1(\gamma,\lambda)$, say $(ß,\zeta)_1 = \mathfrak{L}(\phi)$, and choose $(Ṃ,\zeta)_1 \in (ß,\zeta)_1$ such that

$$(\Psi,\zeta)_1 \cap (Ṃ,\zeta)_1 = \phi,$$

which is possible by definition of γ, $\mathfrak{I}((\Psi,\zeta)_\infty) = (Ṃ,\zeta)_\infty$ which is reasonable to be the first step for TWO in $NBSG_1(\gamma,\lambda)$.

In the second step, Player ONE in $NBSĢ(\Psi,\zeta,\vartheta,N)$ chooses an NBS compact subset $(\Psi,\zeta)_2$ of (Ψ,ζ,ϑ,N). TWO in $NBSĢ(\Psi,\zeta,\vartheta,N)$, aims to make the second step for player ONE in $NBSG_1(\gamma,\lambda)$, say $(ß,\zeta)_2 = \mathfrak{L}((ß,\zeta)_1,(Ṃ,\zeta)_1)$, and choose $(Ṃ,\zeta)_2 \in (ß,\zeta)_2$ such that

$$(\Psi,\zeta)_2 \cap (Ṃ,\zeta)_2 = 0$$

and $\mathfrak{I}((\Psi,\zeta)_1,(Ṃ,\zeta)_1,(\Psi,\zeta)_2) = (Ṃ,\zeta)_2$ is reasonable to be the second step for TWO in $NBSG_1(\gamma,\lambda)$.

In the υ-thinning, Player ONE in $NBSĢ(\Psi,\zeta,\vartheta,N)$ chooses an NBS compact subset $(\Psi,\zeta)_\upsilon$ of (Ψ,ζ,ϑ,N). TWO in $NBSĢ(\Psi,\zeta,\vartheta,N)$ aims to make the υ-th step for player ONE in $NBSG_1(\gamma,\lambda)$, say

$$(ß,\zeta)_\upsilon = \mathfrak{L}((ß,\zeta)_2,(Ṃ,\zeta)_1,\cdots,(ß,\zeta)_{\upsilon-1},(Ṃ,\zeta)_{\upsilon-1})$$

and choose $(Ṃ,\zeta)_\upsilon \in (ß,\zeta)_\upsilon$ such that

$$(\Psi,\zeta)_\upsilon \cap (Ṃ,\zeta)_\upsilon = 0$$

and $\mathfrak{I}((\Psi,\zeta)_1,(Ṃ,\zeta)_1,...,(\Psi,\zeta)_{\upsilon-1},(Ṃ,\zeta)_{\upsilon-1},(\Psi,\zeta)_\upsilon = (Ṃ,\zeta)_\upsilon$ is reasonable to be the υ-th step for TWO in $NBSG_1(\gamma,\lambda)$. Since $\mathfrak{L}$ is a winning strategy for ONE in $NBSG_1(\gamma,\lambda)$, then $\{(Ṃ,\zeta)_\upsilon : \upsilon \in \mathbb{N}\}$ is not a locally finite family. Therefore, $\mathfrak{L}$ is a winning strategy for TWO in $NBSĢ(\Psi,\zeta,\vartheta,N)$. □

6. Conclusion

In this work, we study and about defining and finding out the properties of N-bipolar soft connected spaces and N-bipolar soft disconnected spaces. We show that the N-bipolar soft union of N-bipolar soft connected spaces are not necessarily NBS connected. Moreover, we define the N-bipolar soft compact space. We introduce a new game by using N-bipolar soft open covering and study some of the characterizations and properties of this game.

Adding this concept will also help strengthen the foundation of the NBS topologies toolkit. It is expected that the results of this paper will be applied to problems in many domains that contain uncertainties and will encourage further research on N-bipolar soft topology to provide a general framework for practical life applications.

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