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THEOREMS ON DIFFERENTIAL SUBORDINATION DEFINED BY MEROMORPHIC MULTIVALENT FUNCTIONS

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Abstract. By using some theorems on differential subordination we get in my work‘s some applications of second order differential subordination defined by meromorphic multivalent functions with integral operator.

1. Introduction

The class Ψ_p denote of the series of the form:

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, \dots\},\ a_k \geqslant 0), \tag{1.1}$$

which are p-valent and analytic in the domain $Y = \{z : z \in \mathbb{C}, 0 < |z| < 1\}$. We called that f is subordinate to g in Y, if f and g are analytic functions in Y. We denote $f \prec g$ or $f(z) \prec g(z)$, if there exists a function called Schwarz

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$w(z)$ in Y, $w(0) = 0$ with $|w(z)|$ less than 1 such that

$$f(z) = g(w(z)), \ (z \in Y).$$

In this case, if the g is univalent function in Y, then $f \prec g$, $g(0) = f(0)$, and $f(Y) \subset g(Y)$ ([1], [14]).

For all function f, we get it through the equation (1.1) and $g \in \psi_p$ it will be in shape

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k,$$

when f and g is Hadamard product of defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} a_k b_k z^k.$$

All functions f are analytic and one to one on $Y/E(f)$, we give it through Q, where

$$E(f) \ = \ \{\zeta \in \partial p : \lim_{z \to \zeta} f(z) = \infty\}$$

such that $\acute{f}(\zeta) \neq 0$ for all $\zeta \in \partial Y \backslash E(f)$ ([15]).

Suppose $\psi : \mathbb{C}^3 \times Y \to \mathbb{C}$, and define the univalent function h in Y with $q \in Q$. At the source ([10]) we assume the problem of determine acceptable conditions functions ψ such that

$$\varkappa(p(z), zp\acute{(}z), z^2 p\acute{\acute{(}}z); z) \prec h(z). \tag{1.2}$$

Leads to $p(z) \prec q(z)$, for all functions $p(z)$ belong to $\mathcal{L}[a, n] = \{f \in \mathcal{L} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$, where $\mathcal{L}$ is linear space of all analytic in Y, $a \in \mathbb{C}$ and n is positive integer that satisfying the differential subordination (1.2).

Furthermore, we will try to find conditions, lets on that q is the last possible function with the property, named in the subordination (1.2) the best dominant.

Suppose $\phi : \mathbb{C}^3 \times Y \to \mathbb{C}$, and $h \in \mathcal{L}$ with $q \in \mathcal{L}[a, n]$. Over a group of researches in the same field ([11], [12]), Miller and Mocanu studied this topic and deepened condition the limited it ϕ such that

$$h(z) \prec \varphi(p(z), zp\acute{(}z), z^2 p\acute{\acute{(}}z); z) \tag{1.3}$$

implies $q(z) \prec p(z)$, for all functions $p \in Q$ it achieves the pervious condition superordination. We also investigated the conditions the lead to the function being formed q the bigger function with this property, named in the superordination (1.3) best subordinant. Look the others [1], [2], [3], [4], [5] and [6], they worked in the same filed of research as ours differential subordination functions for different classes.

Now, we use the operator $\Psi_p^m : \Psi_p \to \Psi_P$ ([14]) define in the form

$$\Psi_p^m f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} [\frac{1}{k+p+1}]^m a_n z^n, \; (m \geq 0; p \in \mathcal{N}). \tag{1.4}$$

It is easily verified from (1.4) that,

$$z(\Psi_p^m f(z))^{'} = \Psi_p^{m-1} f(z) - (P=1)\Psi_p^m f(z). \tag{1.5}$$

Through (1.5), we will notice

$$(\varkappa+T)z(\Psi_p^{m+2} f(z))^{'} = (\varphi+p)(\Psi_p^{m+1} f(z)) - (\Psi + p(1-(\varkappa+T)))(\Psi_p^{m+2} f(z)). \tag{1.6}$$

In research, we identify some important properties on the acceptable functions to get by the operator $\Psi_p^m(m, z, \varkappa)$.

2. Preliminaries

To get some useful results, we use the important lemmas.

Lemma 2.1. ([9]) *Suppose q is a univalent function in $Y, \zeta \in \mathbb{C}$ and let*

$$Re\{\frac{z\acute{\acute{p}}(z)}{\acute{p}(z)} + 1\} > \max\{0, -Re(\frac{1}{\zeta})\}. \tag{2.1}$$

If $p(z)$ is an analytic function in Y, and $p(0) = q(0)$ and

$$p(z) + \zeta z\acute{p}(z) \prec q(z) + \zeta z\acute{q}(z), \tag{2.2}$$

then $p(z) \prec q(z)$, and best dominant is $q(z)$.

Lemma 2.2. ([13]) *Suppose $q(z)$ is univalent in Y, and suppose the analytic functions θ, φ in domain D contain in $q(Y)$ and $\varphi(w) \neq 0$ where w is long to $q(Y)$. Let $Q(z) = zq(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Let*

(1) *Q is the starlike univalent in Y.*

(2) *$Re\{\frac{zh^{'}(z)}{(Q(z))}\}$ is grater than zero for all $z \in Y$.*

If the analytic p and $p(0) = q(0), p(Y) \subseteq D$ and

$$\theta(p(z)) + zp^{'}(z)\varphi(p(z)) \prec \theta(q(z)) + zq^{'}(z)\varphi(q(z)), \tag{2.3}$$

then $p \prec q$, and the best dominant is $q(z)$.

Lemma 2.3. ([7]) *Suppose $q(z)$ is a convex function in $Y, q(0) = a$ and $\zeta \in \mathbb{C}, Re(\zeta) > 0$. If $p \in \mathcal{L}[a, 1]$ and $p(z) + \gamma z q^{'}(z)$ is univalent function in Y then*

$$q(z) + \zeta q^{'}(z) \prec p(z) + \zeta p^{'}(z). \tag{2.4}$$

Leads to $q(z) \prec p(z)$, it best subordinant is $q(z)$.

Lemma 2.4. ([8]) *Suppose $q(z)$ is a convex univalent function in the disk Y and suppose θ, φ are analytic in D contain $q(Y)$. Let*

(1) $Re\{\frac{\theta^{'}(q(z))}{\varphi(q(z))}\} > 0, \ \ \forall z \in Y,$

(2) *$zq^{'}(z) + \varphi(q(z))$ is univalent starlike function in Y.*

If $p(z) \in \mathcal{L}[q(0), 1] \cap Q$, $p(Y) \subseteq D$, and $\theta(p(z)) + zp^{'}z)\varphi(p(z))$ is univalent in Y, and

$$\theta(q(z)) + zq^{'}(z)\varphi(q(z)) \prec \theta(p(z)) + zp^{'}(z)\varphi(p(z)), \tag{2.5}$$

then $q(z) \prec p(z)$ and the best subordinant is $q(z)$.

3. Main results

Suppose in the mention of this research that $\varkappa > 0$, $\eta, T \geq 0$; $p \in \mathcal{N}$, $q \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}$; $z \in Y$ and the powers are understood as principle values.

Theorem 3.1. *Suppose $q(z)$ is a univalent function in Y with $q(0) = 0, \gamma > 0$ and let*

$$Re\Big\{1 + \frac{zq^{''}(z)}{q^{'}(z)}\Big\} > \max\Big\{0, -Re(\frac{\sigma(\eta + p)}{T + \varkappa})\Big\}. \tag{3.1}$$

If $f \in \Psi_p$ satisfies the subordination

$$\Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma} + \Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma}\Big(\frac{\Psi_p^{m+1} f(z)}{\psi_p^{m+2} f(z)} - 1\Big) \prec q(z) + \frac{T + \varkappa}{\sigma(\eta + p)} zq^{'}(z), \tag{3.2}$$

then

$$\Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma} \prec q(z)$$

and the best dominant is $q(z)$.

Proof. Regarding the analytic function

$$\Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma}, \ \ \sigma > 0, \ \ z \in Y. \tag{3.3}$$

Differentiating (3.3) and by using some logarithmically with respect to z and use the equation (1.6) in the resulting and we get

$$\frac{zp^{'}(z)}{p(z)} = \frac{\sigma(\eta + p)}{\varkappa + T}\Big(\frac{\Psi_p^{m+1} f(z)}{\Psi_p^{m+2} f(z)} - 1\Big), \tag{3.4}$$

that is,

$$\frac{\varkappa + T}{\sigma(\eta + p)} zp^{'}(z) = \Big(\frac{\psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma}\Big(\frac{\psi_p^{m+1} f(z)}{\psi_p^{m+2} f(z)} - 1\Big).$$

Then, the (3.2) is equivalent to

$$p(z) + \frac{T + \varkappa}{\sigma(\eta + p)} zp^{'}(z) \prec q(z) + \frac{T + \varkappa}{\sigma(\eta + p)} zq^{'}(z). \tag{3.5}$$

By using Lemma 2.1, with $\zeta = \frac{T+\varkappa}{\sigma(\eta+p)}$. □

We choose the convex function $f(z) = \frac{1+A_1 z}{1+A_2 z}$, in (3.1), we get the following corollary.

Corollary 3.2. *Suppose $A_1, A_2 \in \mathbb{C}$, $A_1 \neq A_2$, $|A_2| < 1$, $Re(\zeta) > 0$ and $\sigma > 0$. If $f(z) \in \psi_p$ satisfies the subordination*

$$\Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma} + \Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma}\Big(\frac{\Psi_p^{m+1} f(z)}{\Psi_p^{m+2} f(z)} - 1\Big) \prec \frac{1 + A_1 z}{1 + A_2 z} + \frac{T + \varkappa(A_1 - A_2 z)}{\sigma(\eta + p)(1 - A_2 z)^2},$$

then

$$\Big(\frac{\Psi_p^{m+2} f(w)}{z^p}\Big)^{\sigma} \prec \frac{1 + A_1 z}{1 + A_2 z}$$

and $\frac{1+A_1 z}{1+A_2 z}$ is the best dominant.

Choose $q = 0$ in (3.1), we get the next corollary.

Corollary 3.3. *Suppose $q(z)$ is a univalent function in the disk Y, $q(0) = 1$, $\sigma > 0$, and let the condition (3.1) satisfy. If $f(z) \in \Psi_p$ hold the subordination*

$$(\frac{\Psi_p^2 f(z)}{z^p})^{\sigma} + (\frac{\Psi_p^2 f(z)}{z^p})^{\sigma}(\frac{\Psi_p^1 f(z)}{\Psi_p^2 f(z)} - 1) \prec q(z) + \frac{T + \varkappa}{\sigma(\varkappa + p)} zq^{'}(z),$$

then

$$\Big(\frac{\Psi_p^2 f(z)}{z^p}\Big)^{\sigma} \prec q(z)$$

and the best dominant is $q(z)$.

Choose $\eta = \varkappa = 1$ in (3.1), we get the corollary.

Corollary 3.4. *Suppose $q(z)$ is a univalent function in Y, with $q(0) = 1$, $\beta \in \mathbb{C}, \eta > 0$, and let (3.1) satisfy. If $f \in \Psi_p$ satisfies the subordination*

$$\Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma} + \Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma}\Big(\frac{\Psi_p^{m+1} f(z)}{\Psi_p^{m+2} f(z)} - 1\Big) \prec q(z) + \frac{1 + T}{\sigma(1 + p)} zq^{'}(z),$$

then

$$\Big(\frac{\Psi_p^{m+2}f(w)}{z^p}\Big)^{\sigma} \prec q(z)$$

and the best dominant is $q(z)$.

Theorem 3.5. *Suppose* $q(z)$ *is meromorphic in* p, $q(0)=1$ *and* $q(z)\neq 0$ *for all* $z\in Y$. *Suppose* $\lambda,\sigma\in\mathbb{C}, f\in\Psi_p$ *and* f, q *satisfy the conditions:*

$$\Big(\frac{\Psi_p^{m+2}f(w)}{z^p}\Big)^{\sigma} \neq 0 \tag{3.6}$$

and

$$Re\Big\{1+\frac{zq^{''}(z)}{q^{'}(z)}-\frac{zq^{'}(z)}{q(z)}\Big\}>0,\ \ (z\in Y). \tag{3.7}$$

If

$$\frac{v\Psi_p^{m+1}f(z)}{\Psi_p^{m+2}f(w)} \prec 1+\frac{(T+\varkappa)zq^{'}(z)}{\sigma(\eta+p)q(z)^{'}}, \tag{3.8}$$

then

$$\Big(\frac{\Psi_p^{m+2}f(w)}{z^p}\Big)^{\sigma} \prec q(z)$$

and the best dominant is $q(z)$.

Proof. Let

$$p(z)=\Big(\frac{\Psi_p^{m+2}f(w)}{z^p}\Big)^{\sigma},\ z\in Y. \tag{3.9}$$

Then $p(z)$ is analytic in Y according to (3.4) and differentiable (3.9) logarithmically with respect to z, we get

$$\frac{zp^{'}(z)}{p(z)}=\frac{\sigma(\Phi+p)}{(T+\varkappa)}\Big(\frac{\Psi_p^{m+1}f(z)}{\Psi_p^{m+2}f(z)}-1\Big). \tag{3.10}$$

To reach the desired result we will use Lemma 2.2. Let

$$\theta(w)=1 \text{ and } \varphi(w)=\frac{T+\varkappa}{\sigma(\eta+p)w}.$$

Then θ is analytic function in $\mathbb{C}$ and $\varphi(w)\neq 0$ is analytic in $\mathbb{C}$.

Also, if we suppose

$$Q(z)=zq^{'}(z)\varphi(q(z))=\frac{(T+\varkappa)zq^{'}(z)}{\sigma(\eta+p)q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=1+\frac{(T+\varkappa)zq^{'}(z)}{\sigma(\eta+p)q(z)}.$$

Then by using (3.7), $Q(z)$ will be starlike function in Y. Also we have

$$Re\{\frac{zh^{'}(z)}{Q(z)}\} = Re\{1 + \frac{zq^{''}(z)}{q^{'}(z)} - \frac{zq^{'}(z)}{q(z)}\} > 0, \quad (z \in Y).$$

So, use Lemma 2.2, then we will get the subordination (3.6), this implies

$$p(z) \prec q(z).$$

This means that $q(z)$ is the best dominant of (3.8). □

Applying $q(z) = \frac{1+A_1 z}{1+A_2 z}(-1 \leq A_2 < A_1 \leq 1)$ in this theorem, we will get the desired result.

Corollary 3.6. *Suppose* $\sigma \in \mathbb{C}$, $f(z) \in \Psi_p$ *and let*

$$\frac{\Psi_p^{m+2} f(z)}{z^p} \neq 0, \quad (z \in Y).$$

If

$$\frac{\Psi_p^{m+1} f(z)}{\Psi_p^{m+2} f(z)} \prec 1 + \frac{(T + \varkappa)z(A_1 - A_2)}{\sigma(\eta + p)(1 + A_1 z)(1 + A_2 z)},$$

then

$$\Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma} \prec \frac{1 + A_1 z}{1 + A_2 z}$$

and the best dominant is $q(z) = \frac{1+A_1 z}{1+A_2 z}$.

Applying $q(z) = \frac{1+A_1 z}{1+A_2 z}$ inside (3.2), we will get the following corollary.

Corollary 3.7. *Suppose* $\sigma \in \mathbb{C}$, $f(z) \in \Psi_p$ *and let*

$$\frac{\Psi_p^{m+2} f(z)}{z^p} \neq 0, \quad z \in Y.$$

If

$$\frac{\Psi_p^{m+1} f(z)}{\Psi_p^{m+2} f(z)} \prec 1 + \frac{2(T + \varkappa)z}{\sigma(\eta + p)(1 - z)(1 + z)},$$

then

$$\Big(\frac{\Psi_p^{m+2} f(z)}{z^p}\Big)^{\sigma} \prec \frac{1 + z}{1 - z}$$

and the best dominant is $q(z) = \frac{1+z}{1-z}$.

Theorem 3.8. *Suppose $q(z)$ is meromorphic in Y with $q(0)=1$, let σ belong to $\mathbb{C}$, $\varkappa, v, \mathcal{U\!U} \in \mathbb{C}$ and $v+\mathcal{U\!U} \neq 0$. Let $f \in \Psi_p$, f and q satisfy the conditions:*

$$\frac{v\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}{(v+\mathcal{U\!U})z^p} \neq 0, \ (z \in Y) \tag{3.11}$$

and

$$Re\{1-\frac{zq^{''}(z)}{q^{'}(z)}\} > max\{0, -Re(\varkappa)\}, \ (z \in Y). \tag{3.12}$$

If

$$\begin{aligned} k(z) = \gamma\Big[&\frac{v\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}{(v+\mathcal{U\!U})z^p}\Big]^{\sigma} \\ &+\sigma\Big[\frac{vz(\Psi_p^{m+1}f(z))^{'}+\mathcal{U\!U}z(\Psi_p^{m+2}f(z))^{'}}{v\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}-p\Big] \end{aligned} \tag{3.13}$$

and

$$k(z) \prec \gamma q(z) + \frac{zq^{'}(z)}{q(z)^{'}}, \tag{3.14}$$

then

$$\Big[\frac{\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}{(v+\mathcal{U\!U})z^p}\Big]^{\sigma} \prec q(z)$$

and the best dominant is $q(z)$ in (3.11).

Proof. Let

$$p(z) = \Big[\frac{v\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}{(v+\mathcal{U\!U})z^p}\Big]^{\sigma}, \ (z \in Y). \tag{3.15}$$

Then $p(z)$ is analytic in Y according to (3.8) and differentiable (3.15) logarithmically to z, and we get

$$\frac{zp^{'}(z)}{p(z)} = \sigma\Big[\frac{vz(\Psi_p^{m+1}f(z))^{'}+\mathcal{U\!U}z(\Psi_p^{m+2}f(z))^{'}}{v\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}-p\Big] \tag{3.16}$$

and

$$\begin{aligned} zp^{'}(z) = \sigma[&\frac{v\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}{(v+\eta)z^p}]^{\sigma} \\ &\times\Big[\frac{vz(\Psi_p^{m+1}f(z))^{'}+\mathcal{U\!U}z(\Psi_p^{m+2}f(z))^{'}}{v\Psi_p^{m+1}f(z)+\mathcal{U\!U}\Psi_p^{m+2}f(z)}-p\Big]. \end{aligned}$$

To reach the desired result, we use Lemma 2.2 and in (2.2) consider

$$\theta(w) = \gamma w \text{ and } \varphi(w) = \frac{1}{w^{'}}$$

then θ is analytic in $\mathbb{C}$ with $\varphi(w) \neq 0$ is analytic in $\mathbb{C}$. And suppose

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma\Big[\frac{vz(\Psi_p^{m+1}f(z))' + \mathcal{U}z(\Psi_p^{m+2}f(z))'}{v\Psi_p^{m+1}f(z) + \mathcal{U}\Psi_p^{m+2}f(z)} - p\Big]$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \gamma\Big[\frac{v\Psi_p^{m+1}f(z) + \mathcal{U}\Psi_p^{m+2}f(z)}{(v+\mathcal{U})z^p}\Big]^{\sigma} \\ &\quad + \sigma\Big[\frac{vz(\Psi_p^{m+1}f(z))' + \mathcal{U}z(\Psi_p^{m+2}f(z))'}{v\Psi_p^{m+1}f(z) + \mathcal{U}\psi_p^{m+2}f(z)} - p\Big]. \end{aligned}$$

Then, by (3.11), $Q(z)$ is a starlike function in Y. And we have

$$Re\Big\{\frac{zh'(z)}{Q(z)}\Big\} = Re\Big\{\gamma + 1 + \frac{zq''(z)}{q'(z)}\Big\} > 0, \ (z \in Y),$$

we deduce that the subordination (3.14) by using Lemma 2.2, this implies

$$p(z) \prec q(z).$$

Use $q(z) = \frac{1+A_1z}{1+A_2z}(-1 \leq A_2 < A_1 \leq 1)$ in this theorem and applying (3.4), then the condition (3.12) will be

$$\max\{0, -Re(\gamma)\} \leq \frac{1+|A_2|}{1+|A_2|}.$$

□

Now, if $v = 1$ *and* $\mathcal{U} = 0$, we will get the following corollary.

Corollary 3.9. *Suppose* $\gamma \in \mathbb{C}$ *and*

$$\max\{0, -Re(\gamma)\} \leq \frac{1+|A_2|}{1+|A_2|}.$$

Suppose $f(z) \in \Psi_p$ *and let*

$$\frac{\Psi_p^{m+1}f(z)}{z^p} \neq 0, \ (z \in Y).$$

If

$$\gamma\Big[\frac{v\Psi_p^{m+1}f(z)}{z^p}\Big]^{\sigma} + \sigma\Big[\frac{vz(\Psi_p^{m+1}f(z))'}{\Psi_p^{m+1}f(z)} - p\Big] \prec \gamma\frac{1+A_1z}{1+A_2z} + \frac{(A_1 - A_2)z}{(1+A_1z)(1+A_2z)},$$

then

$$\Big[\frac{v\Psi_p^{m+1}f(z)}{z^p}\Big]^{\gamma} \prec \frac{1+A_1z}{1+A_2z}$$

and the best dominant is $q(z) = \frac{1+A_1 z}{1+A_2 z}$.

Now, we use $p = v = q = 1$, $\mathcal{U}\!\mathcal{U} = 0$ and $q(z) = \frac{1+z}{1-z}$ by Theorem 3.8, we will get following corollary.

Corollary 3.10. *Suppose* $f(z) \in \Psi_p$,

$$\frac{\Psi_p^2 f(z)}{z^p} \neq 0, \ (z \in Y)$$

and $\sigma \in \mathbb{C}$. *If*

$$\gamma\Big[\frac{\Psi_p^2 f(z)}{z}\Big]^{\sigma} + \sigma\Big[\frac{z\big(\Psi_p^2 f(z)\big)'}{\Psi_p^2 f(z)} - 1\Big] \prec \gamma\frac{1+z}{1-z} + \frac{2z}{(1+z)(1-z)},$$

then

$$\Big[\frac{\Psi_p^2 f(z)}{z}\Big]^{\gamma} \prec \frac{1+z}{1-z}$$

and the best dominant is $q(z) = \frac{1+z}{1-z}$.

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