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FUNCTIONAL EQUATIONS RELATED TO GP MAPPINGS OF DEGREE AT MOST n

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Abstract. Let V and W be real vector spaces, in this paper, we prove that a mapping $f : V \to W$ satisfies a specific functional equation (1.1) if and only if there exist mappings $f_1, f_2, \ldots, f_n : V \to W$ satisfying $f(x) = \sum_{k=1}^{n} f_k(x)$ and $f_k(ax) = a_k f_k(x)$ for all $x \in V$ and $k \in \{1, 2, \ldots, n\}$, where a is a real number with $a > 1$ and $0 < a_1 < a_2 < \cdots < a_n$.

1. Introduction

Albert and Baker mentioned in their paper [1] the following theorem, which was later proved by Djoković [3]:

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Theorem 1.1. (Albert and Baker) *Let V and W be some real vector spaces. For every $n \in \mathbb{N}$ and mapping $f : V \to W$, the following are equivalent.*

(i) $\underset{y}{\overset{n+1}{\Delta}} f(x) = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^{n+1-i} f(x+iy) = 0$ *for all $x, y \in V$.*

(ii) *There exist mappings $f_0, f_1, \ldots, f_n$ such that $f(x) = \sum_{k=0}^{n} f_k(x)$ and $\underset{y}{\overset{k}{\Delta}} f_k(x) - k! f_k(y) = 0$ for all $x, y \in V$ and all $k \in \{0, 1, \ldots, n\}$.*

Every solution to the functional equation $\underset{y}{\overset{n}{\Delta}} f(x) = 0$ is said to be a generalized polynomial mapping (or GP mapping) of degree at most $n-1$ (see [2]) and every nonzero solution mapping to the functional equation $\underset{y}{\overset{n}{\Delta}} f(x) - n! f(y) = 0$ is said to be a monomial mapping of degree n.

Throughout this paper, let V and W be real vector spaces, n be a fixed positive integer, a be a real number greater than 1, and $A = \{a_1, a_2, \ldots, a_n\}$ satisfy $0 < a_1 < a_2 < \cdots < a_n$.

We now present the main theorem of this paper, which is a direct consequence of Corollary 2.7 and Lemma 2.9.

Theorem 1.2. *For every $n \in \mathbb{N}$ and mapping $f : V \to W$, the following are equivalent.*

(i) *A mapping $f : V \to W$ satisfies the functional equation*

$$f(a^n x) + \sum_{m=1}^{n} (-1)^m \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} a_{i_1} a_{i_2} \cdots a_{i_m} f(a^{n-m} x) = 0 \tag{1.1}$$

for all $x \in V$.

(ii) *There exist mappings $f_1, f_2, \ldots, f_n : V \to W$ such that $f(x) = \sum_{k=1}^{n} f_k(x)$ and*

$$f_k(ax) = a_k f_k(x) \tag{1.2}$$

for any $x \in V$ and $k \in \{1, 2, \ldots, n\}$.

In particular, the mappings $f_1, f_2, \ldots, f_n$ can be expressed as

$$f_k(x) = \frac{f\left(a^{n-1} x\right) + \sum_{m=1}^{n-1} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \ldots, i_m \neq k}} a_{i_1} a_{i_2} \cdots a_{i_m} f\left(a^{n-m-1} x\right)}{\prod_{1 \le i \le n,\, i \neq k} (a_k - a_i)} \tag{1.3}$$

for all $x \in V$ and $k \in \{1, 2, \ldots, n\}$.

Remark: *If f_k satisfies the functional equation $\underset{y}{\overset{k}{\Delta}} f_k(x) - k! f_k(y) = 0$ for all $x, y \in V$ and all $k \in \{0, 1, \ldots, n\}$, then $f_k(rx) = r^k f_k(x)$ for all $x \in V$ and all rational numbers $r \in \mathbb{Q}$ (see Theorem 3 and Corollary 3 in* [3] *and the introductory section of* [2]*).*

Therefore, by Theorem 1.1 and Remark, if a mapping $f : V \to W$ satisfies the functional equation $\underset{y}{\overset{n+1}{\Delta}} f(x) = 0$ for all $x, y \in V$ and $f(0) = 0$, then there exist mappings $f_1, f_2, \ldots, f_n$ such that $f(x) = \sum_{k=1}^{n} f_k(x)$ and $f_k(rx) = r^k f_k(x)$ whenever $x \in V$ and $r \in \mathbb{Q}$. And, by our main theorem, f satisfies the functional equation

$$f(r^n x) + \sum_{m=1}^{n} (-1)^m \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} r^{i_1} r^{i_2} \cdots r^{i_m} f(r^{n-m} x) = 0$$

for all $x \in V$ and rational numbers $r > 1$.

As a result, if a mapping $f : V \to W$ satisfies the functional equation $\underset{y}{\overset{n+1}{\Delta}} f(x) = 0$ for all $x, y \in V$ and $f(0) = 0$, then we can have $f(x) = \sum_{k=1}^{n} f_k(x)$, where

$$f_k(x) = \frac{f\big(r^{n-1}x\big) + \sum_{m=1}^{n-1} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \ldots, i_m \ne k}} r^{i_1} r^{i_2} \cdots r^{i_m} f\big(r^{n-m-1}x\big)}{\prod_{1 \le i \le n,\, i \ne k} \big(r^k - r^i\big)}$$

for all $x \in V$ and for any fixed rational constant $r > 1$.

The advantage of our paper is that it provides clearer information about the function $f_k(x)$ than previous papers.

2. Main result

From now on, let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of real numbers satisfying $0 < a_1 < a_2 < \cdots < a_n$, and let $a > 1$.

For convenience, we first introduce the abbreviations and definitions that will be used in this paper.

Definition 2.1. Let $\omega_k(A)$ be the real numbers defined by

$$\omega_k(A) := \frac{(-1)^{n-k}}{\omega(A)} \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ a_1 & \cdots & a_{k-1} & a_{k+1} & \cdots & a_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-2} & \cdots & a_{k-1}^{n-2} & a_{k+1}^{n-2} & \cdots & a_n^{n-2} \end{vmatrix} \tag{2.1}$$

for every $k \in \{1, 2, \dots, n\}$, where the symbol $\omega(A)$ denotes the Vandermonde determinant defined by

$$\omega(A) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} (a_i - a_j).$$

For a given mapping $f : V \to W$, we define the mappings $\Gamma_{a,A} f, f_{k,a,A} : V \to W$ by

$$\Gamma_{a,A} f(x) := f(a^n x) + \sum_{m=1}^{n} (-1)^m \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} a_{i_1} a_{i_2} \cdots a_{i_m} f(a^{n-m} x) \tag{2.2}$$

and

$$\begin{aligned} f_{k,a,A}(x) := \omega_k(A) \Bigg(& f\big(a^{n-1}x\big) + \sum_{m=1}^{n-1} (-1)^m \\ & \times \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \dots, i_m \ne k}} a_{i_1} a_{i_2} \cdots a_{i_m} f\big(a^{n-m-1}x\big) \Bigg) \end{aligned} \tag{2.3}$$

for all $x \in V$ and $k \in \{1, 2, \dots, n\}$.

Lemma 2.2. *If there exist mappings $f_1, f_2, \cdots, f_n : V \to W$ such that $\sum_{k=1}^{n} f_k(x) = 0$ and each f_k has the property $f_k(ax) = a_k f_k(x)$ for all $x \in V$, then $f_k(x) = 0$ for all $x \in V$ and $k \in \{1, 2, \dots, n\}$.*

Proof. We note that $\sum_{k=1}^{n} a_k^i f_k(x) = \sum_{k=1}^{n} f_k(a^i x) = 0$ for all $x \in V$, $k \in \{1, 2, \dots, n\}$, and $i \in \{0, 1, 2, \dots, n-1\}$. Thus, $(f_1(x), f_2(x), \dots, f_n(x))$ is the solution to the system of linear equations

$$\begin{cases} f_1(x) + f_2(x) + \cdots + f_n(x) = 0, \\ a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x) = 0, \\ \quad \vdots \\ a_1^{n-1} f_1(x) + a_2^{n-1} f_2(x) + \cdots + a_n^{n-1} f_n(x) = 0. \end{cases}$$

Due to the Cramer's rule, the mapping f_k is determined by

$$f_k(x) = \frac{1}{\omega(A)} \begin{vmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ a_1 & \cdots & a_{k-1} & 0 & a_{k+1} & \cdots & a_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & \cdots & a_{k-1}^{n-1} & 0 & a_{k+1}^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = 0$$

for all $x \in V$ and $k \in \{1, 2, \dots, n\}$, which completes the proof. □

Lemma 2.3. *If* $m \in \{0, 1, 2, \dots, n-2\}$, *then*

$$\sum_{k=1}^{n} a_k^m \omega_k(A) = 0 \tag{2.4}$$

and

$$\sum_{k=1}^{n} a_k^{n-1} \omega_k(A) = 1. \tag{2.5}$$

In particular, the identity

$$\omega_k(A) = \frac{1}{\prod\limits_{1 \le i \le n,\, i \ne k} (a_k - a_i)} \tag{2.6}$$

holds for all $k \in \{1, 2, \dots, n\}$.

Proof. We note that

$$\begin{aligned}
\omega(A) \sum_{k=1}^{n} a_k^m \omega_k(A) &= \sum_{k=1}^{n} a_k^m \cdot (-1)^{n-k} \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ a_1 & \cdots & a_{k-1} & a_{k+1} & \cdots & a_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-2} & \cdots & a_{k-1}^{n-2} & a_{k+1}^{n-2} & \cdots & a_n^{n-2} \end{vmatrix} \\
&= \sum_{k=1}^{n} \begin{vmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ a_1 & \cdots & a_{k-1} & 0 & a_{k+1} & \cdots & a_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-2} & \cdots & a_{k-1}^{n-2} & 0 & a_{k+1}^{n-2} & \cdots & a_n^{n-2} \\ 0 & \cdots & 0 & a_k^m & 0 & \cdots & 0 \end{vmatrix} \\
&= \sum_{k=1}^{n} \begin{vmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1 & \cdots & a_{k-1} & a_k & a_{k+1} & \cdots & a_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-2} & \cdots & a_{k-1}^{n-2} & a_k^{n-2} & a_{k+1}^{n-2} & \cdots & a_n^{n-2} \\ 0 & \cdots & 0 & a_k^m & 0 & \cdots & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_n^{n-2} \\ a_1^m & a_2^m & \cdots & a_n^m \end{vmatrix} \\
&= \begin{cases} 0 & (\text{for } m \in \{0, 1, \dots, n-2\}), \\ \omega(A) & (\text{for } m = n-1). \end{cases}
\end{aligned}$$

Moreover, using (2.1) and the identity

$$\prod_{\substack{1\le j<i\le n\\ i,j\ne k}} (a_i - a_j) = \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ a_1 & \cdots & a_{k-1} & a_{k+1} & \cdots & a_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-2} & \cdots & a_{k-1}^{n-2} & a_{k+1}^{n-2} & \cdots & a_n^{n-2} \end{vmatrix},$$

we easily get

$$\omega_k(A) = \frac{(-1)^{n-k} \prod_{\substack{1\le j<i\le n\\ i,j\ne k}} (a_i - a_j)}{\prod_{1\le j<i\le n} (a_i - a_j)} = \frac{(-1)^{n-k}}{\prod_{1\le j<k} (a_k - a_j) \prod_{k<i\le n} (a_i - a_k)},$$

which completes the proof. □

Example 2.4. For $A = \{a_1, a_2, a_3\} = \{2, 3, 7\}$, we have

$$\begin{aligned} \omega_1(A) &= \frac{1}{(2-3)(2-7)} = \frac{1}{5}, \\ \omega_2(A) &= \frac{1}{(3-2)(3-7)} = -\frac{1}{4}, \\ \omega_3(A) &= \frac{1}{(7-2)(7-3)} = \frac{1}{20} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \sum_{k=1}^{3} a_k^0 \omega_k(A) &= 2^0 \cdot \frac{1}{5} - 3^0 \cdot \frac{1}{4} + 7^0 \cdot \frac{1}{20} = 0, \\ \sum_{k=1}^{3} a_k^1 \omega_k(A) &= 2^1 \cdot \frac{1}{5} - 3^1 \cdot \frac{1}{4} + 7^1 \cdot \frac{1}{20} = 0, \\ \sum_{k=1}^{3} a_k^2 \omega_k(A) &= 2^2 \cdot \frac{1}{5} - 3^2 \cdot \frac{1}{4} + 7^2 \cdot \frac{1}{20} = 1. \end{aligned}$$

Theorem 2.5. *If $f : V \to W$ is a mapping, then*

$$f(x) = \sum_{k=1}^{n} f_{k,a,A}(x)$$

and

$$f_{k,a,A}(ax) - a_k f_{k,a,A}(x) = \omega_k(A) \Gamma_{a,A} f(x) \tag{2.8}$$

for all $k \in \{1, 2, \ldots, n\}$ and $x \in V$.

Proof. It follows from (2.3) and (2.4) that

$$\begin{aligned}\sum_{k=1}^{n} f_{k,a,A}(x) &= \sum_{k=1}^{n} \omega_k(A)\Bigg(f\big(a^{n-1}x\big) + \sum_{m=1}^{n-1} (-1)^m \\ &\qquad \times \sum_{\substack{1\le i_1<i_2<\cdots<i_m\le n\\ i_1,i_2,\ldots,i_m\ne k}} a_{i_1}a_{i_2}\cdots a_{i_m} f\big(a^{n-m-1}x\big)\Bigg) \\ &= \sum_{m=1}^{n-1}(-1)^m \sum_{k=1}^{n}\omega_k(A) \\ &\qquad \times \sum_{\substack{1\le i_1<i_2<\cdots<i_m\le n\\ i_1,i_2,\ldots,i_m\ne k}} a_{i_1}a_{i_2}\cdots a_{i_m} f\big(a^{n-m-1}x\big)\end{aligned} \tag{2.9}$$

for all $x \in V$.

In the case of $m \in \{1, 2\}$ we use (2.4) to obtain

$$\begin{aligned}\sum_{k=1}^{n}\omega_k(A) \sum_{\substack{1\le i_1\le n\\ i_1\ne k}} a_{i_1} &= \sum_{k=1}^{n}\omega_k(A)\Bigg(\sum_{i=1}^{n} a_i - a_k\Bigg) \\ &= -\sum_{k=1}^{n}\omega_k(A)a_k \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{n}\omega_k(A) \sum_{\substack{1\le i_1<i_2\le n\\ i_1,i_2\ne k}} a_{i_1}a_{i_2} &= \sum_{k=1}^{n}\omega_k(A) \sum_{1\le i_1<i_2\le n} a_{i_1}a_{i_2} \\ &\quad - \sum_{k=1}^{n}\omega_k(A)a_k \sum_{\substack{1\le i_1\le n\\ i_1\ne k}} a_{i_1} \\ &= \sum_{k=1}^{n}\omega_k(A) \sum_{\substack{1\le i_1<i_2\le n\\ i_1,i_2\ne k}} a_{i_1}a_{i_2} \\ &\quad - \sum_{k=1}^{n}\omega_k(A)a_k \sum_{1\le i_1\le n} a_{i_1} + \sum_{k=1}^{n}\omega_k(A)a_k^2 \\ &= 0.\end{aligned}$$

On the other hand, for $3 \leq m \leq n-1$, we again use (2.4) and (2.5) for calculation

$$\begin{aligned}
&(-1)^m \sum_{k=1}^{n} \omega_k(A) \sum_{\substack{1\leq i_1<i_2<\cdots<i_m\leq n\\ i_1,i_2,\ldots,i_m\neq k}} a_{i_1}a_{i_2}\cdots a_{i_m}\\
&\quad= (-1)^m \sum_{k=1}^{n} \omega_k(A) \sum_{1\leq i_1<i_2<\cdots<i_m\leq n} a_{i_1}a_{i_2}\cdots a_{i_m}\\
&\qquad+ (-1)^{m-1} \sum_{k=1}^{n} \omega_k(A)a_k \sum_{\substack{1\leq i_1<i_2<\cdots<i_{m-1}\leq n\\ i_1,i_2,\ldots,i_{m-1}\neq k}} a_{i_1}a_{i_2}\cdots a_{i_{m-1}}\\
&\quad= (-1)^{m-1} \sum_{k=1}^{n} \omega_k(A)a_k \sum_{1\leq i_1<i_2<\cdots<i_{m-1}\leq n} a_{i_1}a_{i_2}\cdots a_{i_{m-1}}\\
&\qquad+ (-1)^{m-2} \sum_{k=1}^{n} \omega_k(A)a_k^2 \sum_{\substack{1\leq i_1<i_2<\cdots<i_{m-2}\leq n\\ i_1,i_2,\ldots,i_{m-2}\neq k}} a_{i_1}a_{i_2}\cdots a_{i_{m-2}}.
\end{aligned}$$

And hence, we further have

$$\begin{aligned}
&(-1)^m \sum_{k=1}^{n} \omega_k(A) \sum_{\substack{1\leq i_1<i_2<\cdots<i_m\leq n\\ i_1,i_2,\ldots,i_m\neq k}} a_{i_1}a_{i_2}\cdots a_{i_m}\\
&= \cdots\\
&= (-1)^2 \sum_{k=1}^{n} \omega_k(A)a_k^{m-2} \sum_{1\leq i_1<i_2\leq n} a_{i_1}a_{i_2} + (-1)^1 \sum_{k=1}^{n} \omega_k(A)a_k^{m-1} \sum_{\substack{1\leq i_1\leq n\\ i_1\neq k}} a_{i_1}\\
&= (-1)^1 \sum_{k=1}^{n} \omega_k(A)a_k^{m-1} \sum_{\substack{1\leq i_1\leq n\\ i_1\neq k}} a_{i_1}\\
&= \sum_{k=1}^{n} \omega_k(A)a_k^m\\
&= \begin{cases} 0 & (\text{for } 3\leq m\leq n-2),\\ 1 & (\text{for } m=n-1). \end{cases}
\end{aligned}$$

By (2.9), we have shown that $\sum_{k=1}^{n} f_{k,a,A}(x) = f(x)$. Moreover, we compute

$$
\begin{aligned}
&\Gamma_{a,A}f(x)\\
&\quad=\sum_{m=1}^{n}(-1)^m\sum_{1\le i_1<i_2<\cdots<i_m\le n}a_{i_1}\cdots a_{i_m}f(a^{n-m}x)+f(a^nx)\\
&\quad=\sum_{m=1}^{n-1}(-1)^m\sum_{\substack{1\le i_1<i_2<\cdots<i_m\le n\\ i_1,i_2,\ldots,i_m\neq k}}a_{i_1}\cdots a_{i_m}f(a^{n-m}x)+f(a^n)\\
&\qquad+\sum_{m=2}^{n}(-1)^m a_k\sum_{\substack{1\le i_1<i_2<\cdots<i_{m-1}\le n\\ i_1,i_2,\ldots,i_{m-1}\neq k}}a_{i_1}\cdots a_{i_{m-1}}f(a^{n-m}x) \qquad (2.10)\\
&\qquad-a_kf\big(a^{n-1}x\big)\\
&\quad=\sum_{m=1}^{n-1}(-1)^m\sum_{\substack{1\le i_1<i_2<\cdots<i_m\le n\\ i_1,i_2,\ldots,i_m\neq k}}a_{i_1}\cdots a_{i_m}f(a^{n-m}x)+f(a^nx)\\
&\qquad-a_k\left(\sum_{\ell=1}^{n-1}(-1)^\ell\sum_{\substack{1\le i_1<i_2<\cdots<i_\ell\le n\\ i_1,i_2,\ldots,i_\ell\neq k}}a_{i_1}\cdots a_{i_\ell}f\big(a^{n-\ell-1}x\big)+f\big(a^{n-1}x\big)\right),
\end{aligned}
$$

from which follows that $\omega_k(A)\Gamma_{a,A}f(x)=f_{k,a,A}(ax)-a_kf_{k,a,A}(x)$ for all $x\in V$. Hence, we obtain (2.8). □

Example 2.6. If $A=\{2,4\}$, then $\omega_1(A)=-\frac{1}{2}$ and $\omega_2(A)=\frac{1}{2}$. For the function $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x)=5x^3+4x^2+8x,$$

we have

$$
\begin{aligned}
f_{1,a,A}(x)&=-\frac{1}{2}\big(f(ax)-4f(x)\big)\\
&=-\frac{5a^3}{2}x^3-2a^2x^2-4ax+10x^3+8x^2+16x,\\
f_{2,a,A}(x)&=\frac{1}{2}\big(f(ax)-2f(x)\big)\\
&=\frac{5a^3}{2}x^3+2a^2x^2+4ax-5x^3-4x^2-8x,\\
\Gamma_{a,A}f(x)&=f\big(a^2x\big)-6f(ax)+8f(x)\\
&=5a^6x^3+4a^4x^2+8a^2x-30a^3x^3-24a^2x^2\\
&\quad-48a^1x+40x^3+42x^2+64x.
\end{aligned}
$$

In this case we can verify the previous theorem as follows:

$$\begin{aligned} f(x) &= f_{1,a,A}(x) + f_{2,a,A}(x), \\ f_{2,a,A}(ax) - 4f_{2,a,A}(x) &= \frac{1}{2}\Gamma_{a,A}f(x), \\ f_{1,a,A}(ax) - 2f_{1,a,A}(x) &= -\frac{1}{2}\Gamma_{a,A}f(x) \end{aligned}$$

for all $x \in \mathbb{R}$. However, the mapping $f_{k,a,A}(x)$ does not have the property $f_{k,a,A}(ax) = a_k f_{k,a,A}(x)$ for all $x \in \mathbb{R}$ and $k \in \{1, 2\}$.

The following corollary is easily derived from Theorem 2.5.

Corollary 2.7. *If f satisfies the functional equation*

$$\Gamma_{a,A}f(x) = 0$$

for all $x \in V$, then we have

$$f(x) = \sum_{k=1}^{n} f_{k,a,A}(x) \text{ and } f_{k,a,A}(ax) = a_k f_{k,a,A}(x)$$

for all $x \in V$ and $k \in \{1, 2, \ldots, n\}$.

Proof. Let $f_{k,a,A} : V \to W$ be the mappings as in (2.3) for $k \in \{1, 2, \ldots, n\}$. According to Theorem 2.5, $f(x) = \sum_{k=1}^{n} f_{k,a,A}(x)$ and $f_{k,a,A}(ax) = a_k f_{k,a,A}(x)$ for all $x \in V$ and $k \in \{1, 2, \ldots, n\}$. □

We introduce an example for Corollary 2.7 when $A = \{2, 3, 7\}$.

Example 2.8. For $A = \{2, 3, 7\}$, let $f : V \to W$ satisfy the functional equation

$$f\big(a^3x\big) - (2 + 3 + 7)f\big(a^2x\big) + (2 \cdot 3 + 3 \cdot 7 + 7 \cdot 2)f(ax) - 2 \cdot 3 \cdot 7f(x) = 0$$

for all $x \in V$. Then we can use (2.7) to get

$$\begin{aligned} f_{1,a,A}(x) &= \frac{1}{5}\big(f\big(a^2x\big) - 10f(ax) + 21f(x)\big), \\ f_{2,a,A}(x) &= -\frac{1}{4}\big(f\big(a^2x\big) - 9f(ax) + 14f(x)\big), \\ f_{3,a,A}(x) &= \frac{1}{20}\big(f\big(a^2x\big) - 5f(ax) + 6f(x)\big) \end{aligned}$$

such that $f(x) = \sum_{k=1}^{3} f_{k,a,A}(x)$ and

$$\begin{aligned} f_{1,a,A}(2x) &= 2f_{1,a,A}(x), \\ f_{2,a,A}(2x) &= 3f_{2,a,A}(x), \\ f_{3,a,A}(2x) &= 7f_{3,a,A}(x) \end{aligned}$$

for all $x \in V$.

From now on, we use f_k to denote an arbitrary mapping independent of the mapping $f_{k,a,A}$, which is defined as (2.3). The following lemma is the inverse of Corollary 2.7.

Lemma 2.9. *If there exist mappings $f_1, f_2, \dots, f_n : V \to W$ such that*

$$f(x) = \sum_{k=1}^{n} f_k(x) \text{ and } f_k(ax) = a_k f_k(x),$$

then f satisfies the functional equation

$$\Gamma_{a,A} f(x) = 0$$

and f_k is equal to $f_{k,a,A}$ given in Definition 2.1 *for all $k \in \{1, 2, \dots, n\}$.*

Proof. Since $f_k(ax) = a_k f_k(x)$ for all $x \in V$, we have

$$\begin{aligned} \Gamma_{a,A} f_k(x) &= \sum_{m=0}^{n} (-1)^m \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} a_{i_1} a_{i_2} \cdots a_{i_m} f_k(a^{n-m}x) \\ &= \sum_{m=0}^{n-1} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \dots, i_m \ne k}} a_{i_1} a_{i_2} \cdots a_{i_m} f_k(a^{n-m}x) \\ &\quad - a_k \sum_{m=0}^{n-1} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \dots, i_m \ne k}} a_{i_1} a_{i_2} \cdots a_{i_m} f_k\big(a^{n-m-1}x\big) \\ &= 0 \end{aligned}$$

for all $k \in \{1, 2, \dots, n\}$ and hence $\Gamma_{a,A} f(x) = \Gamma_{a,A} \sum_{k=1}^{n} f_k(x) = 0$. By Corollary 2.7, we have

$$f(x) = \sum_{k=1}^{n} f_{k,a,A}(x) \text{ with } f_{k,a,A}(ax) = a_k f_{k,a,A}(x)$$

for all $x \in V$, and so, according to Lemma 2.2, f_k is equal to $f_{k,a,A}$ for $k \in \{1, 2, \dots, n\}$. □

Example 2.10. For $A=\{2,3,7\}$, let $f:\mathbb{R}\to\mathbb{R}$ be a function defined as

$$f(x)=5|x|^{\log_a 7}-4|x|^{\log_a 3}+8|x|^{\log_a 2}.$$

Put $f_1(x)=8|x|^{\log_a 2}$, $f_2(x)=-4|x|^{\log_a 3}$ and $f_3(x)=5|x|^{\log_a 7}$ for all $x\in\mathbb{R}$, then we have

$$f_1(ax)=2f_1(x),\ f_2(ax)=3f_2(x),\ f_3(ax)=7f_3(x)$$

and

$$\Gamma_{a,A}f(x)=f\big(a^3x\big)-12f\big(a^2x\big)+41f(ax)-42f(x)=0$$

for all $x\in\mathbb{R}$. Together with (2.7), we can check that

$$\begin{aligned}
f_1(x)&=\frac{1}{5}\big(f\big(a^2x\big)-10f(ax)+21f(x)\big)=f_{1,a,A}(x),\\
f_2(x)&=-\frac{1}{4}\big(f\big(a^2x\big)-9f(ax)+14f(x)\big)=f_{2,a,A}(x),\\
f_3(x)&=\frac{1}{20}\big(f\big(a^2x\big)-5f(ax)+6f(x)\big)=f_{3,a,A}(x)
\end{aligned}$$

for all $x\in\mathbb{R}$.

The following main theorem is a consequence of Corollary 2.7 and Lemma 2.9.

Theorem 2.11. *A mapping $f:V\to W$ is a solution to the functional equation*

$$\Gamma_{a,A}f(x)=0$$

if and only if there exist mappings $f_1,f_2,\ldots,f_n:V\to W$ such that

$$f(x)=\sum_{k=1}^{n}f_k(x)\ \textit{and}\ f_k(ax)=a_kf_k(x)$$

for all $x\in V$ and $k\in\{1,2,\ldots,n\}$. In particular, each $f_k:V\to W$ is the same as $f_{k,a,A}$ given in Definition 2.1 *for $k\in\{1,2,\ldots,n\}$.*

3. Applications

For any given mapping $f:V\to W$, we will use the following notations:

$$\tilde{f}(x):=f(x)-f(0),\ f_o(x):=\frac{1}{2}\big(f(x)-f(-x)\big),\ f_e(x):=\frac{1}{2}\big(f(x)+f(-x)\big)$$

for all $x\in V$.

For convenience, some definitions of Definition 2.1 can be slightly modified.

Definition 3.1. For any given mapping $f : V \to W$, we define

$$\begin{aligned} &f_{k,a,n}(x) \\ &\quad := f_{k,a,\{a^1,a^2,\ldots,a^n\}}(x) \\ &\quad = \frac{f\big(a^{n-1}x\big) + \sum\limits_{m=1}^{n-1}(-1)^m \sum\limits_{\substack{1\le i_1<\cdots<i_m\le n\\ i_1,\ldots,i_m\ne k}} a^{i_1+\cdots+i_m} f\big(a^{n-m-1}x\big)}{\prod\limits_{1\le i\le n,\, i\ne k}\big(a^k - a^i\big)}, \\ &(f_o)_{2k-1,a,n}(x) \\ &\quad := (f_o)_{k,a,\{a^1,a^3,\ldots,a^{2n-1}\}}(x) \\ &\quad = \frac{f_o\big(a^{n-1}x\big) + \sum\limits_{m=1}^{n-1}(-1)^m \sum\limits_{\substack{1\le i_1<\cdots<i_m\le n\\ i_1,\ldots,i_m\ne k}} a^{2(i_1+\cdots+i_m)-m} f_o\big(a^{n-m-1}x\big)}{\prod\limits_{1\le i\le n,\, i\ne k}\big(a^{2k-1} - a^{2i-1}\big)} \end{aligned}$$

for all $x \in V$ and $k \in \{1, 2, \ldots, n\}$.

For any given mapping $f : V \to W$, we further define

$$\begin{aligned} &(f_e)_{2k,a,n}(x) \\ &\quad := (f_e)_{k,a,\{a^2,a^4,\ldots,a^2n\}}(x) \\ &\quad = \frac{f_e\big(a^{n-1}x\big) + \sum\limits_{m=1}^{n-1}(-1)^m \sum\limits_{\substack{1\le i_1<\cdots<i_m\le n\\ i_1,\ldots,i_m\ne k}} a^{2(i_1+\cdots+i_m)} f_e\big(a^{n-m-1}x\big)}{\prod\limits_{1\le i\le n,\, i\ne k}\big(a^{2k} - a^{2i}\big)}, \\ &\Gamma_{a^{1,2,\ldots,n}} f(x) \\ &\quad := \Gamma_{a,\{a^1,a^2,\ldots,a^n\}} f(x) \\ &\quad = f(a^n x) + \sum_{m=1}^{n}(-1)^m \sum_{1\le i_1<\cdots<i_m\le n} a^{i_1+i_2+\cdots+i_m} f(a^{n-m}x), \\ &\Gamma_{a^{1,3,\ldots,2n-1}} f_o(x) \\ &\quad := \Gamma_{a^{1,3,\ldots,2n-1}} f_o(x) \\ &\quad = f_o(a^n x) + \sum_{m=1}^{n}(-1)^m \sum_{1\le i_1<\cdots<i_m\le n} a^{2(i_1+\cdots+i_m)-m} f_o(a^{n-m}x), \end{aligned}$$

$$\begin{aligned}&\Gamma_{a^{2,4,\ldots,2n}} f_e(x)\\ &\quad := \Gamma_{a^{2,4,\ldots,2n}} f_e(x)\\ &\quad = f_e(a^n x) + \sum_{m=1}^{n} (-1)^m \sum_{1 \le i_1 < \cdots < i_m \le n} a^{2(i_1+\cdots+i_m)} f_e(a^{n-m}x)\end{aligned}$$

for all $x \in V$ and $k \in \{1, 2, \ldots, n\}$.

Example 3.2. In the case of $a = 2$ and $k = 3$, we get

$$\begin{aligned}
f_{1,2,3}(x) &= \frac{1}{(2^1-2^2)(2^1-2^3)}\big(f(4x) - 12f(2x) + 32f(x)\big),\\
f_{2,2,3}(x) &= \frac{1}{(2^2-2^1)(2^2-2^3)}\big(f(4x) - 10f(2x) + 16f(x)\big),\\
f_{3,2,3}(x) &= \frac{1}{(2^3-2^1)(2^3-2^2)}\big(f(4x) - 6f(2x) + 8f(x)\big),\\
(f_o)_{1,2,3}(x) &= \frac{1}{(2^{2-1}-2^{4-1})(2^{2-1}-2^{6-1})}\big(f_o(4x) - 40f_o(2x) + 256f_o(x)\big),\\
(f_e)_{2,2,3}(x) &= \frac{1}{(2^2-2^4)(2^2-2^6)}\big(f_e(4x) - 80f_e(2x) + 1024f_e(x)\big),\\
(f_o)_{3,2,3}(x) &= \frac{1}{(2^{4-1}-2^{2-1})(2^{4-1}-2^{6-1})}\big(f_o(4x) - 34f_o(2x) + 64f_o(x)\big),\\
(f_e)_{4,2,3}(x) &= \frac{1}{(2^4-2^2)(2^4-2^6)}\big(f_e(4x) - 68f_e(2x) + 256f_e(x)\big),\\
(f_o)_{5,2,3}(x) &= \frac{1}{(2^{6-1}-2^{2-1})(2^{6-1}-2^{4-1})}\big(f_o(4x) - 10f_o(2x) + 16f_o(x)\big),\\
(f_e)_{6,2,3}(x) &= \frac{1}{(2^6-2^2)(2^6-2^4)}\big(f_e(4x) - 20f_e(2x) + 64f_e(x)\big),
\end{aligned}$$

$$\Gamma_{2^{1,2,3}} f(x) = f(4x) - 14f(4x) + 56f(2x) - 64f(x),$$

$$\Gamma_{2^{1,3,5}} f_o(x) = f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x),$$

$$\Gamma_{2^{2,4,6}} f_e(x) = f_e(8x) - 84f_e(4x) + 1344f_e(2x) - 4096f_e(x)$$

for all $x \in V$.

For the set $A = \{a^1, a^2, \ldots, a^n\}$, Theorem 2.11 can be replaced by the following theorem using Definition 3.1.

Theorem 3.3. *A mapping $f : V \to W$ is a solution to the functional equation*

$$\Gamma_{a^{1,2,\ldots,n}} f(x) = 0$$

if and only if there exist mappings $f_1, f_2, \ldots, f_n : V \to W$ such that

$$f(x) = \sum_{k=1}^{n} f_k(x) \text{ and } f_k(ax) = a^k f_k(x)$$

for all $x \in V$ and $k \in \{1, 2, \ldots, n\}$. In particular, f_k is the same as $f_{k,a,n}$ given in Definition 3.1 for each $k \in \{1, 2, \ldots, n\}$.

Example 3.4. Let $f_1, f_2, f_3, f : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f_1(x) := \begin{cases} 3x & (\text{for } x \in \mathbb{Q}), \\ -x & (\text{for } x \notin \mathbb{Q}), \end{cases}$$
$$f_2(x) := \begin{cases} x^2 & (\text{for } x \in \mathbb{Q}), \\ 0 & (\text{for } x \notin \mathbb{Q}), \end{cases}$$
$$f_3(x) := \begin{cases} x^3 & (\text{for } x > 0), \\ 0 & (\text{for } x \le 0) \end{cases}$$

and $f(x) := \sum_{k=1}^{3} f_k(x)$ for all $x \in \mathbb{R}$. Then we see that $\overset{4}{\underset{y}{\Delta}} f(x) \neq 0$ for all $x, y \in \mathbb{R}$. However, since $f_k(rx) = r^k f_k(x)$ for all $x \in \mathbb{R}$ and $r \in \mathbb{Q}$ with $r > 1$, f satisfies the functional equation

$$f\big(r^3x\big) - \big(r + r^2 + r^3\big) f\big(r^2x\big) + \big(r^3 + r^4 + r^5\big) f(ax) - r^6 f(x) = 0$$

as well as the equations

$$f_1(x) = \frac{f(r^2x) - (r^2 + r^3) f(rx) + r^5 f(x)}{(r - r^2)(r - r^3)},$$
$$f_2(x) = \frac{f(r^2x) - (r + r^3) f(rx) + r^4 f(x)}{(r^2 - r)(r^2 - r^3)},$$
$$f_3(x) = \frac{f(r^2x) - (r + r^2) f(rx) + r^3 f(x)}{(r^3 - r)(r^3 - r^2)}$$

hold for all $x \in \mathbb{R}$ and $r \in \mathbb{Q}$ with $r > 1$.

Example 3.5. If $f : V \to W$ is a solution to the functional equation $\overset{n+1}{\underset{y}{\Delta}} f(x) = 0$ (for all $x, y \in V$) and $f(0) = 0$, then there exist mappings $f_1, f_2, \ldots, f_n : V \to W$ such that

$$f(x) = \sum_{k=1}^{n} f_k(x) \quad \text{and} \quad f_k(rx) = r^k f_k(x)$$

for any $r \in \mathbb{Q}$ with $r > 1$. According to Theorem 2.11, the mapping $f : V \to W$ satisfies the functional equation $\Gamma_{r^{1,2,\ldots,n}} f(x) = 0$ and f_k's are given by

$$f_k(x) := \frac{f(r^{n-1}x) + \sum_{m=1}^{n-1}(-1)^m \sum_{\substack{1\le i_1<i_2<\cdots<i_m\le n \\ i_1,i_2,\ldots,i_m\ne k}} r^{i_1+i_2+\cdots+i_m} f\big(r^{n-m-1}x\big)}{\prod_{1\le i\le n,\, i\ne k}\big(r^k - r^i\big)}$$

for all $x \in V$.

Corollary 3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function of degree at most n with $f(0) = 0$, that is, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $f(x) = \sum_{k=1}^{n} \alpha_k x^k$ for all $x \in \mathbb{R}$. Then f satisfies the functional equation $\Gamma_{a^{1,2,\ldots,n}} f(x) = 0$ and f has the properties*

$$\alpha_k x^k = f_{k,a,n}(x)$$

for all $x \in \mathbb{R}$ and $a > 1$.

Proof. If we put $f_k(x) := \alpha_k x^k$ for all $x \in \mathbb{R}$, $x \in \mathbb{R}$, and for each $k \in \{1, 2, \ldots, n\}$, then it is obvious that

$$f(x) = \sum_{k=1}^{n} f_k(x) \quad \text{and} \quad f_k(ax) = a^k f_k(x)$$

for any $a \in \mathbb{R}$ with $a > 1$. According to Theorem 3.3, we obtain the desired results. □

For $A = \{a^1, a^3, \ldots, a^{2n-1}\}$, Theorem 2.11 is transformed as follows:

Theorem 3.7. *An odd mapping $f_o : V \to W$ satisfies functional equation*

$$\Gamma_{a^{1,3,\ldots,2n-1}} f_o(x) = 0$$

if and only if there exist mappings $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n : V \to W$ such that

$$f_o(x) = \sum_{k=1}^{n} \hat{f}_k(x) \quad \text{and} \quad \hat{f}_k(ax) = a^{2k-1}\hat{f}_k(x).$$

In particular, $\hat{f}_k$ is equal to $(f_o)_{2k-1,a,n}$ given in Definition 3.1 for every $k \in \{1, 2, \ldots, n\}$.

Corollary 3.8. *Let $f_o : \mathbb{R} \to \mathbb{R}$ be an odd polynomial function of degree at most $2n-1$, that is, there exist $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n \in \mathbb{R}$ such that $f_o(x) = \sum_{k=1}^{n} \hat{\alpha}_k x^{2k-1}$ for all $x \in \mathbb{R}$. Then f_o satisfies the functional equation*

$$\Gamma_{a^{1,3,\ldots,2n-1}} f_o(x) = 0$$

and f_o has the properties

$$\hat{\alpha}_k x^{2k-1} = (f_o)_{2k-1,a,n}(x)$$

hold for all $x \in \mathbb{R}$ and $a > 1$.

The following example is for $n = 3$.

Example 3.9. Consider a function $f_o : \mathbb{R} \to \mathbb{R}$ defined by

$$f_o(x) = -4x^5 + 5x^3 + 8x.$$

Then we have

$$\begin{aligned}\Gamma_{a^{1,3,5}} f_o(x) &= f_o\big(a^3 x\big) - \big(a + a^3 + a^5\big) f_o\big(a^2 x\big) \\ &\quad + \big(a^4 + a^6 + a^8\big) f_o(ax) - a^9 f_o(x) \\ &= 0\end{aligned}$$

for all $x \in \mathbb{R}$ and any fixed $a > 1$. Thus, it follows from Corollary 3.8 with $n = 3$ that there exist functions $\hat{f}_1, \hat{f}_2, \hat{f}_3 : \mathbb{R} \to \mathbb{R}$ such that

$$f_o(x) = \sum_{k=1}^{3} \hat{f}_k(x) \quad \text{and} \quad \hat{f}_k(ax) = a^{2k-1} \hat{f}_k(x)$$

for each $k \in \{1, 2, 3\}$.

It is easy to show that $\hat{f}_1(x) = 8x$, $\hat{f}_2(x) = 5x^3$ and $\hat{f}_3(x) = -4x^5$, and

$$\begin{aligned}(f_o)_{1,a,3}(x) &= \frac{f_o(a^2x) - (a^3 + a^5) f_o(ax) + a^8 f_o(x)}{(a^3 - a)(a^5 - a)} = 8x = \hat{f}_1(x), \\ (f_o)_{3,a,3}(x) &= -\frac{f_o(a^2x) - (a + a^5) f_o(ax) + a^6 f_o(x)}{(a^3 - a)(a^5 - a^3)} = 5x^3 = \hat{f}_2(x), \\ (f_o)_{5,a,3}(x) &= \frac{f_o(a^2x) - (a + a^3) f_o(ax) + a^4 f_o(x)}{(a^5 - a)(a^5 - a^3)} = -4x^5 = \hat{f}_3(x)\end{aligned}$$

for all $x \in \mathbb{R}$.

On the other hand, for the set $A = \{a^2, a^4, \ldots, a^{2n}\}$, the following functional equation can be considered.

Theorem 3.10. *An even mapping $f_e : V \to W$ satisfies the functional equation*

$$\Gamma_{a^{2,4,\ldots,2n}} f_e(x) = 0 \ (x \in V)$$

if and only if there exist mappings $\check{f}_1, \check{f}_2, \ldots, \check{f}_n : V \to W$ such that

$$f_e(x) = \sum_{k=1}^{n} \check{f}_k(x) \quad \text{and} \quad \check{f}_k(ax) = a^{2k} \check{f}_k(x)$$

for all $x \in V$. In particular, $\check{f}_k$ is equal to $(f_e)_{2k,a,n}$ for each $k \in \{1, 2, \ldots, n\}$.

Corollary 3.11. *Let $f_e : \mathbb{R} \to \mathbb{R}$ be an even polynomial function of degree at most $2n$, and let $f(0) = 0$, that is, there exist $\breve{\alpha}_1, \breve{\alpha}_2, \ldots, \hat{\alpha}_n \in \mathbb{R}$ such that $f_e(x) = \sum_{k=1}^{n} \breve{\alpha}_k x^{2k}$ for all $x \in \mathbb{R}$. Then f_e satisfies the functional equation $\Gamma_{a^{2,4,\ldots,2n}} f_e(x) = 0$ and f_e has the properties*

$$\breve{\alpha}_k x^{2k} = (f_e)_{2k,a,n}(x)$$

for all $x \in \mathbb{R}$ and $a > 1$.

We introduce an example for the case of $n = 3$ in Corollary 3.11.

Example 3.12. We consider an even function $f_e : \mathbb{R} \to \mathbb{R}$ defined by

$$f_e(x) = x^6 - 4x^2$$

and then

$$\begin{aligned}\Gamma_{a^{2,4,6}} f_e(x) &= f_e\big(a^3 x\big) - \big(a^2 + a^4 + a^6\big) f_e\big(a^2 x\big) \\ &\quad + \big(a^6 + a^8 + a^{10}\big) f_e(ax) - a^{12} f_e(x) \\ &= 0\end{aligned}$$

for any fixed $a > 1$. We can find the mappings $\breve{f}_1, \breve{f}_2, \breve{f}_3 : \mathbb{R} \to \mathbb{R}$:

$$\breve{f}_1(x) = -4x^2, \ \breve{f}_2(x) = 0 \quad \text{and} \quad \breve{f}_3(x) = x^6$$

which satisfy

$$f_e(x) = \sum_{k=1}^{3} \breve{f}_k(x) \quad \text{and} \quad \breve{f}_k(ax) = a^{2k} \breve{f}_k(x)$$

for each $k \in \{1, 2, 3\}$. We verify that

$$\begin{aligned}(f_e)_{2,a,3}(x) &= \frac{f_e(a^2x) - (a^4 + a^6) f_e(ax) + a^{10} f_e(x)}{(a^4 - a^2)(a^6 - a^2)} = -4x^2 = \breve{f}_1(x), \\ (f_e)_{4,a,3}(x) &= -\frac{f_e(a^2x) - (a^2 + a^6) f_e(ax) + a^8 f_e(x)}{(a^4 - a^2)(a^6 - a^4)} = 0 = \breve{f}_2(x), \\ (f_e)_{6,a,3}(x) &= \frac{f_e(a^2x) - (a^2 + a^4) f_e(ax) + a^6 f_e(x)}{(a^6 - a^2)(a^6 - a^4)} = x^6 = \breve{f}_3(x)\end{aligned}$$

for all $x \in \mathbb{R}$.

Now we can easily prove the following theorem using Theorems 3.7 and 3.10.

Theorem 3.13. *Let $f : V \to W$ be a mapping.*

(i) *f satisfies the functional equations*

$$\Gamma_{a^{1,3,\dots,2n-1}} f_o(x) = 0 \text{ and } \Gamma_{a^{2,4,\dots,2n}} f_e(x) = 0$$

if and only if there exist mappings $f_1, f_2, \dots, f_{2n} : V \to W$ such that

$$f = \sum_{\ell=1}^{2n} f_\ell, \; f_\ell(ax) = a^\ell f_\ell(x), \text{ and } f_\ell(-x) = (-1)^\ell f_\ell(x)$$

for all $x \in V$. In particular, f_{2k-1} is equal to $(f_o)_{2k-1,a,n}$ and f_{2k} is equal to $(f_e)_{2k,a,n-1}$ for each $k \in \{1, 2, \dots, n\}$.

(ii) *f satisfies the functional equations*

$$\Gamma_{a^{1,3,\dots,2n-1}} f_o(x) = 0 \text{ and } \Gamma_{a^{2,4,\dots,2n-2}} f_e(x) = 0$$

if and only if there exist mappings $f_1, f_2, \dots, f_{2n-1} : V \to W$ such that

$$f = \sum_{\ell=1}^{2n-1} f_\ell, \; f_\ell(ax) = a^\ell f_\ell(x), \text{ and } f_\ell(-x) = (-1)^\ell f_\ell(x)$$

for all $x \in V$. In particular, f_{2k-1} is equal to $(f_o)_{2k-1,a,n}$ and f_{2k} is equal to $(f_e)_{2k,a,n-1}$ for each $k \in \{1, 2, \dots, n-1\}$.

Corollary 3.14. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function.*

(i) *If f is a polynomial function of degree at most $2n$ and $f(0) = 0$, i.e., there exist $\alpha_1, \alpha_2, \dots, \alpha_{2n} \in \mathbb{R}$ such that $f(x) = \sum_{\ell=1}^{2n} \alpha_\ell x^\ell$ for $x \in \mathbb{R}$, then f satisfies the functional equation $\Gamma_{a^{1,2,3,\dots,2n}} f(x) = 0$ and the equations*

$$\alpha_{2k-1} x^{2k-1} = (f_o)_{2k-1,a,n}(x) \text{ and } \alpha_{2k} x^{2k} = (f_e)_{2k,a,n}(x)$$

for all $x \in \mathbb{R}$, $a > 1$, and for any $k \in \{1, 2, \dots, n\}$.

(ii) *If f is a polynomial function of degree at most $2n-1$ with $f(0) = 0$, i.e., there exist $\alpha_1, \alpha_2, \dots, \alpha_{2n-1} \in \mathbb{R}$ such that $f(x) = \sum_{\ell=1}^{2n-1} \alpha_\ell x^\ell$ for all $x \in \mathbb{R}$, then f satisfies the functional equation $\Gamma_{a^{1,2,3,\dots,2n-1}} f(x) = 0$ and the equations*

$$\alpha_{2k-1} x^{2k-1} = (f_o)_{2k-1,a,n}(x) \text{ and } \alpha_{2k'} x^{2k'} = (f_e)_{2k',a,n-1}(x)$$

for all $x \in \mathbb{R}$, $a > 1$, $k \in \{1, 2, \dots, n\}$, and for all $k' \in \{1, 2, \dots, n-1\}$.

If we put $a = 2$ and $n = 3$ in Corollary 3.14 (i), then we obtain the following example.

Example 3.15. We consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^6 - 4x^5 + 5x^3 - 4x^2 + 8x$$

for all $x \in \mathbb{R}$. Then $f_o(x) = -4x^5 + 5x^3 + 8x$ and $f_e(x) = x^6 - 4x^2$. In Examples 3.9 and 3.12, if we put $f_{2k-1}(x) = \hat{f}_k(x)$ and $f_{2k}(x) = \check{f}_k(x)$ for $k \in \{1, 2, 3\}$, then there exist functions $f_1, f_2, \ldots, f_6$ such that

$$f(x) = f_o(x) + f_e(x) = \sum_{\ell=1}^{6} f_\ell(x) \quad \text{and} \quad f_\ell(2x) = 2^\ell f_\ell(x)$$

for each $\ell \in \{1, 2, \ldots, 6\}$. In particular, f_{2k-1} is equal to $(f_o)_{2k-1,a,3}$ and f_{2k} is equal to $(f_e)_{2k,a,3}$ for each $k \in \{1, 2, 3\}$.

Example 3.16. If $f : V \to W$ satisfies the functional equation $\overset{2n+1}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$, then there exist mappings $f_1, f_2, \ldots, f_{2n} : V \to W$ such that

$$f_o(x) = \sum_{k=1}^{n} f_{2k-1}(x), \quad f_{2k-1}(rx) = r^{2k-1} f_{2k-1}(x),$$

$$\tilde{f}_e(x) = \sum_{k=1}^{n} f_{2k}(x), \qquad f_{2k}(rx) = r^{2k} f_{2k}(x)$$

for any $r \in \mathbb{Q}$ with $r > 1$, where $\tilde{f}(x) := f(x) - f(0)$ for all $x \in V$. According to Theorem 3.7, the mapping $f : V \to W$ satisfies functional equations $\Gamma_{r^{1,3,\ldots,2n-1}} f_o(x) = 0$ and $\Gamma_{r^{2,4,\ldots,2n}} \tilde{f}_e(x) = 0$, and f_ℓ's are given by

$$f_{2k-1}(x) := \frac{f_o\big(r^{n-1}x\big) + \sum_{m=1}^{n-1} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \ldots, i_m \ne k}} r^{2(i_1+i_2+\cdots+i_m)-m} f_o\big(r^{n-m-1}x\big)}{\prod_{1 \le i \le n,\, i \ne k} \big(r^{2k-1} - r^{2i-1}\big)}$$

and

$$f_{2k}(x) := \frac{\tilde{f}_e\big(r^{n-1}x\big) + \sum_{m=1}^{n-1} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \ldots, i_m \ne k}} r^{2(i_1+i_2+\cdots+i_m)} \tilde{f}_e\big(r^{n-m-1}x\big)}{\prod_{1 \le i \le n,\, i \ne k} \big(r^{2k} - r^{2i}\big)}$$

for all $x \in V$.

Example 3.17. If $f : V \to W$ satisfies the functional equation $\overset{2n}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$, then there exist mappings $f_1, f_2, \dots, f_{2n-1} : V \to W$ such that

$$f_o(x) = \sum_{k=1}^{n} f_{2k-1}(x), \quad f_{2k-1}(rx) = r^{2k-1} f_{2k-1}(x),$$
$$\tilde{f}_e(x) = \sum_{k=1}^{n-1} f_{2k}(x), \qquad f_{2k}(rx) = r^{2k} f_{2k}(x)$$

for any $r \in \mathbb{Q}$ with $r > 1$. Hence, $f : V \to W$ satisfies both functional equations $\Gamma_{r^{1,3,\dots,2n-1}} f_o(x) = 0$ and $\Gamma_{r^{2,4,\dots,2n-2}} \tilde{f}_e(x) = 0$. Additionally, $f_1, f_2, \dots,$ f_{2n-1} are given by

$$f_{2k-1}(x) := \frac{f_o\big(r^{n-1}x\big) + \sum_{m=1}^{n-1} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n \\ i_1, i_2, \dots, i_m \ne k}} r^{2(i_1+i_2+\cdots+i_m)-m} f_o\big(r^{n-m-1}x\big)}{\prod_{1 \le i \le n,\, i \ne k} \big(r^{2k-1} - r^{2i-1}\big)}$$

and

$$f_{2k}(x) := \frac{\tilde{f}_e\big(r^{n-2}x\big) + \sum_{m=1}^{n-2} (-1)^m \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_m \le n-1 \\ i_1, i_2, \dots, i_m \ne k}} r^{2(i_1+i_2+\cdots+i_m)} \tilde{f}_e\big(r^{n-m-2}x\big)}{\prod_{1 \le i \le n-1,\, i \ne k} \big(r^{2k} - r^{2i}\big)}$$

for all $x \in V$.

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