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CONVERGENCE AND STABILITY OF FIBONACCI SR-ITERATION PROCESS FOR MONOTONE NON-LIPSCHITZIAN MAPPING

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Abstract. In this paper, we introduce a new iteration process (called the Fibonacci SR-iteration process) for monotone non-Lipschitzian mapping (that is, nearly asymptotically nonexpansive mapping) in partially ordered hyperbolic metric space and prove strong and Δ-convergence theorem. Further, we construct a numerical example to demonstrate that our iteration process is faster than the Fibonacci Mann iteration process [4]. Our results generalize, extend, and unify the corresponding results of Agrawal et. al. [3, 4], Alfuraidan and Khamsi [5], and many more results in this direction.

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1. Introduction

Metric fixed point theory is one of the important branches of nonlinear analysis. In 1922, a milestone result of metric fixed point theory was given by Banach, which is known as the Banach contraction principle (BCP), the principle not only provided the existence and uniqueness of fixed points but also it provided simplest iteration process to approximate fixed point. After the BCP, several generalizations came into the picture.

In 1965, Browder [8, 9], Gohde [15], and Kirk [20] independently gave the existence theorem for nonexpansive mapping. The existence theorem of Kirk [20] was slightly more general than the theorem of Browder's and Gohde's existence theorem. In BCP the Picard iteration process (PIP) ($x_{n+1} = Tx_n, n \geq 0$) to approximate the fixed point for contraction mapping, When we work with slightly weaker mapping, then PIP does not converge. So many iterations like Mann [26], Ishikawa [17], Agarwal et al. [2], Noor [28], Abbas and Nazir[1], an accelerated iteration [10] and Thakur et al., [38] iteration processes came into the picture to sort out this problem(see [11, 18, 19]).

In 1972, Goebel and Kirk [14] introduced the asymptotically nonexpansive mappings. On the other hand, in 2005, the class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian type mappings was introduced by Sahu [33].

The two important extensions to partially ordered metric space ($POMS$) was given by Ran and Reuring [31] and Nieto and López [27]. In [31] applied their results to solve matrix equations while in [27] applied them to solve differential equations.

In 2016, Dehaish and Khamsi [12] introduced the notion of monotone nonexpansive mapping by extending Browder's and Gohde's fixed point theorem. In 1991 Schu [35] introduced a modified Mann iteration based on the good behavior of the Lipschitz constant associated with the iterates of involved mappings. The modified Mann iteration scheme does not converge for monotone mapping. Therefore, Alfuraidan and Khamsi [5] introduced the Fibonacci-Mann iteration scheme and proved strong and weak convergence in partially ordered Banach space for monotone asymptotically nonexpansive mapping.

In 2019, Aggarwal et al. [3] studied the existence and convergence of fixed points for monotone nearly asymptotically nonexpansive in hyperbolic space using the S iteration process [2]. In the same year Aggarwal et al. [4] proved strong convergence and Δ-convergence of Fibonacci-Mann iteration for a monotone non-Lipschitzian mapping (that is, nearly asymptotically nonexpansive mapping) in partially ordered hyperbolic metric space and proved stability of Fibonacci-Mann iteration and construct a numerical example to illustrate support the results in [4].

Motivated by the above results in this paper, we introduce a new iteration process (called the Fibonacci SR-iteration process) for monotone non-Lipschitzian mapping in partially ordered hyperbolic metric space and prove strong and Δ-convergence theorem. Further, we construct a numerical example to demonstrate that our iteration process is faster than the Fibonacci Mann iteration process [4]. Our results generalized, extend and unify the corresponding results of Agrawal et al. [3, 4], Alfuraidan and Khamsi [5], and many more results in this direction.

2. Preliminaries

Let us collect the following definition.

Definition 2.1. ([22]) Let (X, d) be a metric space. Then triplet (X, d, W) is referred to as hyperbolic metric space (HMS) if there is a mapping $W : X \times X \times [0, 1] \to X$ such that for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$ the following hold:

(i) $d(z, W(x, y, \alpha)) \leq (1 - \alpha)d(z, x) + \alpha d(z, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$,
(iii) $W(x, y, \alpha) = W(x, y, 1 - \alpha)$,
(iv) $d(W(x, y, \alpha), W(z, w, \alpha)) \leq (1 - \alpha)d(x, z) + \alpha d(y, w)$.

Example 2.2. ([34]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$. Define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1x_2 - y_1y_2|$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X. Now for $\alpha \in [0, 1]$, define a function $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \alpha) = \left(\alpha x_1 + (1 - \alpha)y_1, \frac{\alpha x_1x_2 + (1 - \alpha)y_1y_2}{\alpha x_1 + (1 - \alpha)y_1}\right).$$

Then we can easily verify that (X, d, W) is a HMS (see in [30]).

Definition 2.3. ([23]) Assume that (X, d, W) is a HMS. Then X is said to be a uniformly convex if there exists a $\delta \in [0, 1)$ such that

$$d(\frac{1}{2}x \oplus \frac{1}{2}y, z) \leq (1 - \delta)r,$$

if for $x, y, z \in X$, $\varepsilon \in (0, 2]$ and $r > 0$, whenever $d(x, z) \leq r$ and $d(y, z) \leq r$ and $d(x, y) \geq \varepsilon r$.

Definition 2.4. Let us assume that (X, d, W) is a HMS. If $a, b \in X$ and $\alpha \in [0, 1]$, then we will use $(1-\alpha)a \oplus \alpha b \in W(a, b, \alpha)$. A subset C of this HMS is called convex if $a, b \in C$ implies that $W(a, b, \alpha) \in C$.

The following equalities hold even for the more general setting of a convex metric space (see [37], Proposition 1.2): $d(b, W(a, b, \alpha) = (1-\alpha)d(a, b)$ and $d(a, W(a, b, \alpha) = \alpha d(a, b)$ for all $a, b \in C$. As a consequence, we obtain $W(a, b, 0) = a$ and $W(a, b, 1) = b$.

Throughout in our paper, we assume that $F(T) = \{x \in C \subset X : Tx = x\}$ is the set of fixed points of $T : C \to C$.

Definition 2.5. A hyperbolic space (X, d) satisfies the property *(R)*, if $\{\tau_n\}$ is a nonincreasing sequence of nonempty, bounded, closed and convex subset of X,

$$\bigcap_{n=1}^{\infty} \tau_n \neq \emptyset.$$

Remark 2.6. Every uniformly convex hyperbolic space ($UCHS$) enjoys the Property (R).

Assume that (X, d, W) is HMS and C is a nonempty subset of X. Assume that $\{\theta_n\}$ is a bounded sequence in C and $\theta \in C$. Then

(a) $r(\{\theta_n\}, \theta) = \limsup_{n\to\infty} d(\theta_n, \theta)$ is known as asymptotic radius of $\{\theta_n\}$ at θ.
(b) $r(\{\theta_n\}, C) = \inf\{r(\{\theta_n\}, \theta) : \theta \in X\}$, is known as asymptotic radius of $\{\theta_n\}$ relative to C.
(c) $A(\{\theta_n\}, C) = \{\theta \in C : r(\{\theta_n\}, C) = r(\{\theta_n\}, \theta\}$ is known as asymptotic center of $\{\theta_n\}$ relative to C.

In 1976 Lim [25] developed the concept of Δ-convergence in a metric space. The idea of Δ-convergence is applied in $CAT(0)$ space by Kirk and Panyanak [21] and showed numerous Banach space results involving weak convergence have precise analogs in this environment.

Definition 2.7. A bounded sequence $\{x_n\}$ in X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Lemma 2.8. ([29]) *Suppose $\{\ell_n\}, \{m_n\}$ and $\{\delta_n\}$ are sequences of nonnegative satisfying*

$$\ell_{n+1} \leq \delta_n \ell_n + m_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty}(\delta_n - 1) < \infty$ *and* $\sum_{n=1}^{\infty} m_n < \infty$, *then* $\lim_{n\to\infty} \ell_n$ *exists.*

Lemma 2.9. ([24]) *Let C be a nonempty, closed and convex subset of complete uniformly convex hyperbolic metric space ($UCHMS$). Then every bounded sequence $\{x_n\} \subset X$ has a unique asymptotic center concerning C.*

Lemma 2.10. ([13]) *Let X be a $UCHS$. Let $R \in [0, \infty)$ be such that*

$$\limsup_{n\to\infty} d(x_n, a) \leq R, \ \ \limsup_{n\to\infty} d(y_n, a) \leq R$$

and

$$\limsup_{n\to\infty} d(W(x_n, y_n, a_n), a) = R,$$

where $\{\alpha_n\} \in [a, b]$ with $0 < a \leq b < 1$. Then we have $\limsup_{n\to\infty} d(x_n, y_n) = 0$.

A metric space (X, d) along with partial ordering $\preccurlyeq$ is denoted by $(X, d, \preccurlyeq)$. Two points x and y in X are comparable whenever $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition 2.11. ([7]) Let $(X, d, \preccurlyeq)$ be a partial ordered metric space ($POMS$). The map $T : X \to X$ is said to be monotone or order-preserving, if

$$x \preccurlyeq y \implies T(x) \preccurlyeq T(y)$$

for any $x, y \in X$.

Definition 2.12. Assume that $(X, d, \preccurlyeq)$ is a $POMS$ endowed with partial order and C is a nonempty subset of X. Then $T : C \to C$ is said to be

(i) a monotone Lipschitz mapping (MLM) [7], if there exist k such that

$$d(T(x), T(y)) \leq kd(x, y),$$

for every comparable element $x, y \in C$. If $k = 1$ the mapping T is said to be order preserving nonexpansive mapping.

(ii) a monotone asymptotically nonexpansive mapping ($MANM$) [6], if there exists a sequence $\{k_n\}$ of positive numbers such that $k_n \to 1$ as $n \to \infty$ and

$$d(T^n(x), T^n(y)) \leq k_n d(x, y),$$

for every comparable element $x, y \in C$.

(iii) a monotone nearly Lipschitzian (MNL) [33] with respect to $\{a_n\}$, if for each $n \in N$, there exist a constant $k_n \geq 0$ such that

$$d(T^n(x), T^n(y)) \leq k_n(d(x, y) + a_n), \tag{2.1}$$

where $a_n \in [0, \infty)$ with $a_n \to 0$ every comparable element $x, y \in C$.

Remark 2.13. The infimum of constant k_n for which the last inequality (2.1) hold, is denoted by $\eta(T^n)$ and called the nearly Lipschitz constant. The MNL mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be monotone nearly asymptotically nonexpansive ($MNAN$) [33], if $\eta(T^n) \geq 1$ for all $n \in N$ and $\lim_{n\to\infty} \eta(T^n) = 1$.

Example 2.14. ([34]) Let $C = [1, 4] \times [1, 4]$ and $T : C \to C$ be a mapping defined by

$$T(x_1, x_2) = \begin{cases} (2, 2) & \text{if } x \in [1, 2) \times [1, 2), \\ (4, 4) & \text{if } x \in [2, 4] \times [2, 4]. \end{cases}$$

Then T is a discontinuous type nearly asymptotically nonexpansive mapping with $a_1 = 14$ and $a_n = 0$ for $n \geq 2$, $k_n = 1$ for all $n \in \mathbb{N}$.

Let X be a partially ordered set with a partial order $\preccurlyeq$ and let $(X, d, \preccurlyeq)$ be a partially ordered hyperbolic metric space (POHMS). The elements $s, t \in X$ are said to be comparable if $s \preccurlyeq t$ or $t \preccurlyeq s$. Denote order interval for $s, t \in X$, as follows:

$$[t, \to) = \{x \in X : t \preccurlyeq x\} \quad \text{and} \quad (\leftarrow, s] = \{x \in X : x \preccurlyeq s\}.$$

Throughout the paper, we will assume that the order interval is closed and convex.

Lemma 2.15. ([24]) *Let (X, d) be a complete uniform convex hyperbolic metric space ($UCHMS$) and C be a nonempty convex and closed subset of X. Then every bounded sequence $\{x_n\} \in X$ has a unique asymptotic center for C.*

Theorem 2.16. ([3]) *Let (X, d) be a complete uniform convex partial ordered hyperbolic metric space ($UCPOHMS$) and C be a nonempty, convex, and closed subset of X which contains more than one point. If $T : C \to C$ is a continuous $MNAN$ mappings and there exists $x_0 \in C$ such that $x_0 \preccurlyeq T(x_0)$. Then T has fixed point.*

Definition 2.17. Let X be a $POMS$. Then, X is said to satisfy the monotone weak Opial condition whenever any monotone sequence $\{p_n\}$ in X which is Δ-convergent to $p \in X$. We have the following :

$$\limsup_{n\to\infty} d(p_n, p) < \limsup_{n\to\infty} d(p_n, q)$$

for every $q \in X$ such that p and q are not equal.

The continuity assumption in Theorem 2.16 was relaxed in [3] by using the weak Opial condition, and proved the following theorem.

Theorem 2.18. ([4]) *Let (X, d) be a complete $UCPOHMS$, and C be a nonempty, convex, and closed subset of X which contains more than one point. Assume that X satisfies the monotone weak Opial condition. If $T : C \to C$ is a $MNAN$ mapping and there exist $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable, then T has a fixed point.*

In 2019, Sajjan Agrawal et al. [4] proved the strong and Δ-convergence of Fibonacci-Mann iteration for a monotone non-Lipschitzian mapping (i.e. nearly asymptotically nonexpansive mapping) in $POHMS$ as follows:

Theorem 2.19. ([4]) *Let X be a complete $UCPOHMS$ and C be a nonempty, convex and closed subset of X and let $T : C \to C$ be a $MNAN$ mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by*

$$x_{n+1} = W(T^{f(n)}x_n, x_n, t_n) \tag{2.2}$$

for any $n \in N$, where f(n) is a Fibonacci sequence defined by $f(0) = f(1) = 1$ and $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$ with $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$), where $0 < a \leq t_n$ and $x_1 \in C$. If $p \preccurlyeq x_1$ (or $x_1 \preccurlyeq p$) for some $p \in F(T)$, then $\{x_n\}$ is Δ-convergent to a fixed point x^* of T.

3. Strong and Δ-convergence theorem

In this section, first, we introduce Fibonacci SR-iteration process ($FSRIP$) and prove strong and Δ-convergence theorem for our iteration process for $NANM$ in the setting of complete $UCPOHM$.

Definition 3.1. Let X be a HMS and C be a nonempty convex subset of X. Let $T : C \to C$ be a mapping. Fix $x_0 \in C$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Then $FSRIP$, $\{x_n\}$ is defined by

$$\begin{cases} y_n = W(T^{f(n)}x_n, x_n, \beta_n), \\ x_{n+1} = W(T^{f(n)}x_n, T^{f(n)}y_n, \alpha_n) \end{cases} \tag{3.1}$$

for any $n \in N$, where $f(n)$ is Fibonacci sequence defined as in Theorem 2.19.

Now we prove the following lemma for our main result.

Lemma 3.2. *Let X be a $POHMS$ and C be a convex and bounded nonempty subset of X. Assume that the map $T : C \to C$ is monotone. Let $x_1 \in C$ be such that $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$) and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] and consider the sequence $\{x_n\} \subset C$ generated by (3.1). Let p be the fixed point of T such that $x_1 \preccurlyeq p$ (or $p \preccurlyeq x_1$). Then*

(i) $T^n(x_1) \preccurlyeq T^{n+1}(x_1)$ *(or $T^{(n+1)}(x_1) \preccurlyeq T^n(x_1)$),*

(ii) $x_1 \preccurlyeq x_n \preccurlyeq p$ *(or* $p \preccurlyeq x_n \preccurlyeq x_1$*)*,
(iii) $T^{f(n)}(x_1) \preccurlyeq T^{f(n)}(x_n) \preccurlyeq p$ *(or* $p \preccurlyeq T^{f(n)}(x_n) \preccurlyeq T^{f(n)}(x_1)$*)*,
(iv) $x_n \preccurlyeq x_{n+1} \preccurlyeq T^{f(n)}(x_n)$ *(or* $T^{f(n)}(x_n) \preccurlyeq x_{n+1} \preccurlyeq x_n$*) for any* $n \in N$,
(v) $x_n \preccurlyeq p$ *(or* $p \preccurlyeq x_n$*), provided that* $\{x_n\}$ *is* Δ*-convergent to a point* $p \in C$ [36].

Proof. Using the convexity of the order intervals and the monotonicity of T, we can easily deduce (i), (ii), and (iii) see in [5]. We prove (iv) by induction. Without loss of generality, we assume that $x_0 \preccurlyeq T(x_0)$. First we note that $x_0 \preccurlyeq x_1 \preccurlyeq T^{f(0)}(x_0) = T(x_0)$. The monotonicity of T implies that $T(x_0) \preccurlyeq T(x_1)$ which yields $x_0 \preccurlyeq x_1 \preccurlyeq T^{f(1)}(x_1)$. Using the convexity of order interval $[x_1, T^{f(1)}(x_1)]$ and (3.1), we have

$$x_1 \preccurlyeq y_1 \preccurlyeq T^{f(1)}(x_1).$$

As T is monotone $T^{f(1)}(x_1) \preccurlyeq T^{f(1)}(y_1)$. Again by convexity of the order interval $[T^{f(1)}(x_1), T^{f(1)}(y_1)]$ and (3.1) we have $T^{f(1)}(x_1) \preccurlyeq x_2 \preccurlyeq T^{f(1)}(y_1)$, it implies that $x_1 \preccurlyeq x_2 \preccurlyeq T^{f(1)}(x_1)$. For fix $n \geq 2$, assume that $x_k \preccurlyeq x_{k+1} \preccurlyeq T^{f(k)}(x_k)$ for any $k \in [0, n-1]$. Now we claim that $x_n \preccurlyeq x_{n+1} \preccurlyeq T^{f(n)}(x_n)$. By the convexity of order intervals this will be hold if we prove that $x_n \preccurlyeq T^{f(n)}(x_n)$.

Our assumption implies

$$x_n \preccurlyeq T^{f(n-1)}(x_{n-1)}) \preccurlyeq T^{f(n-1)+f(n-2)}(x_{n-2}) = T^{f(n)}(x_{n-2}),$$

where we used the monotonicity of T, $x_{n-1} \preccurlyeq T^{f(n-2)}(x_{n-2})$ and the definition of Fibonacci sequence. Since $x_{n-2} \preccurlyeq x_{n-1} \preccurlyeq x_n$, the monotonicity of T implies that

$$x_n \preccurlyeq T^{f(n)}(x_n).$$

The induction argument is complete. □

Now we develop the following lemma for our main theorem.

Lemma 3.3. *Let* $(X, d, \preccurlyeq)$ *be a complete* $UCPOHMS$, C *be nonempty convex and closed subset of* X *and* $T : C \to C$ *be a* $MNAN$ *mapping with sequence* $\{a_{f(n)}, \eta(T^{f(n)})\}$ *and* $F(T) \neq \emptyset$ *such that* $\sum_{n=1}^{\infty} a_{f(n)} < \infty$ *and* $\sum_{n=1}^{\infty}(\eta(T^{f(n)}) - 1) < \infty$. *Let the sequence* $\{x_n\}$ *be defined by* (3.1) *with* $x_1 \preccurlyeq T(x_1)$ *(or* $T(x_1) \preccurlyeq x_1$*), where* $0 < a < \alpha_n, \beta_n \leq 1$ *and* $x_1 \in C$. *If* $p \preccurlyeq x_1$ *(or* $x_1 \preccurlyeq p$*) for some* $p \in F(T)$, *then following holds:*

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists.*
(ii) $\lim_{n\to\infty} d(T^{f(n)}(x_n), x_n) = 0$.

Proof. Let $p \in F(T)$. It follows form Lemma 3.2, $T^{f(n)}(x_n) \preccurlyeq p$. From (3.1), we get

$$\begin{aligned} d(y_n, p) &= d(W(T^{f(n)}x_n, x_n, \beta_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(T^{f(n)}x_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n \eta(T^{f(n)})\ d(x_n, p) + \eta(T^{f(n)})\ \beta_n\ a_{f(n)} \\ &\leq (1-\beta_n + \beta_n\ \eta(T^{f(n)}))\ d(x_n, p) + \eta(T^{f(n)})\ \beta_n\ a_{f(n)}. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(T^{f(n)}x_n, T^{f(n)}y_n, \alpha_n), p) \\ &\leq (1-\alpha_n)\ d(T^{f(n)}x_n, p) + \alpha_n\ d(T^{f(n)}y_n, p) \\ &\leq (1-\alpha_n)\Big(\eta(T^{f(n)})\ d(x_n, p) + \eta(T^{f(n)})\ a_{f(n)}\Big) \\ &\quad + \alpha_n\Big(\eta(T^{f(n)})d(y_n, p) + \eta(T^{f(n)})a_{f(n)}\Big) \\ &\leq (1-\alpha_n)\ \eta(T^{f(n)})\ d(x_n, p) + \alpha_n\ \eta(T^{f(n)})\ d(y_n, p) + \eta(T^{f(n)})a_{f(n)} \\ &\leq (1-\alpha_n)\ \eta(T^{f(n)})\ d(x_n, p) + \eta(T^{f(n)})a_{f(n)} \\ &\quad + \alpha_n\eta(T^{f(n)})\Big((1-\beta_n+\beta_n\ \eta(T^{f(n)}))d(x_n, p) + \eta(T^{f(n)})\ \beta_n\ a_{f(n)}\Big) \\ &\leq (\eta(T^{f(n)}) - \alpha_n\beta_n\ \eta(T^{f(n)}) + \alpha_n\beta_n\ \eta(T^{f(n)})^2)\ d(x_n, p) \\ &\quad + \eta(T^{f(n)})a_{f(n)}(1 + \alpha_n\beta_n\ \eta(T^{f(n)})). \end{aligned} \tag{3.3}$$

Also, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\Big(\eta(T^{f(n)}) - \alpha_n\ \beta_n\eta(T^{f(n)}) + \alpha_n\ \beta_n\eta(T^{f(n)})^2 - 1\Big) \\ &\quad = \sum_{n=1}^{\infty}(\eta(T^{f(n)}) - 1)(1 + \alpha_n\beta_n\eta(T^{f(n)})) \\ &\quad \leq \sup_{1 \leq n \leq \infty}(1 + \beta_n\alpha_n\eta(T^{f(n)}))\sum_{n=1}^{\infty}(\eta(T^{f(n)}) - 1) \\ &\quad < \infty. \end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \eta(T^{f(n)}) a_{f(n)} (1 + \alpha_n \ \beta_n \ \eta(T^{f(n)})) \\
&\quad \leq \sup_{1 \leq n \leq \infty} \eta(T^{f(n)}) (1 + \alpha_n \ \beta_n \ \eta(T^{f(n)})) \sum_{n=1}^{\infty} a_{f(n)} \\
&\quad < \infty.
\end{aligned}$$

It follows from Lemma 2.8 that $\lim_{n\to\infty} d(x_n, p)$ exists.

Next, we prove that $\lim_{n\to\infty} d(T^{f(n)}(x_n), x_n) = 0$. For this, let

$$\lim_{n\to\infty} d(x_n, p) = R \geq 0.$$

If $R = 0$, then it is obvious that

$$\lim_{n\to\infty} d(T^{f(n)}(x_n), x_n) = 0.$$

Let us assume that $\lim_{n\to\infty} d(x_n, p) = R > 0$.

$$\begin{aligned}
\limsup_{n\to\infty} d(T^{f(n)}(x_n), p) &\leq \limsup_{n\to\infty} (\eta(T^{f(n)}) d(x_n, p) + a_{f(n)} \eta(T^{f(n)})) \\
&\leq \limsup_{n\to\infty} d(x_n, p) = R. \qquad (3.4)
\end{aligned}$$

From (3.1), we have

$$\begin{aligned}
\limsup_{n\to\infty} d(y_n, p) &\leq \limsup_{n\to\infty} ((1 - \beta_n) d(x_n, p) + \beta_n d(T^{f(n)}(x_n), p)) \\
&\leq \limsup_{n\to\infty} (1 - \beta_n + \beta_n \eta(T^{f(n)})) d(x_n, p) + \limsup_{n\to\infty} \eta(T^{f(n)}) a_{f(n)} \\
&\leq \limsup_{n\to\infty} \eta(T^{f(n)}) d(x_n, p) + \limsup_{n\to\infty} \eta(T^{f(n)}) a_{f(n)} \\
&\leq R. \qquad (3.5)
\end{aligned}$$

Using (3.5), we have

$$\begin{aligned}
\limsup_{n\to\infty} d(T^{f(n)}(y_n), p) &\leq \limsup_{n\to\infty} (\eta(T^{f(n)}) d(y_n, p) + a_{f(n)} \eta(T^{f(n)})) \\
&\leq R. \qquad (3.6)
\end{aligned}$$

By (3.4) and (3.6), and application of Lemma 2.10, we have

$$\limsup_{n\to\infty} d(T^{f(n)}(x_n), T^{f(n)}(y_n)) = 0. \qquad (3.7)$$

Using (3.1) and (3.7), we have

$$\begin{aligned}\lim_{n\to\infty} d(x_{n+1}, T^{f(n)}(y_n)) &= \lim_{n\to\infty} d(W(T^{f(n)}(x_n), T^{f(n)}(y_n), \alpha_n), T^{f(n)}(y_n)) \\ &\leq \lim_{n\to\infty} \alpha_n d(T^{f(n)}(x_n), T^{f(n)}(y_n)) \\ &\to 0. \end{aligned} \tag{3.8}$$

Using (3.8) and triangle inequality, we have

$$\begin{aligned}\liminf_{n\to\infty} d(x_{n+1}, p) &\leq \liminf_{n\to\infty}\ d(x_{n+1}, T^{f(n)}(y_n)) + \liminf_{n\to\infty}\ d(T^{f(n)}(y_n), p) \\ &\leq \liminf_{n\to\infty}\ d(x_{n+1}, T^{f(n)}(y_n)) \\ &\quad + \liminf_{n\to\infty}\ (\eta(T^{f(n)})d(y_n, p) + \eta(T^{f(n)})a_{f(n)}), \\ R &\leq \liminf_{n\to\infty}\ d(y_n, p). \end{aligned} \tag{3.9}$$

Combining (3.5) and (3.9), we have

$$\begin{aligned} R &= \lim_{n\to\infty} d(y_n, p) \\ &= \lim_{n\to\infty} d(W(T^{f(n)}(x_n), x_n, p). \end{aligned}$$

Using Lemma 2.10, we have $\lim_{n\to\infty} d(T^{f(n)}x_n, (x_n)) = 0$. □

Theorem 3.4. *Let $(X, d, \preccurlyeq)$ be a complete $UCPOHMS$, C, T and $\{x_n\}$ be defined as in Lemma 3.3 with $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$), where $0 < a < \alpha_n, \beta_n \leq 1$ and $x_1 \in C$. If $p \preccurlyeq x_1$ (or $x_1 \preccurlyeq p$) for some $p \in F(T)$. Then $\{x_n\}$ is Δ-convergent to fixed point p of T.*

Proof. From Lemma 3.3-(i), $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F(T)$ and Lemma 3.3-(ii), the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} d(T^{f(n)}(x_n), x_n) = 0.$$

By Lemma 2.15, $\{x_n\}$ have unique asymptotic center. Let $A(\{x_n\}) = x^*$ and $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{u_n\}) = u$.

Now we claim that $x^* = u$. On contrary suppose that $x^* \neq u$. Then

$$\begin{aligned}\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x^*) \\ &\leq \limsup_{n\to\infty} d(x_n, x^*) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

which is a contradiction and hence $\Delta - \lim_{n\to\infty} x_n = x^*$.

Now we claim that $x^* \in F(T)$. From Lemma 3.2 $x_n \preccurlyeq x^*$.

$$\begin{aligned}\limsup_{n\to\infty} d(T^{f(n)}x^*, x_n) &\leq \limsup_{n\to\infty} d(T^{f(n)}(x_n), T^{f(n)}(x^*)) \\ &\quad + \limsup_{n\to\infty} d(T^{f(n)}(x_n), x_n) \\ &\leq \limsup_{n\to\infty} (\eta(T^{f(n)})d(x_n, x^*) + \eta(T^{f(n)})a_{f(n)}) \\ &\quad + \limsup_{n\to\infty} d(T^{f(n)}(x_n), x_n) \\ &\leq \limsup_{n\to\infty} d(x^*, x_n), \\ r(T^{f(n)}x_n, \{x_n\}) &\leq r(x^*, \{x_n\}). \end{aligned} \tag{3.10}$$

Since $\Delta - \lim_{n\to\infty} x_n = x^*$, therefore

$$r(x^*, \{x_n\}) \quad \leq \quad r(T^{f(n)}(x_n), \{x_n\}). \tag{3.11}$$

Combining (3.10) and (3.11), we have $T^{f(n)}x^* = x^*$, which completes the proof. $\square$

Theorem 3.5. *Let $(X, d, \preccurlyeq)$ be a complete $UCPOHMS$, C, T and $\{x_n\}$ be defined as in Lemma 3.3 with $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$), where $0 < a < \alpha_n, \beta_n \leq 1$ and $x_1 \in C$. If $p \preccurlyeq x_1$ (or $x_1) \preccurlyeq p$ for some $p \in F(T)$, then $\{x_n\}$ is convergent strongly to fixed point x^* of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

Proof. It is easy to see that if $\{x_n\}$ converges to a point $x^* \in F(T)$, then $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. For converse part, suppose that

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

From the proof of Theorem 3.4, $\lim_{n\to\infty} d(x_n, x^*)$ exist. But as it is given in the hypothesis that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, therefore,

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Thus, for a given $\epsilon > 0$ there exist a $K(\epsilon) \in N$ such that

$$d(x_n, F(T)) < \frac{\epsilon}{2} \text{ for } n > K(\epsilon).$$

Particularly, $\inf\{d(x_k, x^*) : x^* \in F(T)\} < \frac{\epsilon}{2}$. So there exist $x^* \in F(T)$ such that $d(x_k, x^*) < \frac{\epsilon}{2}$. Now, for $n, m > K(\epsilon)$

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x^*, x_m) < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset of X then

$$\lim_{n\to\infty} x_n = x^* \in C.$$

□

Theorem 3.6. *Let $(X, d, \preccurlyeq)$ be a complete $UCPOHMS$, C be a nonempty, convex and compact subset of X and $T : C \to C$ be a continuous $MNAM$ with sequence $\{a_{f(n)}, \eta(T^{f(n)})\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_{f(n)} < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^{f(n)}) - 1) < \infty$. Let the sequence $\{x_n\}$ be defined by* (3.1) *with $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$), where $0 < a < \alpha_n, \beta_n \leq 1$ and $x_1 \in C$. If $p \preccurlyeq x_1$ (or $x_1 \preccurlyeq p$) for some $p \in F(T)$, then $\{x_n\}$ is convergent strongly to fixed point x^* of T.*

Proof. From Theorem 3.3-(ii), we have

$$\lim_{n\to\infty} d(T^{f(n)}(x_n), x_n) = 0.$$

Since C is compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges strongly to $y \in C$. Therefore,

$$\lim_{k\to\infty} d(x_{n_k}, y) = 0.$$

Now

$$\begin{aligned} d(x_{n_k}, T^{f(n_k)}p) &\leq d(x_{n_k}, T^{f(n_k)}(x_{n_k})) + d(T^{f(n_k)}(x_{n_k}), T^{f(n_k)}(p)) \\ &\leq d(x_{n_k}, T^{f(n_k)}(x_{n_k})) + \eta(T^{f(n_k)})d(x_{n_k}, p) + \eta(T^{f(n_k)})a_{f(n_k)}. \end{aligned}$$

Taking limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{n_k}, T^{f(n_k)}(x_{n_k})) = 0.$$

By uniqueness of limit, we obtain

$$T^{f(n_k)}(p) = p.$$

That is $p \in F(T)$. Since $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$. Hence, $\{x_n\}$ converges strongly to $p \in F(T)$. □

Corollary 3.7. *Let $(X, d, \preccurlyeq)$ be a complete $UCPOHMS$, C be a nonempty convex subset of X and $T : C \to C$ be a $MANM$ with sequence $\{a_{f(n)}, \eta(T^{f(n)})\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_{f(n)} < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^{f(n)}) - 1) < \infty$. Let sequence $\{x_n\}$ be defined by* (3.1) *with $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$), where*

$0 < a < \alpha_n, \beta_n \leq 1$ and $x_1 \in C$. If $p \preccurlyeq x_1$ (or $x_1 \preccurlyeq p$) for some $p \in F(T)$, then $\{x_n\}$ is Δ-convergent to fixed point p of T.

Corollary 3.8. *Let $(X, d, \preccurlyeq)$ be a complete $UCPOHMS$, C, T and $\{x_n\}$ be defined as in Corollary 3.7 with $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$), where $0 < a < \alpha_n, \beta_n \leq 1$ and $x_1 \in C$. If $p \preccurlyeq x_1$ (or $x_1 \preccurlyeq p$) for some $p \in F(T)$, then $\{x_n\}$ is convergent strongly to fixed point x^* of T.*

Remark 3.9. Corollaries 3.7 and 3.8 are the generalization of Theorem 3.6 and Theorem 3.11 of [5] in two ways, first $FMIP$ defined in 2.2 to $FSRIP$ and second from $UCBS$ to more general space $UCPOHMS$.

4. Weak w^2-stability result

We know that a fixed point iteration is numerically stable if small perturbation (due to approximation, rounding errors, etc.) during computation will produce small changes in the approximate value of the fixed point computed by methods. In this direction, in 1988, Harder and Hicks [16] gave the formal definition of stability and proved some stability results for Picard, Mann, and Kirk fixed point iteration procedures under various contractive conditions. Let us, first we define stability.

Definition 4.1. ([16]) Let (X, d) be a metric space, T be a self-mapping on X and $\{x_n\}$ be iterative sequence produced by the mapping T such that

$$\begin{cases} x_1 \in X, \\ x_{n+1} = f(T(x_n)), \end{cases} \tag{4.1}$$

where x_1 is an initial approximation and f is a function. Assume that $\{x_n\}$ converges strongly to $p \in F(T)$. If for an arbitrary sequence $\{y_n\} \subset X$,

$$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} y_n = p,$$

then the iterative sequence $\{x_n\}$ is said to be stable with respect to T or simply stable.

Definition 4.2. ([32]) Let (X, d) be a metric space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X. We say that the sequences are equivalent if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

The following definition of w^2-stability was given by Timis [39] in 2010.

Definition 4.3. ([39]) Let (X, d) be a metric space, T be a self-mapping on X and $\{x_n\} \subset X$ be an iterative sequence given by (3.1). Suppose that $\{x_n\}$ converges strongly to $p \in F(T)$. If for any equivalent sequence $\{y_n\} \subset X$ of $\{x_n\}$

$$\lim_{n\to\infty} d(y_{n+1}, f(T(y_n)) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} y_n = p,$$

then iterative $\{x_n\}$ is said to be weak w^2-stable with respect to T.

Theorem 4.4. *Let $(X, d, \preccurlyeq)$ be a complete $UCPOHMS$, C be a nonempty convex subset of X and $T : C \to C$ be a continuous $MNAN$ mapping with $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_{f(n)} < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^{f(n)}) - 1) < \infty$. If the sequence $\{x_n\}$ is defined by (3.1) with $x_1 \preccurlyeq T(x_1)$ (or $T(x_1) \preccurlyeq x_1$), where $0 < a < \alpha_n, \beta_n \leq 1$ and $x_1 \in C$. If $p \preccurlyeq x_1$ (or $x_1 \preccurlyeq p$) for some $p \in F(T)$, and $\{p_n\}$ is any equivalent sequence of $\{x_n\}$ with $x_n \preccurlyeq p_n$ (or $p_n \preccurlyeq x_n$) defined by*

$$\begin{cases} p_1 \in C, \\ p_{n+1} = W(T^{f(n)}p_n, T^{f(n)}q_n, \alpha_n), \\ q_n = W(T^{f(n)}p_n, p_n, \beta_n), \end{cases} \tag{4.2}$$

then iteration process (3.1) *is weak w^2-stable with respect to T.*

Proof. Since $x_n \preccurlyeq p_n$ then by monotonicity of T, $T^{f(n)}(x_n) \leq T^{f(n)}(p_n)$. Now, we compute by using (3.1) and (4.2), we have

$$\begin{aligned} d(y_n, q_n) &= d(W(T^{f(n)}x_n, (x_n), \beta_n), W(T^{f(n)}(p_n), p_n, \beta_n) \\ &\leq (1 - \beta_n)d(x_n, p_n) + \beta_n d(T^{f(n)}(x_n), T^{f(n)}(p_n)) \\ &\leq (1 - \beta_n)d(x_n, p_n) + \beta_n \eta(T^{f(n)})d(x_n, p_n) + \beta_n a_{f(n)}\eta(T^{f(n)}) \\ &\leq (1 - \beta_n + \beta_n \eta(T^{f(n)}))d(x_n, p_n) + \beta_n a_{f(n)}\eta(T^{f(n)}). \end{aligned} \tag{4.3}$$

Next, we compute by using (3.1) and (4.2), we have

$$\begin{aligned} d(p_{n+1}, x_{n+1}) &\leq d(W(T^{f(n)}(p_n), T^{f(n)}(q_n), \alpha_n), W(T^{f(n)}(x_n), T^{f(n)}(y_n), \alpha_n)) \\ &\quad + (1-\alpha_n)d(T^{f(n)}(p_n)), T^{f(n)}(x_n)) + \alpha_n d(T^{f(n)}(q_n), T^{f(n)}(y_n)) \\ &\leq (1 - \alpha_n)(\eta(T^{f(n)})d(p_n, x_n) + a_{f(n)}\eta(T^{f(n)})) \\ &\quad + \alpha_n(\eta(T^{f(n)})d(y_n, p_n) + \eta(T^{f(n)})a_{f(n)}) \\ &\leq (1 - \alpha_n)\eta(T^{f(n)})d(p_n, x_n) \\ &\quad + \alpha_n \eta(T^{f(n)})d(y_n, p_n) + \eta(T^{f(n)})a_{f(n)}. \end{aligned} \tag{4.4}$$

Using (4.3) and (4.4), we have

$$\begin{aligned} d(p_{n+1}, x_{n+1}) &\leq (1-\alpha_n)\eta(T^{f(n)})d(p_n, x_n) + \eta(T^{f(n)})a_{f(n)} \\ &\quad + \alpha_n\eta(T^{f(n)})((1-\beta_n + \beta_n\eta(T^{f(n)}))d(x_n, p_n) \\ &\quad + \beta_n a_{f(n)}\eta(T^{f(n)})) \\ &\leq R_n d(x_n, p_n) + S_n a_{f(n)}\eta(T^{f(n)}), \end{aligned} \tag{4.5}$$

where

$$R_n = (\eta(T^{f(n)}) - \alpha_n\beta_n\ \eta(T^{f(n)}) + \alpha_n\beta_n\ \eta(T^{f(n)})^2)$$

and

$$S_n = 1 + \alpha_n\beta_n\ \eta(T^{f(n)}).$$

Now set

$$\varepsilon_n = d(p_{n+1}, W(T^{f(n)}(p_n), T^{f(n)}(q_n), \alpha_n). \tag{4.6}$$

Suppose that $\varepsilon_n \to 0$ as $n \to \infty$. Then

$$\begin{aligned} d(p_{n+1}, x^*) &\leq d(p_{n+1}, W(T^{f(n)}(p_n), T^{f(n)}(q_n), \alpha_n) \\ &\quad + d(p_{n+1}, x_{n+1}) + d(x_{n+1}, x^*). \end{aligned} \tag{4.7}$$

Taking $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty} d(p_{n+1}, x^*) = 0.$$

Thus $\{x_n\}$ is weak w^2-stable with respect to T. □

5. Numerical example

Let $X = R,\ C = [0, 4]$ and define a mapping $T : C \to C$ as

$$T(x) = \begin{cases} 2 & \text{if } x \in [0, 2), \\ 4 & \text{if } x \in [2, 4]. \end{cases} \tag{5.1}$$

is $MNAN$ mapping with discontinuity. However, T is not asymptotically nonexpansive mapping. The sequence $\{a_n\}$ with $a_1 = 2$ is eventually constant sequence which converges to 0, we have

$$\|Tx - Ty\| \leq \|x - y\| + a_1$$

and

$$T^n x = 4$$

for all $x \in [0, 4]$ and $n > 1$.

Figure 1, Figure 2, Figure 3, and Figure 4 shows the convergence behavior of the $FSRIP$ and $FMIP$ for $t_n = .5, t_n = .55, t_n = .8$ and $t_n = \frac{1}{\sqrt{n^3+1}}$ for different initial values.

FIGURE 1. Convergence table of ***FSRIP*** and ***FMIP*** for t_n=.5

Iteration No	FSRI For x = 0	FMI For x = 0	FSRI For x = 1	FMI For x = 1	FSRI For x = 2	FMI For x = 2
1	0	0	1	1	4	2
2	2	1	2	1.5	4	3
3	4	1.5	4	1.75	4	3.5
4	4	1.75	4	1.875	4	3.75
5	4	2.875	4	2.9375	4	3.875
10	4	3.96484	4	3.9668	4	3.99609
11	4	3.98242	4	3.9834	4	3.99805
14	4	3.9978	4	3.99792	4	3.99976
18	4	3.99986	4	3.99987	4	3.99998
19	4	3.99993	4	3.99994	4	3.99999
20	4	3.99997	4	3.99997	4	4
21	4	3.99998	4	3.99998	4	4
22	4	3.99999	4	3.99999	4	4
23	4	4	4	4	4	4

FIGURE 2. Convergence graph of ***FSRIP*** and ***FMIP***for t_n=.5

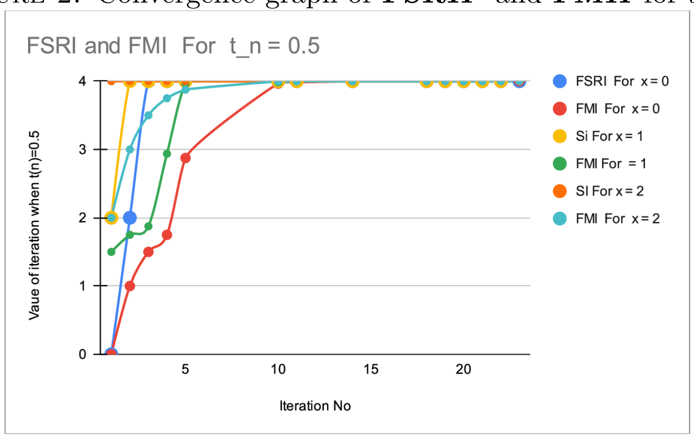


FIGURE 3. Convergence table of ***FSRIP*** and ***FMIP*** for t_n=.55

Iteration No	FSRI For x = 0	FMI For x = 0	FSRI For x = 1	FMI For x = 1	FSRI For x = 2	FMI For x = 2
1	0	0	1	1	4	2
2	2	1.1	2	1.55	4	3.1
3	4	1.595	4	1.7975	4	3.595
4	4	1.81775	4	1.90888	4	3.81775
5	4	3.01799	4	3.05899	4	3.91799
10	4	3.98188	4	3.98264	4	3.99849
11	4	3.99185	4	3.99219	4	3.99932
14	4	3.99926	4	3.99929	4	3.99994
18	4	3.99997	4	3.99997	4	4
19	4	3.99999	4	3.99999	4	4
20	4	3.99999	4	3.99999	4	4
21	4	4	4	4	4	4
22	4	4	4	4	4	4
23	4	4	4	4	4	4

FIGURE 4. Convergence graph of ***FSRIP*** and ***FMIP*** for $t_n = \frac{1}{\sqrt{n^3+1}}$

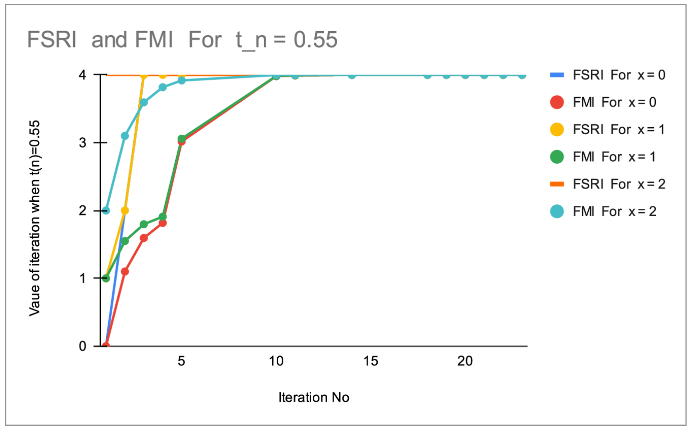


FIGURE 5. Convergence table of ***FSRIP*** and ***FMIP*** for t_n=.8

Iteration No	FSRI For x = 0	FMI For x = 0	FSRI For x = 1	FMI For x = 1	FSRI For x = 2	FMI For x = 2
1	0	0	1	1	4	2
2	2	1.6	2	1.8	4	3.6
3	4	1.92	4	1.96	4	3.92
4	4	1.984	4	1.992	4	3.984
5	4	3.5968	4	3.5984	4	3.9968
10	4	3.99987	4	3.99987	4	4
11	4	3.99997	4	3.99997	4	4
14	4	4	4	4	4	4
18	4	4	4	4	4	4
19	4	4	4	4	4	4
20	4	4	4	4	4	4
21	4	4	4	4	4	4
22	4	4	4	4	4	4
23	4	4	4	4	4	4

FIGURE 6. Convergence graph of ***FSRIP*** and ***FMIP*** for t_n=.8

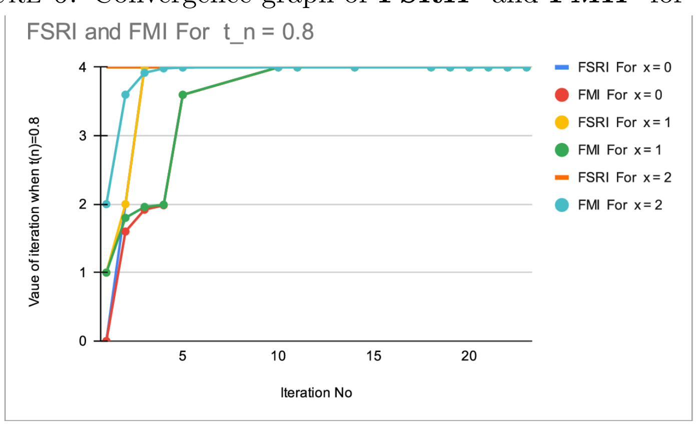
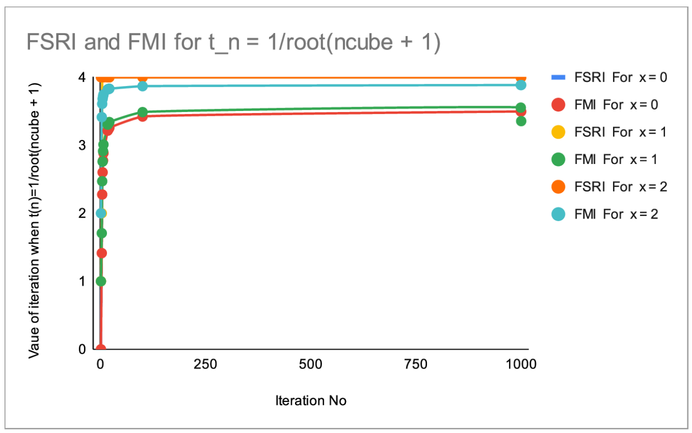


FIGURE 7. Convergence table of ***FSRIP*** and ***FMIP*** for $t_n = \frac{1}{\sqrt{n^3+1}}$

Iteration No	FSRI For x = 0	FMI For x = 0	FSRI For x = 1	FMI For x = 1	FSRI For x = 2	FMI For x = 2
1	0	0	1	1	4	2
3	2	1.41421	2	1.70711	4	3.41421
4	4	2.27614	4	2.4714	4	3.60948
5	4	2.60192	4	2.76028	4	3.68328
6	4	2.77533	4	2.91405	4	3.72256
7	4	2.88443	4	3.01079	4	3.74728
17	4	3.21468	4	3.30363	4	3.82209
18	4	3.22695	4	3.31451	4	3.82487
19	4	3.23798	4	3.32429	4	3.82737
20	4	3.24795	4	3.33314	4	3.82963
21	4	3.25703	4	3.34119	4	3.83169
100	4	3.42253	4	3.48794	4	3.86918
999	4	3.49707	4	3.55404	4	3.88607
1000	4	3.49709	4	3.35533	4	3.88607

FIGURE 8. Convergence graph of ***FSRIP*** and ***FMIP*** for $t_n = \frac{1}{\sqrt{n^3+1}}$



REFERENCES

[1] M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mat. Vesnik, **66** (2014), 223-–234.

[2] R.P. Agarwal, D.O. Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., **8** (2007), 61.

[3] S. Aggarwal, I. Uddin and J.J. Nieto, *A fixed-point theorem for monotone nearly asymptotically nonexpansive mappings*, J. Fixed Point Theory Appl., **21** (2019), 91.

[4] S. Aggarwal and I. Uddin, *Convergence and stability of Fibonacci-Mann iteration for a monotone non-Lipschitzian mapping*, Demonstr. Math., **2019**(52) (2019),388-–396.

[5] M.R. Alfuraidan and M.A. Khamsi, *Fibonacci-Mann iteration for monotone asymptotically nonexpansive mappings*, Bull. Aust. Math. Soc., **96** (2017), 307-–316.

[6] M.R. Alfuraidan and M.A. Khamsi, *A fixed Point Theorem for monotone asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **146** (2018), 2451-–2456.

[7] M. Bachar and M.A. Khamsi, *Fixed points of monotone mappings and application to integral equations*, Fixed Point Theory Appl., **2015** (2015), Article ID 110.

[8] F.E. Browder, *Fixed point theorem for noncompact mappings in Hilbert space*, Proc. Net. Acad. Sci. USA, **53** (1965), 1272-–1276.

[9] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Net. Acad. Sci. USA, **54** (1965), 1041-–1044.

[10] A. Chen and D.J. Wen, *Convergence analysis of an accelerated iteration for monotone generalized α-nonexpansive mappings with a partial order*, J. Funct. Spaces, **2019** (2019), Article ID 2789819, 8 pages.

[11] S. Dahputre, R. Tiwari and J. Shrivas, *Approximating fixed points for generalized α-nonexpansive mapping in $CAT(0)$ space via new iterative algorithm*, Nonlinear Funct. Anal. Appl., **29**(1) (2024), 69-81.

[12] B.A.B. Dehaish and M.A. Khamsi, *Browder and Gohde fixed point theorem for monotone nonexpansive mappings*, Fixed Point Theory Appl., **2016** (2016), 20.

[13] H. Fukhar-ud-din and M.A. Khamsi, *Approximating common fixed point in hyperbolic spaces*, Fixed Point Theory Appl., **2014** (2014), Article ID 113.

[14] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35** (1972), 171-–174.

[15] D. Gohde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr., **30** (1965), 251-–258.

[16] A.M. Harder and T.L. Hicks, *Stability result for fixed point iteration procedures*, Math. Japonica, **33** (1988), 693-–706.

[17] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., **44**(1974), 147-–150.

[18] J.K. Kim, Samir Dashputre, Padmavati and Rashmi Verma, *Convergence theorems for generalized α-nonexpansive mappings in uniformly hyperbolic spaces*, Nonlinear Funct. Anal. Appl., **29**(1) (2024), 1–14.

[19] J.K. Kim, S. Dashputre, Padmavati and K. Sakure, *Generalized α-nonexpansive mappings in Hyperbolic spaces,* Nonlinear Funct. Anal. Appl., **27**(3) (2022), 449-469.

[20] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.

[21] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68** (2008), 3689—3696.

[22] U. Kohlenbach, *Some logical theorems with application in functional analysis*, Trans. Amer. Math. Soc., **357** (2005), 89-–128.

[23] L. Leustean, *A quadratic rate of asymptotic regularity for CAT(0) space*, J. Math. Anal. Appl., **325** (2007), 386-–399.

[24] L. Leustean, *Nonexpansive iterations in uniformly convex W-hyperbolic spaces*, Nonlinear Anal. Optim.: Nonlinear Analysis, **513** (2010), 193–209.

[25] T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., **60** (1976), 179-–182.

[26] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506—510.

[27] J.J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation*, Order, **22**(3) (2005), 223-–239.

[28] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251** (2000), 217-–229.

[29] M.O. Osilike and S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Comput. Modellings, **32** (2000), 1181-–1191.

[30] R. Pandey, R. Pant, and A. al Rawashdesh, *Fixed point results for a class of monotone nonexpansive type mappings in hyperbolic space*, J. Funct. Space (2018), 11 pages.

[31] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2005), 1435-–1443.

[32] P. Rubbioni and T. Cardinali, *A generalization of the Caristi fixed point theorem in metric space*, Fixed Point Theory, **11** (2010), 3—10.

[33] D.R. Sahu, *Fixed point of demi continuous nearly Lipschitzian mappings in Banach space*, Comment. Math. Uni. Carolin., **46** (2005), 653-–666.

[34] G.S. Saluja and H.K. Nashine, *Convergence of an implicit iteration process for a finite family of asymptotically quasi nonexpansive mappings in convex metric spaces*, Opusc. Math., **30** (2010), 331-–340.

[35] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **58** (1991), 407-–413.

[36] R. Shukla, R. Pant and P. Kumam, *On the α-nonexpansive mapping in partially ordered hyperbolic metric spaces*, J. Math. Anal., **8** (2017), 1-–15.

[37] W. Takahashi, *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep., **22** (1970), 142-–149.

[38] B.S. Thakur, D. Thakur and M. Postolache, *A new iteration scheme for approximating fixed points of nonexpansive mappings*, Filomat, **30** (2016), 2711-–2720.

[39] I. Timis, *On the weak stability of Picard iteration for some contractive type mappings*, Annal. Uni. Craiova, Math. Comput. Sci. Series, **37** (2010), 106-–114.