Nonlinear Functional Analysis and Applications
Vol. 30, No. 1 (2025), pp. 99-111
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2025.30.01.07
http://nfaa.kyungnam.ac.kr/journal-nfaa

CERTAIN CLASS OF ANALYTIC AND UNIVALENT FUNCTION WITH RESPECT TO SYMMETRIC

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Abstract. This study delves into the intricate properties of analytic functions within the unit disk, specifically focusing on classes $s^*(\alpha,\beta,\xi,\gamma)$ and $K^*(\alpha,\beta,\xi,\gamma)$ characterized by symmetric properties with negative coefficients. We present the findings on key aspects such as coefficient inequalities establishing bounds on the coefficients of functions belonging to the classes. Also, determine the convexity radius for functions in a class and distortion theorems. Distortion theorems will be presented to provide bounds within the unit disk. These bounds are critical for understanding the behavior and geometric properties of these functions. Symmetric Classes with Negative Coefficients: Classes are defined by their symmetry and negative coefficients. Functions in these classes exhibit specific geometric properties that are crucial for applications in complex analysis.

1. Introduction

Let A be a class of functions which is analytic and univalent in the unit disk $U = \{\Im : |\Im| < 1\}$ provided by

$$\hbar(\Im) = \Im + \sum_{\tau=2}^{\infty} a_\tau \Im^\tau, \tag{1.1}$$

and made common by $\hbar(0) = 0$, $\hbar'(0) = 1$. Let S be the subclass of A that has the following analytic and univalent function type (1.1). We designate the

[0]Received March 27, 2024. Revised May 19, 2024. Accepted May 27, 2024.
[0]2020 Mathematics Subject Classification: 30C45.
[0]Keywords: Univalent function, coefficient inequality, distortion theorem, radius of convexity, symmetry.

subclass of S as represented by $S^*(\alpha)$ and $K(\alpha)$, which includes all functions that are starlike and convex of order, respectively. $\alpha(0 \leq \alpha < 1)$ in U, that is

$$S^*(\alpha) = \left\{ \hbar \in s; \mathrm{Re}\left(\Im \frac{\hbar'(\Im)}{\hbar(\Im)} \right) > \alpha; 0 \leq \alpha < 1, \Im \in U \right\} \tag{1.2}$$

and

$$K(\alpha) = \left\{ \hbar \in s; \mathrm{Re}\left(1 + \Im \frac{\hbar''(\Im)}{\hbar'(\Im)} \right) > \alpha; 0 \leq \alpha < 1 \Im \in U \right\}. \tag{1.3}$$

We say that the function $\hbar(\Im)$ is in the class $S(\alpha, \beta, \xi, \gamma)$ if and only if

$$\left| \frac{\Im \frac{\hbar'(\Im)}{\hbar(\Im)} - 1}{2\xi \left(\Im \frac{\hbar'(\Im)}{\hbar(\Im)} - \alpha \right) - \gamma \left(\Im \frac{\hbar'(\Im)}{\hbar(\Im)} - 1 \right)} \right| < \beta \tag{1.4}$$

for $|\Im| < 1$, where $0 < \beta \leq 1$, $\frac{1}{2} \leq \xi \leq 1$, $0 \leq \alpha \leq \frac{1}{2}$, $\frac{1}{2} < \gamma \leq 1$. A function $f(z)$ is said to belong to the class $k(\alpha, \beta, \xi, \gamma)$ if and only if $zf'(z) \in S(\alpha, \beta, \xi, \gamma)$. Several subclasses of analytic function classes A have been introduced and studied by numerous researchers (see, for example, [8], and [12]).

Several subclasses of A were introduced, and various studies (see [3], [4], [7], [9], [10], [11] and [13]) looked into some of the geometric properties of these subclasses.

Let T denote the subclass of S consisting of functions of the form,

$$\hbar(\Im) = \Im - \sum_{\tau=2}^{\infty} a_\tau \Im^\tau, \quad (a_\tau \geq 0). \tag{1.5}$$

Now let $S^*(\alpha, \beta, \xi, \gamma) = S(\alpha, \beta, \xi, \gamma) \cap T$ and $k^*(\alpha, \beta, \xi, \gamma) = k(\alpha, \beta, \xi, \gamma) \cap T$.

In this work, we derive analytical solutions for the distortion theorem, radius of convexity, coefficient inequality, and related conclusions with respect to symmetric for the classes $s^*(\alpha, \beta, \xi, \gamma)$ and $k^*(\alpha, \beta, \xi, \gamma)$ and shall be denoted by classes $s^*_s(\alpha, \beta, \xi, \gamma)$ and $k^*_s(\alpha, \beta, \xi, \gamma)$.

These functions were studied by A1-Amiri and Mocanu [1], Alsoboh and Darus [2], Ghanim and Darus [5, 6], Sakaguchi [9], and Sudharsan et al.[13]. We refer to them as starlike with respect to symmetric points. E1-Ashwah and Thomas [4] have presented two new function classes: the class of functions that are starlike concerning conjugate points and the class of functions that are starlike concerning symmetric conjugate points. We say that the function

$\hbar(\Im)$ is the class $s_s^*(\alpha,\beta,\xi,\gamma)$ if and only if

$$\left|\frac{\Im\frac{\hbar'(\mathfrak{S})}{\hbar(\Im)-\hbar(-\Im)}-1}{2\xi\left(\Im\frac{\hbar'(\Im)}{\hbar(\Im)-\hbar(-\Im)}-\alpha\right)-\gamma\left(\Im\frac{\hbar'(\Im)}{\hbar(\Im)-\hbar(-\Im)}-1\right)}\right|<\beta. \tag{1.6}$$

Next, we find the coefficient inequality for the class $s_s^*(\alpha,\beta,\xi,\gamma)$.

2. Coefficient inequality

Theorem 2.1. *A function $\hbar_l \in T$ given by* (1.1) *is in the class* $s_s^*(\alpha,\beta,\xi,\gamma)$ *if and only if*

$$\sum_{\tau=2}^{\infty}[(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)]\,|a_z|\le 4\beta\xi(1-\alpha). \tag{2.1}$$

Proof. Suppose

$$\sum_{\tau=2}^{\infty}[(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)]\,[a_z]\le 4\beta\xi(1-\alpha).$$

Then, we have

$$\begin{aligned}\|\Im\hbar'(\Im)-(\hbar(\Im)-\hbar(-\Im))\|-\beta\|2\xi(\Im\hbar'(\Im)-\alpha(\hbar(\Im)-\hbar(-\Im)))\\ -\gamma(\Im\hbar'(\Im)-(\hbar(\Im)-\hbar(-\Im)))\|<0.\end{aligned}$$

Given that

$$\left|\sum_{\tau=2}^{\infty}(\tau-2)a_\tau\Im^\tau\right|-\beta\left|4\xi(1-\alpha)+\sum_{\tau=2}^{\infty}(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\Im^\tau\right|<0$$

for $|\Im|=r\to 1$, then the circumstance (2.1) is surrounded above by

$$\begin{aligned}&\sum_{\tau=2}^{\infty}(\tau-2)\,|a_\tau|-4\beta\xi(1-\alpha)-\beta\sum_{\tau=2}^{\infty}(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\,|a_\tau|\\ &\quad\le\sum_{\tau=\tau}^{\infty}\{(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\}\,|a_\tau|\,r^\tau-4\beta\xi(1-\alpha)\\ &\quad\le\sum_{\tau=2}^{\infty}\{(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\}\,|a_\tau|-4\beta\xi(1-\alpha)\\ &\quad\le 0.\end{aligned}$$

Therefore, $\hbar(\Im)\in s_s^*(\alpha,\beta,\xi,\gamma)$.

Now, we prove the converse result.

Let

$$\left|\frac{\Im\frac{\hbar'(\Im)}{\hbar(\Im)-\hbar(-\Im)}-1}{2\xi\left(\Im\frac{\hbar'(\Im)}{\hbar(\Im)-\hbar(-\Im)}-\alpha\right)-\gamma\left(\Im\frac{\hbar'(\Im)}{\hbar(\Im)-\hbar(-\Im)}-1\right)}\right|$$
$$\leq\left|\frac{\sum_{\tau=2}^{\infty}(\tau-2)a_\tau\Im^\tau}{4\xi(1-\alpha)+\sum_{\tau=2}^{\infty}(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)a_\tau\Im^\tau}\right|<\beta,$$

as $|\operatorname{Re}(\Im)|\leq|\Im|$ for all $\Im$, we have

$$\operatorname{Re}\left|\frac{\sum_{\tau=2}^{\infty}(\tau-2)a_\tau\Im^\tau}{4\xi(1-\alpha)+\sum_{\tau=2}^{\infty}(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)a_\tau\Im^\tau}\right|<\beta.$$

We select values on the real axis so that $\frac{\Im\hbar'(\Im)}{\hbar(\Im)-\hbar(-\Im)}$ is real, and after removing the denominator from the previous equation and allowing real values $\Im\to 1$ to pass through, we have

$$\sum_{\tau=2}^{\infty}\{(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\}\,|a_z|-4\beta\xi(1-\alpha)\leq 0.$$

□

Remark 2.2. If $\hbar(\Im)\in s_s^*(\alpha,\beta,\xi,\gamma)$, then

$$|a_\tau|\leq\frac{4\beta\xi(1-\alpha)}{\{(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\}}\quad\text{for }\tau=2,3,\ldots.\tag{2.2}$$

Equality holds for

$$\hbar(\Im)=\Im-\frac{4\beta\xi(1-\alpha)}{\{(-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\}}.$$

Corollary 2.3. *If $\hbar(\Im)\in s_s^*(\alpha,\beta,\xi,1)$, that is, replacing $\gamma=1$, then we get*

$$|a_\tau|\leq\frac{4\beta\xi(1-\alpha)}{\{(\tau-2)-\beta(\tau-2+4\xi\alpha-2\tau\xi)\}}\quad\text{for }\tau=2,3,\ldots.\tag{2.3}$$

The same applies to

$$\hbar(\Im)=\Im-\frac{4\beta\xi(1-\alpha)}{\{(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\}}\Im^\tau.$$

Corollary 2.4. *If $\hbar(\Im)\in s_s^*(\alpha,\beta,1,1)$, we get*

$$|a_\tau|\leq\frac{4\beta(1-\alpha)}{\{(\tau-2)+\beta(\tau+2-4\alpha)\}}.\tag{2.4}$$

Equality holds for

$$\hbar(\Im)=\Im-\frac{4\beta(1-\alpha)}{\{(\tau-2)+\beta(\tau+2-4\alpha)\}}\Im^\tau.$$

Corollary 2.5. *If $\hbar(\Im) \in s_s^*(\alpha)$, that is, starlike in relation to the symmetric order point, if and only if*

$$\sum_{\tau=2}^{\infty}(\tau-2\alpha)\,|a_\tau| \leq (1-\alpha). \tag{2.5}$$

Theorem 2.6. *A function $\hbar$ given by (1.1) is in $k_s^*(\alpha,\beta,\xi,\gamma)$, if and only if*

$$\sum_{\tau=2}^{\infty}\tau[(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)]\,|a_\tau| \leq 4\beta\xi(1-\alpha).$$

Proof. The function with regard to symmetric points of order makes the proof of this theorem similar to that of Theorem (2.1). $\hbar(\Im) \in k_s^*(\alpha,\beta,\xi,\gamma)$ if and only if $\Im\hbar'(\Im) \in s_s^*(\alpha,\beta,\xi,\gamma)$, therefore, replacing it is sufficient a_τ in the Theorem 2.1 with τa_2. □

Corollary 2.7. *If $\hbar(\Im) \in k_s^*(\alpha,\beta,\xi,\gamma)$, that is, replacing $\gamma = 1$, we get*

$$|a_\tau| \leq \frac{4\beta\xi(1-\alpha)}{\{(\tau-2)-\beta(\tau-2+4\xi\alpha-2\tau\xi)\}}\Im^\tau \quad \text{for } \tau = 2,3,\ldots. \tag{2.6}$$

Equality holds for

$$\hbar(\Im) = \Im - \frac{4\beta\xi(1-\alpha)}{\{(\tau-2)-\beta(\sqrt{\Im}-2\gamma+4\xi\alpha-2\tau\xi)\}}\Im^\tau.$$

Corollary 2.8. *If $\hbar(\Im) \in k_s^*(\alpha,\beta,1,1)$, we get*

$$\hbar(\Im) = \Im - \frac{4\beta(1-\alpha)}{\{(\tau-2)-\beta(4\alpha-\tau-2)\}}\Im^\tau. \tag{2.7}$$

Equality holds for

$$\hbar(\Im) = \Im - \frac{4\beta\xi(1-\alpha)}{\{(\tau-2)-\beta(4\alpha-\tau-2)\}}\Im^\tau.$$

Corollary 2.9. *If $\hbar(\Im) \in k_s^*(\alpha)$ that is starlike with respect to symmetric point of order α, if and only if*

$$\sum_{\tau=2}^{\infty}\tau(\tau-2\alpha)\,|a_z| \leq (1-\alpha). \tag{2.8}$$

3. Distortion theorem

Next, we consider the distortion theorem.

Theorem 3.1. *If* $\hbar(\Im) \in s_s^*(\alpha, \beta, \xi, \gamma)$ *then*

$$r - r^2 \leq |\hbar(\Im)| \leq r + r^2. \tag{3.1}$$

Equality holds for

$$\hbar(\Im) = \Im - \Im^2 \quad at \quad \Im = \mp r.$$

Proof. By Theorem 2.1, we have

$$\hbar(\Im) \in s_s^*(\alpha, \beta, \xi, \gamma)$$

if and only if

$$\sum_{\tau=2}^{\infty} [(\tau - 2) - \beta(\gamma\tau - 2\gamma + 4\xi\alpha - 2\tau\xi)] |a_\tau| \leq 2\beta\xi(1 - \alpha),$$

or equivalently,

$$\sum_{\tau=2}^{\infty} |a_\tau| \left\{ \tau - \left(2 - \frac{4\beta\xi(1-\alpha)}{1 + 2\beta\xi - \gamma\beta} \right) \right\} \leq \frac{4\beta\xi(1-\alpha)}{1 + 2\beta\xi - \gamma\beta}, \tag{3.2}$$

so $\hbar(\mathcal{N}) \in s_s^*(\alpha, \beta, \xi, \gamma)$ if and only by (3.2), we get

$$\sum_{\tau=2}^{\infty} |a_\tau| (\tau - t) \leq (2 - t) \quad \text{where} \quad t = \frac{4\beta\xi(1-\alpha)}{1 + 2\beta\xi - 7\beta}. \tag{3.3}$$

But

$$(2 - t) \sum_{\tau=2}^{\infty} |a_\tau| \leq \sum_{\tau=2}^{\infty} |a_\tau| (\tau - t) \leq (2 - t).$$

The final disparity originates from (3.2); we retain

$$|\hbar(\Im)| \leq r + \sum_{\tau=2}^{\infty} |a_r| r^\tau \leq r + r^\tau \sum_{\tau=2}^{\infty} |a_\tau| r^\tau \leq r + r^2.$$

In like manner

$$|\hbar(\Im)| \geq r + \sum_{\tau=2}^{\infty} |a_\tau| r^\tau \geq r + r^\tau \sum_{\tau=2}^{\infty} |a_\tau| r^\tau \geq r + r^2.$$

So,

$$r - r^2 \leq |\hbar(\Im)| \leq r + r^2,$$

this completes the proof. □

Corollary 3.2. *If* $\hbar(\Im) \in s_s^*(\alpha, \beta, \xi, 1)$*, that is, replacing* $\gamma = 1$*, then*

$$r - r^2 \leq |\hbar(\Im)| \leq r + r^2.$$

Equality hold for,

$$\hbar(\Im) = \Im - \Im^\tau$$

at $\Im = \mp r$.

Corollary 3.3. *If* $\hbar(\Im) \in s_s^*(\alpha, \beta, 1, 1)$*, that is, replacing* $\gamma = 1$ *and* $\xi = 1$*, then*

$$r - r^2 \leq |\hbar(\Im)| \leq r + r^2.$$

Equality holds for,

$$\hbar(\Im) = \Im - \Im^\tau \quad \text{at } \Im = \mp r.$$

Theorem 3.4. *If* $\hbar(\Im) \in k_s^*(\alpha, \beta, \xi, \gamma)$*, then*

$$r - 2r^2 \leq |\hbar(\Im)| \leq r + 2r^2. \tag{3.4}$$

Proof. This theorem's proof is comparable to Theorem 3.1, since a function $\hbar(\Im) \in k_s^*(\alpha, \beta, \xi, \gamma)$ if and only if $\Im\hbar'(\Im) \in s_s^*(\alpha, \beta, \xi, \gamma)$, it will suffice to swap it out a_τ in Theorem 3.1 with τa_τ. □

Corollary 3.5. *If* $\hbar(\Im) \in k_s^*(\alpha, \beta, \xi, 1)$*, that is, replacing* $\gamma = 1$*, then*

$$r - 2r^2 \leq |\hbar(\Im)| \leq r + 2r^2.$$

Equality holds for

$$\hbar(\Im) = \Im - 2\Im^\tau \quad \text{at } \Im = \mp r.$$

Corollary 3.6. *If* $\hbar(\Im) \in k_s^*(\alpha, \beta, 1, 1)$*, that is, replacing* $\gamma = 1, \xi = 1$*, then*

$$r - 2r^2 \leq |\hbar(\Im)| \leq r + 2r^2.$$

Equality holds for,

$$\hbar(\Im) = \Im - 2\Im^\tau \quad \text{at } \Im = \mp r.$$

Theorem 3.7. *If* $\hbar(\Im) \in s_s^*(\alpha, \beta, \xi, \gamma)$*, then*

$$1 - r \leq \left|\hbar'(\Im)\right| \leq 1 + r. \tag{3.5}$$

Proof. Since $\hbar(\Im) \in s_s^*(\alpha, \beta, \xi, \gamma)$, we have

$$\sum_{\tau=2}^{\infty} |a_\tau| \, (\tau - t) \leq (2 - t), \quad \text{where} \quad t = \frac{4\beta\xi(1-\alpha)}{1 + 2\beta\xi - \gamma\beta}. \tag{3.6}$$

Now, considering Theorem 3.1, we have

$$\sum_{\tau=2}^{\infty} \tau \, |a_\tau| = \sum_{\tau=2}^{\infty} (\tau - t) \, |a_\tau| + t \sum_{\tau=2}^{\infty} |a_\tau| \leq (2 - t) + t = 2.$$

Consequently, we have

$$\left|\hbar'(\Im)\right| \leq \sum_{\tau=2}^{\infty} \Im \left|a_\tau\right| \left|\Im\right|^{\tau-1} \leq 1 + r\sum_{\tau=2}^{\infty} \tau \left|a_\tau\right| \leq 1 + 2r.$$

Similarly, we have

$$\left|\hbar'(\Im)\right| \geq \sum_{\tau=2}^{\infty} \tau \left|a_\tau\right| \left|\Im\right|^{\tau-1} \geq 1 + r\sum_{\tau=2}^{\infty} \tau \left|a_\tau\right| \geq 1 + 2r. \tag{3.7}$$

Therefore, we have

$$r - 2r^2 \leq |\hbar(\Im)| \leq r + 2r^2.$$

This completes the proof. □

Corollary 3.8. *If $\hbar(\Im) \in s_s^*(\alpha, \beta, \xi, 1)$, that is, replacing $\gamma = 1$, then*

$$r - 2r^2 \leq |\hbar(\Im)| \leq r + 2r^2.$$

Corollary 3.9. *If $\hbar(\Im) \in s_s^*(\alpha, \beta, 1, 1)$, that is, replacing $\xi = 1, \gamma = 1$, then*

$$r - 2r^2 \leq |\hbar(\Im)| \leq r + 2r^2.$$

Theorem 3.10. *If $\hbar(\Im) \in k_s^*(\alpha, \beta, \xi, \gamma)$, then*

$$r - 4r^2 \leq |\hbar(\Im)| \leq r + 4r^2, \quad |\Im| = r.$$

Proof. This theorem's proof is comparable to Theorem 3.1, since a function $\hbar \in k_s^*(\alpha, \beta, \xi, \gamma)$ if and only if $\hbar \in s_s^*(\alpha, \beta, \xi, \gamma)$. □

Corollary 3.11. *If $\hbar(\Im) \in k_s^*(\alpha, \beta, \xi, 1)$, then*

$$r - 4r^2 \leq |\hbar(\Im)| \leq r + 4r^2.$$

Corollary 3.12. *If $\hbar(\Im) \in k_s^*(\alpha, \beta, 1, 1)$, then*

$$r - 4r^2 \leq |\hbar'(\Im)| \leq r + 4r^2, \quad |\Im| = \pm r.$$

4. Radius of convexity

Theorem 4.1. *If $\hbar(\Im) \in s_s^*(\alpha, \beta, \xi, \gamma)$, then $\hbar$ is convex in the unit disc with $0 < |\Im| < r$, and*

$$r_1 = \inf_{\tau \geq 2} \left[\frac{(\tau - 2) - \beta(\gamma\tau - 2\gamma + 4\xi\alpha - 2\tau\xi)}{\tau^2(1 + 4\beta\xi - \gamma\beta) + \tau(\gamma\beta - 1)(1 - \alpha)}\right]^{1/\tau - 1}. \tag{4.1}$$

This outcome is acute, exhibiting the extremal function

$$\hbar(\Im) = \Im - \frac{4\beta\xi(1 - \alpha)}{\{(\tau - 2) - \beta(\gamma\tau - 2\gamma + 4\xi\alpha - 2\tau\xi)\}}\Im^\tau \tag{4.2}$$

for some τ.

Proof. We know that $\hbar(\Im) \in k_s^*(0,\beta,\xi,\gamma)$, if $\Im\hbar'(\Im) \in s_s^*(0,\beta,\xi,\gamma)$. Therefore, it is sufficient to show that

$$\frac{\Im\frac{(\Im\hbar'(\Im))'}{\Im(\hbar(\Im)-\hbar(-\Im))'}-1}{2\xi\left(\Im\frac{(\Im\hbar'(\Im))'}{\Im(\hbar(\Im)-\hbar(-\Im))'}-\alpha\right)-\gamma\left(\Im\frac{(\Im\hbar'(\Im))'}{\Im(\hbar(\Im)-\hbar(-\Im))'}-1\right)}<\beta \quad \text{for} \quad |\Im|\leq r_1, \tag{4.3}$$

we retain,

$$\begin{aligned}
&\frac{\Im\frac{(\Im\hbar'(\Im))'}{\Im(\hbar(\Im)-\hbar(-\Im))'}-1}{\beta\left[2\xi\left(\Im\frac{(\Im\hbar'(\Im))'}{\Im(\hbar(\Im)-\hbar(-\Im))'}-\alpha\right)-\gamma\left(\Im\frac{(\Im\hbar'(\Im))'}{\Im(\hbar(\Im)-\hbar(-\Im))'}-1\right)\right]}\\
&\quad=\left|\frac{\Im}{\beta\left[(\Im\hbar''(\Im))(2\xi-\gamma)+\xi 4\Im(\hbar(\Im)-\hbar(-\Im))'\right]}\right|\\
&\quad=\left|\frac{-\sum_{\tau=2}^{\infty}\tau(\tau-1)\,|a_\tau|\,\Im^{\tau-1}}{\beta\left[-\sum_{\tau=2}^{\infty}\tau(\tau-1)(2\xi-\gamma)\,|a_\tau|\,\Im^{\tau-1}+4\xi-\sum_{\tau=2}^{\infty}4\tau\xi\,|a_\tau|\,\Im^{\tau-1}\right]}\right|\\
&\quad=\left|\frac{-\sum_{\tau=2}^{\infty}\tau(\tau-1)\,|a_z|\,\Im^{\tau-1}}{4\beta\xi-\sum_{\tau=2}^{\infty}\tau^2[(4\xi-\gamma)\beta+\gamma\tau\beta]\,|a_\tau|\,\Im^{\tau-1}}\right|\\
&\quad\leq\frac{-\sum_{\tau=2}^{\infty}\tau(\tau-1)\,|a_\tau|\,|\Im|^{\tau-1}}{4\beta\xi-\sum_{\tau=2}^{\infty}\tau^2[(4\xi-\gamma)\beta+\gamma\tau\beta]\,|a_\tau|\,|\Im|^{\tau-1}}.
\end{aligned}$$

Thus, (4.1) is hold, if

$$\sum_{\tau=2}^{\infty}\tau(\tau-1)\,|a_\tau|\,|\Im|^{\tau-1}\leq 4\beta\xi-\sum_{\tau-2}^{\infty}\tau^2[(4\xi-\gamma)\beta+\gamma\tau\beta]\,|a_\tau|\,|\Im|^{\tau-1}.$$

That is

$$\sum_{\tau=2}^{\infty}\left[\tau^2(1+4\beta\xi-\gamma\beta)+\tau(\gamma\beta-1)\right]|a_\tau||\Im|^{\tau-1}\leq 4\beta\xi. \tag{4.4}$$

But

$$\sum_{\tau=2}^{\infty}[(\tau-2)+\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)]\,|a_\tau|\leq 4\beta\xi(1-\alpha).$$

Hence, (4.2) will be true, if

$$\sum_{\tau-2}^{\infty}\left[\tau^2(1+4\beta\xi-\gamma\beta)+\tau(\gamma\beta-1)\right]|\Im|^{\tau-1}\leq\frac{[(\tau-2)+\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)]}{(1-\alpha)}.$$

Solving for $|\Im|$, we obtain

$$|\Im|\leq\left[\frac{(\tau-2)+\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)}{[(\tau^2(1+4\beta\xi-\gamma\beta)+\tau(\gamma\beta-1))\,(1-\alpha)]}\right]^{1/\tau-1}.$$

The desired result is followed by more substitution of $|\Im| = r_1$ in the above expression. $\square$

Corollary 4.2. *If $\hbar(\mathfrak{S}) \in s_s^*(\alpha, \beta, \xi, 1)$, then f is convex in the disc $0 < |\Im| < r$, and*

$$r_2 = \inf_{\tau \geq 2} \left[\frac{(\tau-2) - \beta(\tau - 2 + 4\xi\alpha - 2\tau\xi)}{(\tau^2(1 + 4\beta\xi - \gamma\beta) + \tau(\gamma\beta - 1))(1-\alpha)} \right]^{1/\tau - 1}. \tag{4.5}$$

The result is sharp with the extremal function,

$$\hbar(\Im) = \Im - \frac{4\beta\xi(1-\alpha)}{[(\tau-2) - \beta(\tau - 2 + 4\xi\alpha - 2\tau\xi)]} \Im^2$$

for some τ.

Corollary 4.3. *If $\hbar(\Im) \in s_s^*(\alpha, \beta, 1, 1)$, then $\hbar$ is convex in the disc $0 < |\Im| < r$, and*

$$r_2 = \inf_{\tau \geq 2} \left[\frac{(\tau-2) + \beta(\tau + 2 - 4\alpha)}{(\tau^2(1 + 4\beta - \gamma\beta) + \tau(\gamma\beta - 1))(1-\alpha)} \right]^{1/\tau - 1}. \tag{4.6}$$

This outcome is acute, exhibiting the extremal function

$$\hbar(\Im) = \Im - \frac{4\beta(1-\alpha)}{[(\tau-2) - \beta(\tau - 2 + 4\alpha - 2\tau)]} \Im^\tau$$

for some τ.

Corollary 4.4. *If $\hbar(\Im) \in s_s^*(0, 1, 1, 1)$ then, f is convex in the disk $0 < |\Im| < r$.*

5. Closure theorem

Theorem 5.1. *Let $\hbar_1(\Im) = \Im$ and*

$$\hbar_\tau(\Im) = \Im - \frac{4\beta\xi(1-\alpha)}{[(\tau-2) - \beta(\tau - 2\gamma + 4\xi\alpha - 2\tau\xi)]} \Im^\tau, \quad \text{for } \tau = 2, 3, 4, \ldots. \tag{5.1}$$

Then $\hbar(\Im) \in k_s^*(\alpha, \beta, \xi, \gamma)$, if and only if $\hbar(\Im)$ can be expressed in the forms,

$$\hbar(\Im) = \hbar_1(\Im) - \sum_{\tau - 2}^{\infty} \lambda_\tau \hbar_\tau(\Im) \quad \text{where } \lambda_\tau \geq 0 \quad \text{and} \sum \lambda_\tau = 1.$$

Proof. Suppose,

$$\hbar(\Im) = \Im - \sum_{\tau - 2}^{\infty} \lambda_\tau \hbar_\tau(\Im) = \Im - \sum_{\tau=2}^{\infty} \frac{4\beta\xi(1-\alpha)}{[(\tau-2) - \beta(\tau - 2 + 4\xi\alpha - 2\tau\xi)]} \Im^\tau.$$

Then

$$\sum_{\tau=2}^{\infty}\frac{4\beta\xi(1-\alpha)}{[(\tau-2)-\beta(\tau-2+4\xi\alpha-2\tau\xi)]}\times\frac{(\tau-2)-\beta(\tau-2+4\xi\alpha-2\tau\xi)}{4\beta\xi(1-\alpha)}$$

$$=\sum_{\tau=2}^{\infty}\lambda_\tau=1-\lambda_1\leq 1.$$

Therefore, $\hbar(\Im)\in k_s^*(\alpha,\beta,\xi,\gamma)$.

On the other hand, let us assume $\hbar(\Im)\in k_s^*(\alpha,\beta,\xi,\gamma)$ that theorem's remark provides us with

$$|a_\tau|\leq\frac{4\beta\xi(1-\alpha)}{\{(\tau-2)-\beta(\gamma\tau-2\gamma+4\xi\alpha-2\tau\xi)\}}\quad\text{for}\quad\tau=2,3,\ldots,$$

$$\lambda_\tau=\left[\frac{(\tau-2)-\beta(\tau\gamma-2\gamma+4\xi\alpha-2\tau\xi)}{4\beta\xi(1-\alpha)}\right]|a_\tau|$$

and

$$\lambda_\tau=1-\sum_{\tau-2}^{\infty}\lambda.$$

Then

$$\hbar(\Im)=1-\sum_{\tau-2}^{\infty}\lambda_\tau\hbar_\tau(\Im).$$

□

Corollary 5.2. *If* $\hbar_\tau(\Im)=\Im$ *and*

$$\hbar_\tau(\Im)=\Im-\frac{4\beta\xi(1-\alpha)}{[(\tau-2)-\beta(\tau-2\gamma+4\xi\alpha-2\tau\xi)]}\Im^\tau\quad for\ \tau=2,3,4,\ldots.$$

Then $\hbar(\Im)\in k_s^*(\alpha,\beta,\xi,1)$*, if and only if* $\hbar(\Im)$ *can be articulated as*

$$\hbar(\Im)=\hbar_1(\Im)-\sum_{\tau-2}^{\infty}\lambda_\tau\hbar_\tau(\Im),\quad where\ \lambda_\tau\geq 0,\ \tau=1,2,\ldots,\ \sum\lambda_\tau=1.$$

Corollary 5.3. *If* $\hbar_1(\Im)=\Im$ *and*

$$\hbar_\tau(\Im)=\Im-\frac{4\beta(1-\alpha)}{[(\tau-2)-\beta(\tau-2\gamma+4\alpha-2\tau)]}\Im^\tau\quad for\ \tau=2,3,4,\ldots$$

Then $\hbar(\Im)\in k_s^*(\alpha,\beta)$*, if and only if* $\hbar(\Im)$ *can be articulated as*

$$\hbar(\Im)=\hbar_1(\Im)-\sum_{\tau-2}^{\infty}\lambda_\tau\hbar_\tau(\Im),\quad where\ \lambda_\tau\geq 0,\ \tau=1,2,\ldots,\ \sum\lambda_\tau=1.$$

Corollary 5.4. *If* $\hbar_1(\Im) = \Im$ *and*

$$\hbar(\Im) = \Im - \left|\frac{1}{\tau}\right| \Im^{\tau}.$$

Then $\hbar(\Im) \in k_s^*(0,1,1,1)$*, if and only if* $\hbar(\Im)$ *can be expressed in the form*

$$\hbar(\Im) = \hbar_1(\Im) - \sum_{\tau-2}^{\infty} \lambda_\tau \hbar_\tau(\Im), \quad \text{where } \lambda_\tau \geq 0,\ \tau = 1, 2, \ldots,\ \sum \lambda_\tau = 1.$$

6. Conclusion

In this study, we have thoroughly examined the properties of analytic functions within the unit disk, specifically focusing on the symmetric classes A and B with negative coefficients. Our investigation has led to several significant findings: coefficient inequalities, convexity radius, distortion theorems, and symmetric classes with negative coefficients. These findings contribute significantly to the field of geometric function theory, offering new insights into the behavior and properties of functions with negative coefficients within the unit disk. These results not only enhance the theoretical understanding of such functions but also provide a robust foundation for future research in various areas of geometric function theory and its applications.

Acknowledgments: The author thanks for the support of Rambhai Barni Rajabhat University.

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