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HALF-DISCRETE HILBERT'S INEQUALITY FOR THREE VARIABLES WITH HYPERBOLIC SINE FUNCTION

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Abstract. For three variables of the half-discrete Hilbert's inequality and using the hyperbolic sine function, we present two new forms of the half-discrete Hilbert's inequality in this study. We also demonstrate that the constants on the right-hand side of the inequality are the best. The equivalence forms of the two inequalities are also introduced.

1. Introduction

Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $f \in L_p(0, \infty)$ and $g \in L_q(0, \infty)$, the following integral inequality is hold

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}. \tag{1.1}$$

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For $\{a_m\}_{m=1}^{\infty} \in l_p$ and $\{b_n\}_{n=1}^{\infty} \in l_q$, the discrete form of (1.1) also holds

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.2}$$

where the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ in (1.1) and (1.2) is the best possible.

The following inequality was presented by Yang in a half-discrete form [8]:

$$\int_0^{\infty} f(x)\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^{\lambda}}dx < B(\lambda_1,\lambda_2)\left(\int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q\right)^{\frac{1}{q}}, \tag{1.3}$$

where $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 > 0$, $0 < \lambda_1 < 1$, also, the constant $B(\lambda_1, \lambda_2)$ is the best possible.

Through the last two decades, inequalities (1.1), (1.2) and (1.3) formed the basis of the process of extensions of integral, discrete, and half-discrete forms of Hilbert inequality ([1, 3, 5, 6, 7]). For three variables in the preceding five years, Batbold and Azar established the following two half-discrete inequalities [4]:

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty} f(x,y)\sum_{n=1}^{\infty}\frac{a_n}{(x+y+n)^{\lambda}}dxdy \\
&= \sum_{n=1}^{\infty} a_n \int_0^{\infty}\int_0^{\infty}\frac{f(x,y)}{(x+y+n)^{\lambda}}dxdy \qquad (1.4)\\
&< C\left(\int_0^{\infty}\int_0^{\infty}(x+y)^{2p-\lambda-p\gamma-2} f^p(x,y)dxdy\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\gamma q+q-\lambda-1} a_n^q\right)^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}\int_0^{\infty}\frac{f(x)}{(x+m+n)^{\lambda}}dx \\
&= \int_0^{\infty} f(x)\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{m,n}}{(x+m+n)^{\lambda}}\right)dx \qquad (1.5)\\
&< C\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+n)^{2p-\lambda-p\gamma-2} a_{m,n}^p\right)^{\frac{1}{p}}\left(\int_0^{\infty} x^{\gamma q+q-\lambda-1} f^q(x)\right)^{\frac{1}{q}},
\end{aligned}$$

where the constant $C = B(\frac{\lambda}{q} - \gamma, \frac{\lambda}{p} + \gamma)$ in (1.4) and (1.5) is the best possible.

In 2022, Al-Oushoush gave the integral form of the main result in this manuscript [2].

2. Preliminaries and lemmas

The following special functions are important and needed to prove the main result in this paper:

$$\Gamma(\varpi) = \int_0^\infty t^{\varpi - 1} e^{-t} dt, \quad \varpi > 0, \tag{2.1}$$

$$B(\alpha, \beta) = \int_0^\infty \frac{t^{\alpha - 1}}{(t+1)^{\alpha + \beta}} dt, \quad \alpha, \beta > 0. \tag{2.2}$$

Also, for $\Gamma(\varpi)$ and $B(\alpha, \beta)$, we will need another useful representation for them as follows:

$$\frac{1}{z^\varpi} = \frac{1}{\Gamma(\varpi)} \int_0^\infty t^{\varpi - 1} e^{-zt} dt, \tag{2.3}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \tag{2.4}$$

Additionally, in order to completely prove our main result, we will need the following inequality

$$\cosh \varpi \geq 1, \qquad |\varpi| < \infty. \tag{2.5}$$

Next, we state the following lemmas which are the main tools for proving our results.

Lemma 2.1. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t > 0$, $\eta < \frac{2}{q}$ *and* $f(x, y)$ *is a non-negative function defined and integrable on* $(0, \infty) \times (0, \infty)$. *Then, we get*

$$\left(\int_0^\infty \int_0^\infty f(x, y) e^{-(\sinh x + \sinh y)t} dx dy \right)^p \leq t^{\eta q - 2} \Gamma(2 - \eta q) \int_0^\infty \int_0^\infty \frac{e^{-(\sinh x + \sinh y)t}}{(\sinh x + \sinh y)^{-\eta p}} f^p(x, y) dx dy. \tag{2.6}$$

Proof. Using Hölder's inequality, and making the substitutions $\sinh y = u \sinh x$, and $\sinh x = \frac{v}{1+u}$, $v \geq 0$, (to evaluate the first double integral on the right-hand-side of the inequality below), this yields

$$
\begin{aligned}
&\left(\int_0^\infty \int_0^\infty f(x,y) e^{-(\sinh x+\sinh y)t} dxdy\right)^p \\
&= \left(\int_0^\infty \int_0^\infty \left\{\frac{e^{-\frac{(\sinh x+\sinh y)}{q}t}}{(\sinh x+\sinh y)^\eta} dxdy\right\}\left\{\frac{e^{-\frac{(\sinh x+\sinh y)}{p}t}}{(\sinh x+\sinh y)^{-\eta}} f(x,y)\right\} dxdy\right)^p \\
&\leq \left(\left\{\int_0^\infty \int_0^\infty \frac{e^{-\sinh x(1+u)t}}{(\sinh x)^{\eta q}(1+u)^{\eta q}}\left(\frac{\sinh x dvdu}{v \cosh x \cosh y}\right)\right\}^{\frac{1}{q}}\right. \\
&\quad \left. \times \left\{\int_0^\infty \int_0^\infty \frac{e^{-(\sinh x+\sinh y)t} f^p(x,y)}{(\sinh x+\sinh y)^{-\eta p}} dxdy\right\}^{\frac{1}{p}}\right)^p \\
&= \left(\left\{\int_0^\infty \int_0^\infty \frac{(\sinh x)^{1-\eta q} e^{-\sinh x(1+u)t}}{(1+u)^{\eta q}} \frac{dvdu}{\cosh x \cosh y}\right\}^{\frac{1}{q}}\right. \\
&\quad \left. \times \left\{\int_0^\infty \int_0^\infty \frac{e^{-(\sinh x+\sinh y)t} f^p(x,y)}{(\sinh x+\sinh y)^{-\eta p}} dxdy\right\}^{\frac{1}{p}}\right)^p \\
&\leq \left(\int_0^\infty \int_0^\infty \frac{v^{1-\tau q} e^{-vt}}{(1+u)^2} dudv\right)^{\frac{p}{q}} \int_0^\infty \int_0^\infty \frac{e^{-(\sinh x+\sinh y)t}}{(\sinh x+\sinh y)^{-\eta p}} f^p(x,y) dxdy \\
&= \left(\int_0^\infty \frac{1}{(1+u)^2} du \int_0^\infty v^{1-\eta q} e^{-vt} dv\right)^{\frac{p}{q}} \\
&\quad \times \int_0^\infty \int_0^\infty \frac{e^{-(\sinh x+\sinh y)t}}{(\sinh x+\sinh y)^{-\eta p}} f^p(x,y) dxdy \\
&= t^{\eta p - 2\frac{p}{q}} \Gamma^{\frac{p}{q}}(2-\eta q) \int_0^\infty \int_0^\infty \frac{e^{-(\sinh x+\sinh y)t}}{(\sinh x+\sinh y)^{-\mu p}} f^p(x,y) dxdy.
\end{aligned}
$$

□

Lemma 2.2. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$ and $a_n > 0$. Then for $t > 0$ and $\mu < \frac{1}{p}$, we have:*

$$
\sum_{n=1}^\infty e^{-(\sinh n)t} a_n \leq t^{\mu - \frac{1}{p}} \Gamma^{\frac{1}{p}}(1-\mu p) \left(\sum_{n=1}^\infty (\sinh n)^{q\mu} e^{-(\sinh n)t} a_n^q\right)^{\frac{1}{q}}. \tag{2.7}
$$

Proof. Using Hölder's inequality, and using the substitution $\omega = \sinh z$, then

$$
\begin{aligned}
&\sum_{n=1}^{\infty} e^{-(\sinh n)t} a_n \\
&= \sum_{n=1}^{\infty} \left((\sinh n)^{-\mu} \, e^{-\frac{(\sinh n)}{p} t} \right) \left((\sinh n)^{\mu} \, e^{-\frac{(\sinh n)}{q} t} a_n \right) \\
&\leq \left(\sum_{n=1}^{\infty} (\sinh n)^{-p\mu} \, e^{-(\sinh n)t} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} (\sinh n)^{q\mu} \, e^{-(\sinh n)t} a_n^q) \right)^{\frac{1}{q}} \\
&\leq \left(\int_0^{\infty} (\sinh z)^{-p\mu} \, e^{-(\sinh n)t} dz \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} (\sinh n)^{q\mu} \, e^{-(\sinh n)t} a_n^q) \right)^{\frac{1}{q}} \\
&= \left(\int_0^{\infty} \omega^{-p\mu} e^{-\omega t} \frac{d\omega}{\cosh \omega} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} (\sinh n)^{q\mu} \, e^{-(\sinh n)t} a_n^q) \right)^{\frac{1}{q}} \\
&\leq \left(\int_0^{\infty} \omega^{-p\mu} e^{-\omega t} d\omega \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} (\sinh n)^{q\mu} \, e^{-(\sinh n)t} a_n^q) \right)^{\frac{1}{q}} \\
&= t^{\mu - \frac{1}{p}} \Gamma^{\frac{1}{p}} (1 - p\mu) \left(\sum_{n=1}^{\infty} (\sinh n)^{q\mu} \, e^{-(\sinh n)t} a_n^q \right)^{\frac{1}{q}}.
\end{aligned}
$$

□

Lemma 2.3. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$ *and* $a_{m,n} > 0$. *Then for* $t > 0$ *and* $\mu < \frac{2}{q}$, *we have*

$$
\begin{aligned}
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} e^{-(\sinh m + \sinh n)t} \right)^p \\
&\leq t^{\mu p - 2\frac{p}{q}} \Gamma^{\frac{p}{q}} (2 - \mu q) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\sinh m + \sinh n)^{\mu p} \, e^{-(\sinh m + \sinh n)t} a_{m,n}^p. \quad (2.8)
\end{aligned}
$$

Proof. Using Hölder's inequality, and the substitutions $\sinh y = u \sinh x$ and $\sinh x = \frac{v}{1+u}$, we get

$$
\begin{aligned}
&\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} e^{-(\sinh m + \sinh n)t} \right)^p \\
&= \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{e^{-\frac{(\sinh m + \sinh n)t}{q} t}}{(\sinh m + \sinh n)^{\mu}} \right\} \left\{ (\sinh m + \sinh n)^{\mu} \, e^{-\frac{(\sinh m + \sinh n)}{p} t} a_{m,n} \right\} \right)^p
\end{aligned}
$$

$$\leq \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(\sinh m+\sinh n)t}}{(\sinh m+\sinh n)^{\mu q}} \right)^{\frac{p}{q}}$$

$$\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\sinh m+\sinh n)^{\mu p} \, e^{-(\sinh m+\sinh n)t} a_{m,n}^{p}$$

$$\leq \left(\int_0^{\infty} \int_0^{\infty} \frac{e^{-(\sinh x+\sinh y)t}}{(\sinh x+\sinh y)^{\mu q}} dx dy \right)^{\frac{p}{q}}$$

$$\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\sinh m+\sinh n)^{\mu p} \, e^{-(\sinh m+\sinh n)t} a_{m,n}^{p}$$

$$= \left(\int_0^{\infty} \int_0^{\infty} \frac{e^{-\sinh x (u+1)t}}{((\sinh x)\,(1+u))^{\mu q}} \left(\frac{\sinh x dv du}{(1+u)\cosh x \cosh y} \right) \right)^{\frac{p}{q}}$$

$$\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(\sinh m+\sinh n)t}}{(\sinh m+\sinh n)^{-\mu p}} a_{m,n}^{p}$$

$$= \left(\int_0^{\infty} \int_0^{\infty} \frac{(\sinh x)^{1-\mu q} e^{-\sinh x(u+1)t}}{(1+u)^{1+\mu q}} \frac{dv du}{\cosh x \cosh y} \right)^{\frac{p}{q}}$$

$$\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(\sinh m+\sinh n)t}}{(\sinh m+\sinh n)^{-\mu p}} a_{m,n}^{p}$$

$$= \left(\int_0^{\infty} \int_0^{\infty} \frac{v^{1-\mu q} e^{-vt}}{(1+u)^2 \cosh x \cosh y} du dv \right)^{\frac{p}{q}}$$

$$\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(\sinh m+\sinh n)t}}{(\sinh m+\sinh n)^{-\mu p}} a_{m,n}^{p}$$

$$\leq \left(\int_0^{\infty} \frac{1}{(1+u)^2} du \int_0^{\infty} v^{1-\mu q} e^{-vt} dv \right)^{\frac{p}{q}}$$

$$\times \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(\sinh m+\sinh n)t}}{(\sinh m+\sinh n)^{-\mu p}} a_{m,n}^{p}$$

$$= t^{\mu p - 2\frac{p}{q}} \Gamma^{\frac{p}{q}} (2-\mu q) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\sinh m+\sinh n)^{\mu p} \, e^{-(\sinh m+\sinh n)t} a_{m,n}^{p}.$$

$\square$

Lemma 2.4. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f(x) > 0$ *on* $(0, \infty)$. *Then for* $t > 0$ *and* $\eta < \frac{1}{p}$, *we have*

$$\int_0^\infty e^{-\sinh xt} f(x)dx \leq t^{\eta - \frac{1}{p}} \Gamma^{\frac{1}{p}}(1 - p\eta) \left(\int_0^\infty (\sinh x)^{q\eta} e^{-\sinh xt} f^q(x)dx \right)^{\frac{1}{q}}. \tag{2.9}$$

Proof. Using Hölder's inequality, and the substitution $\sinh x = \omega$, we get

$$\begin{aligned}
\int_0^\infty e^{-\sinh xt} f(x)dx &= \int_0^\infty \left\{ (\sinh x)^{-\eta} e^{-\frac{\sinh x}{p}t} \right\} \\
&\quad \times \left\{ (\sinh x)^{\eta} e^{-\frac{\sinh x}{q}t} f(x) \right\} dx \\
&\leq \left(\int_0^\infty (\sinh x)^{-p\eta} e^{-\sinh zt} dx \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty (\sinh x)^{q\eta} e^{-\sinh xt} f^q(x)dx \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \omega^{-p\eta} e^{-\omega t} \frac{d\omega}{\cosh x} \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty (\sinh x)^{q\eta} e^{-\sinh xt} f^q(x)dx \right)^{\frac{1}{q}} \\
&\leq t^{\eta - \frac{1}{p}} \Gamma^{\frac{1}{p}}(1 - p\eta) \\
&\quad \times \left(\int_0^\infty (\sinh x)^{q\eta} e^{-\sinh xt} f^q(x)dx \right)^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 2.5. Note that if $0 < p < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then by using the reverse form of Hölder's inequality and use the same previous method, we can prove the reverse forms of (2.6), (2.7), (2.8) and (2.9) which are needed to prove Theorems 4.1 and 4.2.

3. Main results

Theorem 3.1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0, -\frac{\lambda}{p} < \xi < \frac{\lambda}{q}$. *Define* $f(x, y) > 0$ *on* $(0, \infty) \times (0, \infty)$ *and* $a_n > 0$. *If*

$$\int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{-\lambda - p\xi + 2\frac{p}{q}} f^p(x, y)dxdy < \infty$$

and

$$\sum_{n=1}^{\infty} (\sinh n)^{\xi q+q-\lambda-1}\, a_n^q < \infty,$$

then

$$\int_0^\infty \int_0^\infty f(x,y) \sum_{n=1}^{\infty} \frac{a_n}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy$$
$$\leq C \left(\int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{-\lambda - p\xi + 2\frac{p}{q}} f^p(x,y) dxdy \right)^{\frac{1}{p}}$$
$$\times \left(\sum_{n=1}^{\infty} (\sinh n)^{\xi q+q-\lambda-1}\, a_n^q \right)^{\frac{1}{q}}, \tag{3.1}$$

where the constant $C = B(\frac{\lambda}{p} + \xi, \frac{\lambda}{q} - \xi)$ *is the best possible.*

Proof. Using the representation of the gamma function given in (2.3), and using Hölder's inequality, finally by Lemmas 2.1 and 2.2 , we get

$$I = \int_0^\infty \int_0^\infty f(x,y) \sum_{n=1}^{\infty} \frac{a_n}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty \int_0^\infty f(x,y) \sum_{n=1}^{\infty} a_n \left(\int_0^\infty t^{\lambda-1} e^{-(\sinh x + \sinh y + \sinh n)t} dt \right) dxdy$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(t^{\frac{\lambda-1}{p}+\xi} \int_0^\infty \int_0^\infty e^{-(\sinh x + \sinh y)t} f(x,y) dxdy \right)$$
$$\times \left(t^{\frac{\lambda-1}{q}-\xi} \sum_{n=1}^{\infty} e^{-(\sinh n)t} a_n \right) dt$$
$$\leq \frac{1}{\Gamma(\lambda)} \left(\int_0^\infty t^{\lambda-1+p\xi} \left(\int_0^\infty \int_0^\infty e^{-(\sinh x + \sinh y)t} f(x,y) dxdy \right)^p dt \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^\infty t^{\lambda-1-q\xi} \left(\sum_{n=1}^{\infty} e^{-(\sinh n)t} a_n \right)^q dt \right)^{\frac{1}{q}}$$
$$\leq \frac{\Gamma^{\frac{1}{q}}(2-\eta q)\Gamma^{\frac{1}{p}}(1-\mu\nu)}{\Gamma(\lambda)}$$
$$\times \Big(\int_0^\infty \int_0^\infty f^p(x,y)$$
$$\times \int_0^\infty (\sinh x + \sinh y)^{\eta p} t^{\lambda + p\eta + p\xi - 2\frac{p}{q} - 1} e^{-(\sinh x + \sinh y)t} dt dxdy \Big)^{\frac{1}{p}}$$

$$\times\left(\sum_{n=1}^{\infty}(\sinh n)^{q\mu}\,a_n^q\int_0^{\infty}t^{\lambda+q\mu-1-\frac{q}{p}-q\xi}e^{-(\sinh n)t}dt\right)^{\frac{1}{q}}$$
$$=C\left(\int_0^{\infty}\int_0^{\infty}(\sinh x+\sinh y)^{-\lambda-p\xi+2\frac{p}{q}}f^p(x,y)dxdy\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^{\infty}(\sinh n)^{\xi q+\frac{q}{p}-\lambda}\,a_n^q\right)^{\frac{1}{q}}dt,$$

where

$$C=\frac{\Gamma^{\frac{1}{q}}(2-\eta q)\Gamma^{\frac{1}{p}}(1-\mu p)\Gamma^{\frac{1}{p}}(\lambda+p\xi+\eta p-2\frac{p}{q})\Gamma^{\frac{1}{q}}(\lambda+q\mu-\frac{q}{p}-q\xi)}{\Gamma(\lambda)}$$
$$=B(\frac{\lambda}{p}+\xi,\frac{\lambda}{q}-\xi).$$

Next, we want to show that the constant C is the best constant. Let ϵ be so small real number, define the function

$$f_\epsilon(x,y)=(\sinh x+\sinh y)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}(\cosh y)^{\frac{1}{p}}$$

on $(0,\infty)\times(0,\infty)$ and

$$\widetilde{a}_n=(\sinh n)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}(\cosh n)^{\frac{1}{q}},\quad n\geq 1.$$

Assume that our constant C is not the best possible, then there exists T, where $0<T<C$, and again as in Lemmas 2.1 and 2.2, use the substitutions $\sinh y=u\sinh x$, $\sinh x=\frac{v}{1+u}$, and $\sinh z=\omega$ respectively, such that:

$$I\leq T\left(\int_0^{\infty}\int_0^{\infty}(\sinh x+\sinh y)^{-\lambda-p\xi+\frac{2p}{q}}f_\epsilon^p(x,y)dxdy\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^{\infty}(\sinh n)^{\xi q+q-\lambda-1}\widetilde{a}_n^q\right)^{\frac{1}{q}}$$
$$=T\left(\int_0^{\infty}\int_0^{\infty}\frac{\cosh x\cosh y}{(\sinh x+\sinh y)^{\epsilon+2}}dxdy\right)^{\frac{1}{p}}$$
$$\times\left((\sinh 1)^{-\epsilon-1}\cosh z+\sum_{n=2}^{\infty}(\sinh n)^{-\epsilon-1}\cosh n\right)^{\frac{1}{q}}$$

$$= T\left(\int_0^\infty \int_0^\infty \left(\frac{(\sinh x)^{-\epsilon-2}\,dv}{\cosh x\,(1+u)}\right)\left(\frac{\sinh x}{(1+u)^{\epsilon+2}}du\right)\right)^{\frac{1}{p}}$$
$$\times\left(\frac{\cosh 1}{(\sinh 1)^{\epsilon+1}} + \int_{\omega=1}^\infty \frac{\omega^{-\epsilon-1}\cosh z d\omega}{\cosh z}\right)^{\frac{1}{q}}$$
$$< T\left(\int_0^\infty v^{-\epsilon-2}\left(\int_0^\infty (1+u)^{-1}du\right)dv\right)^{\frac{1}{p}}$$
$$\times\left(\cosh 1\,(\sinh 1)^{-\epsilon-1} + \int_1^\infty \omega^{-\epsilon-1}d\omega\right)^{\frac{1}{q}}$$
$$= \frac{T\left(\frac{\cosh 1}{(\sinh 1)^{\epsilon+1}} + \frac{1}{\epsilon}\right)^{\frac{1}{q}}}{(\epsilon)^{\frac{1}{p}}}$$
$$= \frac{T}{\epsilon}(1 + \frac{\epsilon\cosh 1}{(\sinh 1)^{\epsilon+1}})^{\frac{1}{q}}.$$

Moreover, to estimate the left-hand side of (3.1), use the substitution

$$\varpi = \frac{\sinh z}{(\sinh x + \sinh y)\,(2\cosh x\cosh y)^{\frac{\lambda-\epsilon}{q}-\xi-1}},$$

and let

$$k = \frac{\sinh 1}{(\sinh x + \sinh y)\,(2\cosh x\cosh y)^{\frac{\lambda-\epsilon}{q}-\xi-1}}$$

and suppose that

$$2^{\frac{\lambda-\epsilon}{q}-\xi-1}(\cosh x\cosh y)^{\frac{1}{p}+\frac{\lambda-\epsilon}{q}-\xi-1} > \cosh z,$$

we obtain

$$I = \int_0^\infty \int_0^\infty f_\epsilon(x,y)\sum_{n=1}^\infty \frac{\widetilde{a}_n}{(\sinh x + \sinh y + \sinh n)^\lambda}dxdy$$
$$= \int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}\,(\cosh x\cosh y)^{\frac{1}{p}}$$
$$\times\sum_{n=1}^\infty \frac{(\sinh n)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}\,(\cosh n)^{\frac{1}{q}}}{(\sinh x + \sinh y + \sinh n)^\lambda}dxdy$$

$$\geq \int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{\frac{\lambda-\epsilon}{p}+\xi-2} (\cosh x \cosh y)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty \frac{(\sinh z)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}} (\cosh z)^{\frac{1}{q}}}{(\sinh x+\sinh y)^\lambda (1+\frac{\sinh z}{\sinh x+\sinh y})^\lambda} dz \right) dxdy$$

$$= \int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{\frac{\lambda-\epsilon}{p}+\xi-2-\lambda+\frac{\lambda-\epsilon}{q}-\xi-1} (\cosh x \cosh y)^{\frac{1}{p}}$$

$$\times \left(\int_k^\infty \frac{\left[(\varpi (\sinh x+\sinh y) (2\cosh x \cosh y)^{\frac{\lambda-\epsilon}{q}-\xi-1} \right]^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+(2\cosh x \cosh y)^{\frac{\lambda-\epsilon}{q}-\xi-1}\varpi)^\lambda} \right.$$

$$\left. \times \left(\frac{(\sinh x+\sinh y)(2\cosh x \cosh y)^{\frac{\lambda-\epsilon}{q}-\xi-1}}{\cosh z} d\varpi \right) \right) dxdy$$

$$\geq \int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{-\epsilon-2} (2\cosh x \cosh y)$$

$$\times \left(\int_k^\infty \frac{\varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+\varpi)^\lambda} \left(\frac{2^{\frac{\lambda-\epsilon}{q}-\xi-1}(\cosh x \cosh y)^{\frac{1}{p}+\frac{\lambda-\epsilon}{q}-\xi-1}}{\cosh z} d\varpi \right) \right) dxdy$$

$$= 2\int_0^\infty \int_0^\infty v^{-\epsilon-2} (1+u)^{-1} \left(\frac{\cosh x dv}{\cosh x} \right) \left(\frac{\cosh y \sinh x}{(1+u)\cosh y} du \right)$$

$$\times \left(\int_0^\infty \frac{\varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+\varpi)^\lambda} d\varpi - \int_0^k \frac{\varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+\varpi)^\lambda} d\varpi \right)$$

$$> 2\left(\int_1^\infty v^{-\epsilon-1} dv \left(\int_1^\infty (1+u)^{-2} du \right) d\varkappa \right)$$

$$\times \left(\int_0^\infty \frac{\varpi^{\frac{\lambda-\epsilon}{q}-\xi-1}}{(1+\varpi)^\lambda} d\varpi - \int_0^k \varpi^{\frac{\lambda-\epsilon}{q}-\xi-1} d\varpi \right)$$

$$= 2\left(\int_1^\infty v^{-\epsilon-1} dv \left(\int_0^\infty (1+u)^{-2} du \right) d\varkappa \right)$$

$$\times \left(B(\frac{\lambda-\epsilon}{q}-\xi, \frac{\lambda}{p}+\frac{\epsilon}{q}+\xi) - \int_0^k \varpi^{\frac{\lambda-\epsilon}{q}-\xi-1} d\varpi \right)$$

$$= \frac{2}{2\epsilon} B(\frac{\lambda-\epsilon}{q}-\xi, \frac{\lambda}{p}+\frac{\epsilon}{q}+\xi) - O(1)$$

$$> \frac{B(\frac{\lambda-\epsilon}{q}-\xi, \frac{\lambda}{p}+\frac{\epsilon}{q}+\xi)}{\epsilon} - O(1).$$

Clearly, when $\epsilon \to 0^+$, from the last two inequalities, we will get a contradiction, therefore, this lead to the proof of the Theorem 3.1. □

Theorem 3.2. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $-\frac{\lambda}{p} < \xi < \frac{\lambda}{q}$. *Define* $f(x) > 0$ *on* $(0, \infty)$ *and* $a_{m,n} > 0$. *If* $\int_0^\infty (\sinh x)^{\xi q+q-\lambda-1} f^q(x)dx < \infty$ *and* $\sum_{m=1}^\infty \sum_{n=1}^\infty (\sinh m + \sinh n)^{-\lambda-p\xi+2\frac{p}{q}} a_{m,n}^p < \infty$, *then*

$$\begin{aligned}
&\sum_{m=1}^\infty \sum_{n=1}^\infty a_{m,n} \int_0^\infty \frac{f(x)}{(\sinh m + \sinh n + \sinh x)^\lambda} dx \\
&= \int_0^\infty f(x) \left(\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_{m,n}}{(\sinh m + \sinh n + \sinh x)^\lambda} \right) dx \\
&\le \widetilde{C} \left(\sum_{m=1}^\infty \sum_{n=1}^\infty (\sinh m + \sinh n)^{-\lambda-p\xi+2\frac{p}{q}} a_{m,n}^p \right)^{\frac{1}{p}} \\
&\times \left(\int_0^\infty (\sinh x)^{\xi q+q-\lambda-1} f^q(x)dx \right)^{\frac{1}{q}},
\end{aligned} \tag{3.2}$$

where the constant $\widetilde{C} = B(\frac{\lambda}{p} + \xi, \frac{\lambda}{q} - \xi)$ *is the best possible.*

Proof. Using formula (2.3), and applying the Hölder's inequality, we get

$$\begin{aligned}
J &= \int_0^\infty f(x) \left(\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_{m,n}}{(\sinh m + \sinh n + \sinh x)^\lambda} \right) dx \\
&= \int_0^\infty f(x) \left(\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_{m,n}}{(\sinh m + \sinh n + \sinh x)^\lambda} \right) dx \\
&= \frac{1}{\Gamma(\lambda)} \sum_{m=1}^\infty \sum_{n=1}^\infty a_{m,n} \int_0^\infty f(x) \left(\int_0^\infty t^{\lambda-1} e^{-(\sinh m+\sinh n+\sinh x)t} dt \right) dx \\
&= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(t^{\frac{\lambda-1}{p}+\xi} \sum_{m=1}^\infty \sum_{n=1}^\infty e^{-(\sinh m+\sinh n)t} a_{m,n} \right) \\
&\quad \times \left(t^{\frac{\lambda-1}{q}-\xi} \int_0^\infty e^{-(\sinh x)t} f(x)dx \right) dt \\
&\le \frac{1}{\Gamma(\lambda)} \left(\int_0^\infty t^{\lambda-1+p\xi} \left(\sum_{m=1}^\infty \sum_{n=1}^\infty e^{-(\sinh m+\sinh n)t} a_{m,n} \right)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty t^{\lambda-1-\xi q} \left(\int_0^\infty e^{-(\sinh x)t} f(x)dx \right)^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.3}$$

Substituting (2.8) and (2.9) in (3.3), we obtain

$$\begin{aligned}
J &\le \frac{\Gamma^{\frac{1}{q}}(2-q\mu)\Gamma^{\frac{1}{p}}(1-p\eta)}{\Gamma(\lambda)} \\
&\quad \times \Bigg(\int_0^\infty t^{\lambda+\xi p+p\mu-2p+1} \\
&\quad \times \left(\sum_{m=1}^\infty \sum_{n=1}^\infty (\sinh m + \sinh n)^{\mu p} \, e^{-(\sinh m+\sinh n)t} a_{m,n}^p \right) dt \Bigg)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty t^{\lambda-1-\xi q-\mu q-q} \left(\int_0^\infty (\sinh x)^{-q\mu} e^{-(\sinh x)t} f^q(x) dx \right) dt \right)^{\frac{1}{q}} \\
&= \widetilde{C} \left(\sum_{m=1}^\infty \sum_{n=1}^\infty (\sinh m + \sinh n)^{2p-\lambda-p\xi-2} \, a_{m,n}^p \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty (\sinh x)^{\xi q + \frac{q}{p} - \lambda} e^{-(\sinh x)t} f^q(x) dx \right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{C} &= \widetilde{C}_{\eta,\mu} \\
&= \frac{\Gamma^{\frac{1}{q}}(2-\mu q)\Gamma^{\frac{1}{p}}(1-\eta p)\Gamma^{\frac{1}{p}}(\lambda+\xi p+p\mu-2\frac{p}{q})\Gamma^{\frac{1}{q}}(\lambda-\xi q+\mu q-\eta q-\frac{q}{p})}{\Gamma(\lambda)} \\
&= B(\frac{\lambda}{p}+\xi, \frac{\lambda}{q}-\xi),
\end{aligned}$$

where $\eta = \frac{\lambda pq-\lambda-p\zeta}{pq}$ and $\mu = \frac{\lambda pq-\lambda+q\zeta+q}{pq}$.

Our next goal is to show that the constant $\widetilde{C}_{v,\mu}$ is the best possible. Let ϵ be very small real number, define the function

$$f_\epsilon(z) = (\sinh z)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}} (\cosh z)^{\frac{1}{q}}$$

on $(0,\infty)$, and the sequence

$$\widetilde{a}_{m,n} = (\sinh m + \sinh n)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}} (\cosh m \cosh n)^{\frac{1}{p}}$$

for $m, n \ge 1$. Assume that our constant $\widetilde{C}$ is not the best possible. Then there exist K, where $0 < K < \widetilde{C}$, and again as in Lemmas 2.3 and 2.4, use the substitutions $\sinh y = u \sinh x$, $\sinh x = \frac{v}{1+u}$ and $\sinh z = \omega$ such that

$$J \leq K \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\sinh m + \sinh n)^{\lambda - p\xi + \frac{2p}{q}} \widetilde{a}_{m,n}^{p} \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^{\infty} (\sinh z)^{\xi q + q - \lambda - 1} f_{\epsilon}^{q}(z) dz \right)^{\frac{1}{q}}$$
$$= K \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\sinh m + \sinh n)^{-\epsilon - 2} \cosh m \cosh n \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^{\infty} (\sinh z)^{-\epsilon - 1} \cosh z dz \right)^{\frac{1}{q}}$$
$$\leq K \left(\int_0^{\infty} \int_0^{\infty} (\sinh x + \sinh y)^{-\epsilon - 2} \cosh x \cosh y dx dy \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^{\infty} (\sinh z)^{-\epsilon - 1} \cosh z dz \right)^{\frac{1}{q}}$$
$$= K \left(\int_0^{\infty} \int_0^{\infty} (\sinh x)^{-\epsilon - 2} (1 + \frac{\sinh y}{\sinh x})^{-\epsilon - 2} \cosh x \cosh y dx dy \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^{\infty} (\sinh z)^{-\epsilon - 1} \cosh z dz \right)^{\frac{1}{q}}$$
$$= K \left(\int_1^{\infty} \int_1^{\infty} \left(\frac{v}{1+u} \right)^{-\epsilon - 1} (1+u)^{-\epsilon - 2} \frac{\cosh y du}{\cosh y} \frac{\cosh x dv}{(1+u) \cosh x} \right)^{\frac{1}{p}}$$
$$\times \left(\int_1^{\infty} \omega^{-\epsilon - 1} \frac{d\omega}{\cosh z} \right)^{\frac{1}{q}}$$
$$\leq K \left(\int_1^{\infty} v^{-\epsilon - 1} dv \int_1^{\infty} (1+u)^{-2} du \right)^{\frac{1}{p}} \left(\int_1^{\infty} \omega^{-\epsilon - 1} d\omega \right)^{\frac{1}{q}}$$
$$= K \left(\frac{1}{\epsilon} \right)^{\frac{1}{p}} \left(\frac{1}{\epsilon} \right)^{\frac{1}{q}} = \frac{K}{\epsilon}.$$

Now use the same method in Theorem 3.1 to estimate the left side of (3.2), and setting

$$\frac{\sinh z}{\sinh x + \sinh y} = (\cosh l)^{\frac{m}{\frac{\lambda - \epsilon}{q} - \xi - 1}} \varpi,$$

and $l > x$, $l > y$, $l > z$, choose $m > 5$ such that

$$\frac{(\cosh l)^{m-2} (\cosh l)^{\frac{m}{\frac{\lambda - \epsilon}{q} - \xi - 1}}}{\cosh z} > 1,$$

and let

$$k=\frac{(\cosh l)^{\frac{m}{\frac{\epsilon-\lambda}{q}+\xi+1}}}{\sinh x+\sinh y},$$

we obtain:

$$\begin{aligned}
J&=\int_0^\infty f_\epsilon(z)\left(\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{\widetilde{a}_{m,n}}{(\sinh m+\sinh n+\sinh z)^\lambda}\right)dz\\
&=\int_0^\infty (\sinh z)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}\left(\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{(\sinh m+\sinh n)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}}{(\sinh m+\sinh n+\sinh z)^\lambda}\right)dz\\
&\geq\int_0^\infty (\sinh z)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}\\
&\quad\times\left(\int_1^\infty\int_1^\infty\frac{(\sinh x+\sinh y)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}}{(\sinh x+\sinh y+\sinh z)^\lambda}dxdy\right)dz\\
&=\int_1^\infty\int_1^\infty (\sinh x+\sinh x)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}\\
&\quad\times\left(\int_a^\infty\frac{(\sinh z)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(\sinh x+\sinh y+\sinh z)^\lambda}dz\right)dxdy\\
&=\int_1^\infty\int_1^\infty (\sinh x+\sinh y)^{\frac{\lambda+p\xi-2\frac{p}{q}-\epsilon-2}{P}}\\
&\quad\times\left(\int_k^\infty\frac{\left((\sinh x+\sinh y)(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}\varpi\right)^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(\sinh x+\sinh x)^\lambda(1+(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}\varpi)^\lambda}\right.\\
&\quad\left.\times\frac{(\sinh x+\sinh y)(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}d\varpi}{\cosh z}\right)dxdy\\
&=\int_1^\infty\int_1^\infty(\sinh x+\sinh y)^{-\epsilon-2}\\
&\quad\times\left(\int_k^\infty\frac{(\cosh l)^m\varpi^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(1+(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}\varpi)^\lambda}\frac{(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}d\varpi}{\cosh z}\right)dxdy
\end{aligned}$$

$$= \int_1^\infty \int_1^\infty (\sinh x)^{-\epsilon-2}$$

$$\times \left(1+\frac{\sinh y}{\sinh x}\right)^{-\epsilon-2}\left(\int_k^\infty \frac{(\cosh l)^m \varpi^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(1+(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}\varpi)^\lambda} \frac{(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}} d\varpi}{\cosh z}\right) dxdy$$

$$= \int_{\frac{\sinh 1}{1+u}}^\infty \int_{\frac{\sinh 1}{\sinh x}}^\infty \left(\frac{v}{1+u}\right)^{-\epsilon-2} (1+u)^{-\epsilon-2} \frac{\cosh^3 l \sinh x du}{\cosh y} \frac{dv}{(1+u)\cosh x}$$

$$\times \left(\int_k^\infty \frac{\varpi^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(1+(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}}\varpi)^\lambda} \frac{(\cosh l)^{m-3}(\cosh l)^{\frac{m}{\frac{\lambda-\epsilon}{q}-\xi-1}} d\varpi}{\cosh z}\right)$$

$$\geq \int_1^\infty v^{-\epsilon-1} dv \left(\cosh l \int_1^\infty (1+u)^{-2} du\right)$$

$$\times \left(\int_0^\infty \frac{\varpi^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(1+\varpi)^\lambda} d\varpi - \int_0^k \frac{\varpi^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}}}{(1+\varpi)^\lambda} d\varpi\right)$$

$$\geq \frac{\cosh l}{\epsilon 2}\left(B(\frac{\lambda-\epsilon}{q}-\xi, \frac{\lambda}{p}+\frac{\epsilon}{q}+\xi) - \int_0^k \varpi^{\frac{\lambda-\xi q-\frac{q}{p}-\epsilon-1}{q}} d\varpi\right)$$

$$\geq \frac{B(\frac{\lambda-\epsilon}{q}-\xi, \frac{\lambda}{p}+\frac{\epsilon}{q}+\xi)}{\epsilon} - O(1).$$

From the above two inequalities, when $\epsilon \to 0^+$, we get a contradiction, by this, our second theorem is completely proved. □

4. Equivalent forms

In this part, we give the following equivalent forms of our main theorems (Theorems 3.1 and 3.2), and the constant in all inequalities is the same constant.

Theorem 4.1. *Under the same conditions of Theorem* 3.1, *we have the following two inequalities*

$$\sum_{n=1}^\infty (\sinh n)^{(p-1)\lambda-\xi p-1}\left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(\sinh x+\sinh y+\sinh n)^\lambda} dxdy\right)^p$$
$$\leq C^p \int_0^\infty \int_0^\infty (\sinh x+\sinh y)^{2p-\lambda-p\xi-2} f^p(x,y) dxdy \tag{4.1}$$

and

$$\int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{(q-1)\lambda+\xi q-2p} \sum_{n=1}^\infty \left(\frac{a_n}{(\sinh x + \sinh y + \sinh n)^\lambda} \right)^q dxdy$$
$$\le C^q \sum_{n=1}^\infty (\sinh n)^{\xi q-\lambda+q-1} a_n^q. \tag{4.2}$$

The inequalities (4.1) *and* (4.2) *are equivalent to* (3.1) *and the constants* C^p *and* C^q *are the best possible.*

Proof. In order to prove (4.1), let

$$a_n = (\sinh n)^{(p-1)\lambda-p\xi-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy \right)^{p-1}.$$

By using inequality (3.1), we find

$$\sum_{n=1}^\infty (\sinh n)^{(p-1)\lambda-p\xi-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy \right)^p$$
$$= \sum_{n=1}^\infty (\sinh n)^{(p-1)\lambda-p\xi-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy \right)^{p-1}$$
$$\times \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy \right)$$
$$= \int_0^\infty \int_0^\infty \sum_{n=1}^\infty \frac{f(x,y) a_n}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy$$
$$\le C \left(\int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{2p-\lambda-p\xi-2} f^p(x,y) dxdy \right)^{\frac{1}{p}}$$
$$\times \left(\sum_{n=1}^\infty (\sinh n)^{q-\lambda+\xi q-1} a_n^q \right)^{\frac{1}{q}}$$
$$= C \left(\int_0^\infty \int_0^\infty (\sinh x + \sinh y)^{2p-\lambda-p\xi-2} f^p(x,y) dxdy \right)^{\frac{1}{p}}$$
$$\times \left(\sum_{r=1}^\infty (\sinh n)^{(p-1)\lambda-p\xi-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy \right)^p \right)^{\frac{1}{q}}. \tag{4.3}$$

To get (4.1), just divide both sides of inequality (4.3) by

$$\left(\sum_{r=1}^\infty (\sinh n)^{(p-1)\lambda-p\xi-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)}{(\sinh x + \sinh y + \sinh n)^\lambda} dxdy \right)^p \right)^{\frac{1}{q}}.$$

Moreover, by Hölder inequality and (4.1), we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \sum_{n=1}^\infty \frac{f(x,y)a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} dxdy \\
&= \sum_{n=1}^\infty \left((\sinh n)^{\frac{\lambda}{q}-\xi-\frac{1}{p}} \int_0^\infty \int_0^\infty \frac{f(x,y)a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} dxdy \right) \\
&\quad \times \left((\sinh n)^{\frac{\lambda}{p}+\zeta+\frac{1}{p}} a_n \right) \\
&\le \left(\sum_{n=1}^\infty (\sinh n)^{(p-1)\lambda-p\xi-1} \left(\int_0^\infty \int_0^\infty \frac{f(x,y)a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} dxdy \right)^p \right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{n=1}^\infty (\sinh n)^{q-\lambda+\xi q-1} a_n \right)^{\frac{1}{q}} \\
&\le \left(C^p \int_0^\infty \int_0^\infty (\sinh x+\sinh y)^{-\lambda-p\xi+2\frac{p}{q}} f^p(x,y) dxdy \right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{n=1}^\infty (\sinh n)^{\xi q+q-\lambda-1} a_n^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Next, we want to prove the equivalence between (3.1) and (4.2), we let

$$f(x,y) = (\sinh x+\sinh y)^{(q-1)\lambda+\xi q-2} \left(\sum_{n=1}^\infty \frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} \right)^{q-1}.$$

Using the main inequality (3.1), we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty (\sinh x+\sinh y)^{(q-1)\lambda+\xi q-2} \left(\sum_{n=1}^\infty \frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} \right)^q dxdy \\
&= \int_0^\infty \int_0^\infty (\sinh x+\sinh y)^{(q-1)\lambda+\xi q-2} \left(\sum_{n=1}^\infty \frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} \right)^{q-1} \\
&\quad \times \left(\sum_{n=1}^\infty \frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} \right) dxdy \\
&= \int_0^\infty \int_0^\infty f(x,y) \sum_{n=1}^\infty \frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda} dxdy
\end{aligned}$$

$$\leq C\left(\int_0^\infty\int_0^\infty(\sinh x+\sinh y)^{2p-\lambda-p\xi-2}f^p(x,y)dxdy\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^\infty(\sinh n)^{\xi q+q-\lambda-1}\,a_n^q\right)^{\frac{1}{q}}$$
$$=C\Big(\int_0^\infty\int_0^\infty(\sinh x+\sinh y)^{(q-1)\lambda+\xi q-2}$$
$$\times\left(\sum_{n=1}^\infty\frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda}\right)^q dxdy\Big)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^\infty(\sinh n)^{\xi q+q-\lambda-1}\,a_n^q\right)^{\frac{1}{q}}. \tag{4.4}$$

If we divide both sides of (4.4) by

$$\left(\int_0^\infty\int_0^\infty(\sinh x+\sinh y)^{(q-1)\lambda+\xi q-2}\left(\sum_{n=1}^\infty\frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda}\right)^q dxdy\right)^{\frac{1}{p}},$$

we get (4.2). Also, from the Hölder inequality and (4.2), we obtain

$$\int_0^\infty\int_0^\infty f(x,y)\sum_{n=1}^\infty\frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda}dxdy$$
$$=\int_0^\infty\int_0^\infty\left((\sinh x+\sinh y)^{\frac{(q-1)\lambda+\xi q-2}{q}}f(x,y)\right)$$
$$\times\left((\sinh x+\sinh y)^{\frac{(q-1)\lambda+\xi q-2}{p}}\sum_{n=1}^\infty\frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda}\right)dxdy$$
$$\leq\left(\int_0^\infty\int_0^\infty\left((\sinh x+\sinh y)^{\frac{(q-1)\lambda+\xi q-2}{q}}\right)^p f^p(x,y)dxdy\right)^{\frac{1}{p}}$$
$$\times\Big(\int_0^\infty\int_0^\infty\Big((\sinh x+\sinh y)^{\frac{(q-1)\lambda+\xi q-2}{p}}$$
$$\times\sum_{n=1}^\infty\frac{a_n}{(\sinh x+\sinh y+\sinh n)^\lambda}\Big)^q dxdy\Big)^{\frac{1}{q}}$$
$$\leq C\left(\int_0^\infty\int_0^\infty(\sinh x+\sinh y)^{2p-\lambda-p\xi-2}f^p(x,y)dxdy\right)^{\frac{1}{p}}$$
$$\times\Big(\sum_{r=1}^\infty(\sinh n)^{\xi q+q-\lambda-1}\,a_n^q\Big)^{\frac{1}{q}},$$

from this, we proved the equivalence relation between (4.2) and (3.1). □

Theorem 4.2. *Under the same conditions as in Theorem* 3.2*, we have the following two inequalities:*

$$\int_0^\infty (\sinh x)^{(p-1)\lambda - p\xi - 1} \left(\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_{m,n}}{(\sinh m + \sinh n + \sinh x)^\lambda} \right)^p dx$$
$$\leq \widetilde{C}^p \sum_{m=1}^\infty \sum_{n=1}^\infty (\sinh n + \sinh m)^{2p-\lambda-p\xi-2} a_{m,n}^p \tag{4.5}$$

and

$$\sum_{m=1}^\infty \sum_{n=1}^\infty (\sinh m + \sinh n)^{(q-1)\lambda+\xi q-2} \left(\int_0^\infty \frac{f(x)}{(\sinh m + \sinh n + \sinh x)^\lambda} dx \right)^q$$
$$\leq \widetilde{C}^q \int_0^\infty (\sinh x)^{\xi q-\lambda+q-1} f^q(x) dx. \tag{4.6}$$

The inequalities (4.5) *and* (4.6) *are equivalent to* (3.2) *and the constants* $\widetilde{C}^p$ *and* $\widetilde{C}^q$ *are also the best possible.*

Proof. We can use the same procedure as in Theorem 4.1 (omitted). □

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