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AN ITERATIVE METHOD FOR A COMMON SOLUTION OF FIXED POINT PROBLEM AND SPLIT GENERALIZED EQUILIBRIUM PROBLEM

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Abstract. The purpose of this paper is to introduce an iterative method for finding a common solution to fixed point problems and split generalized equilibrium problems of demimetric mappings in real Hilbert spaces. We are motivated by the convergence properties of the proposed method and establish the strong convergence of the sequence generated by our algorithm. Additionally, we present a numerical example to illustrate the significance and efficiency of our method. Our results develop and unify several optimization results found in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $||\cdot||$. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point x in a Hilbert space H by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is well known in [16] that a Hilbert space H satisfies *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the

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inequality

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y|| \tag{1.1}$$

holds for every $y \in H$ with $y \neq x$.

Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$, $F : C \times C \to \mathbb{R}$ be two bifunctions. The *generalized equilibrium problem* (GEP) is to find a point $x^* \in C$ such that

$$F(x^*, y) + \phi(x^*, y) \geq 0, \ \forall \ y \in C. \tag{1.2}$$

The solution set of the GEP is denoted by $GEP(F, \phi)$. In particular, If we set $\phi = 0$ in (1.2), then the GEP reduces to the classical equilibrium problem (EP), which is to find a point $x^* \in C$ such that $F(x^*, y) \geq 0$, for all $y \in C$. The solution set of EP is denoted by $EP(F)$. The EP is a generalization of many mathematical models such as variational inequality problems (VIPs), fixed point problems (FPPs), certain optimization problems (OPs), Nash EPs, minimization problems, (MPs) and others; see [9, 18]. Many authors have studied and proposed several iterative algorithms for solving EPs and related OPs, see [1, 2, 3, 4, 20, 22].

In 2013, Kazmi and Rizvi [11] introduced and studied the following split generalized equilibrium problem (SGEP): Let $C \subseteq H_1$ and $Q \subseteq H_2$, where H_1 and H_2 are real Hilbert spaces. Let $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions, and $A : H_1 \to H_2$ be a bounded linear operator. The *SGEP* is defined as follows: Find $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x^*, x) \geq 0, \ \forall \ x \in C \tag{1.3}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + \phi_2(y^*, y) \geq 0, \ \forall \ y \in Q. \tag{1.4}$$

We denote the solution set of SGEP (1.3)-(1.4) by

$$SGEP(F_1, \phi_1, F_2, \phi_2) := \{x^* \in C : x^* \in GEP(F_1, \phi_1) \text{ such that } Ax^* \in GEP(F_2, \phi_2)\}.$$

Furthermore, an iterative algorithm was also presented by the authors for approximating the solution of SGEP in a real Hilbert space. If $\phi_1 = 0$ and $\phi_2 = 0$, in (1.3) and (1.4) then the SGEP reduces to split equilibrium problem (SEP), which is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall \ x \in C \tag{1.5}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \quad \forall \ y \in Q. \tag{1.6}$$

Observe that (1.5) is the classical EP. Therefore, the inequalities (1.5) and (1.6) comprise a pair of EPs which involves finding the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of (1.5) in H_1, which is the solution of (1.6) in H_2. The solution set of SEP (1.5)-(1.6) is denoted by $SEP(F_1, F_2) := \{z \in EP(F_1) : Az \in EP(F_2)\}$.

Another important problem in fixed point theory is the fixed point problem (FPP), which is defined as follows:

$$\text{Find a point } x^* \in C \text{ such that } Tx^* = x^*, \tag{1.7}$$

where $T : C \to C$ is a nonlinear operator. We denote the set of fixed points of T by $Fix(T)$. The fixed point theory for demimetric mapping can be utilized in various areas such as game theory, control theory, mathematical economics, etc.

In 2016, Suantai et al. [19] introduced the following iterative scheme for solving SEP and FPP of nonspreading multi-valued mapping in Hilbert spaces:

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ u_n = T_{r_n}^{F_1}\big(I - \gamma A^*(I - T_{r_n}^{F_2})A\big)x_n, \\ x_{n+1} \in \alpha_n x_n + (1-\alpha_n)Su_n, \end{cases} \tag{1.8}$$

for all $n \geq 1$, where C is a nonempty closed convex subset of a real Hilbert space $H, \{\alpha_n\} \subset (0,1), r_n \subset (0,\infty)$, S is a nonspreading multi-valued mapping, and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of the bounded linear operator A. Under the following conditions on the control sequences

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$,

the authors proved that the sequence $\{x_n\}$ defined by (1.8) converges weakly to a point $p \in F(S) \cap SEP(F_1, F_2) \neq \emptyset$.

In this article, we are interested in studying the problem of finding a common solution for both the SGEP (1.3)-(1.4) and the common FPP for demimetric mappings. The motivation for studying such problems is in their potential application to mathematical models whose constraints can be expressed as FPP and SGEP. This occurs, in particular, in practical problems such as signal processing, network resource allocation and image recovery. A scenario is in network bandwidth allocation problem for two services in heterogeneous wireless access networks in which the bandwidth of the services are mathematically related (see, for instance, [7, 13] and the references therein).

Motivated by the above results and the ongoing research interest in this direction, in this article, we introduce an iterative algorithm of a common solution of fixed point problem and split generalized equilibrium problem of demimetric mappings in real Hilbert spaces. We establish a strong convergence of the sequence generated by the proposed algorithm. We present a numerical example to illustrate the significance and efficient performance of our method.

Subsequent sections of this work are organized as follows: In Section 2, we recall some basic definitions and lemmas that are relevant in establishing our main results. In Section 3, we prove some lemmas that are useful in establishing the strong convergence of our proposed algorithm and also prove the strong convergence theorem for the algorithm.

2. Preliminaries

Definition 2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be:

(1) directed if

$$\|Tx - x^*\|^2 \leq \|x - x^*\| - \|Tx - x\|, \ \forall \ x \in C, x^* \in Fix(T),$$

(2) β-demicontractive if there exists a constant $\beta \in [0, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\| + \beta\|Tx - x\|, \ \forall \ x \in C, x^* \in Fix(T),$$

(3) k-demimetric if there exists a constant $k \in (-\infty, 1)$ such that

$$\langle x - x^*, x - Tx\rangle \geq \frac{1-k}{2}\|x - Tx\|^2, \ \forall \ x \in C, x^* \in Fix(T). \tag{2.1}$$

Clearly, (2.1) is equivalent to the following:

$$\|Tx - x^*\|^2 \leq \|x - x^*\| + k\|Tx - x\|, \ \forall \ x \in C, x^* \in Fix(T).$$

The class of demimetric mappings is fundamental because many common types of mappings arise in optimization belong to this class, see for example [6, 23] and references therein. The demimetric mappings include the directed mappings and the demicontractive mappings as special cases. More so, this class mapping contains the classes of strict pseudo-contractions, firmly-quasi-nonexpansive mappings, 2-generalized hybrid mappings and quasi-nonexpansive mappings.

The following results will be needed in the sequel:

Lemma 2.2. ([21]) *In a real Hilbert space H, the following inequalities hold for all $x, y \in H$:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$;
(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$;
(iii) $||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2$.

Lemma 2.3. ([24]) *Let a_n, b_n and γ_n be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\gamma_n)a_n + b_n, \ n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exist.

Lemma 2.4. ([15]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x, y, z \in H$ and a real number α, the set*

$$\{u \in C : \|y-u\|^2 \leq \|x-u\|^2 + \langle z, u\rangle + \alpha\}$$

is closed and convex.

Assumption 2.5. Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C\times C \to \mathbb{R}$ and $\phi_1 : C\times C \to \mathbb{R}$ be two bifunctions satisfying the following conditions:

(A1) $F_1(x,x) = 0$ for all $x \in C$,
(A2) F_1 is monotone, that is, $F_1(x,y) + F_1(y,x) \leq 0$ for all $x, y \in C$,
(A3) F_1 is upper hemicontinuous, that is, for all $x, y, z \in C$,

$$\lim_{t\downarrow 0} F_1\big(tz + (1-t)x, y\big) \leq F_1(x,y),$$

(A4) for each $x \in C, y \mapsto F_1(x,y)$ is convex and lower semicontinuous,
(A5) $\phi_1(x,x) \geq 0$ for all $x \in C$,
(A6) for each $y \in C, x \mapsto \phi_1(x,y)$ is upper semicontinuous,
(A7) for each $x \in C, \mapsto \phi_1(x,y)$ is convex and lower semicontinuous

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F_1(y,x) + \phi_1(y,x) + \frac{1}{r}\langle y-x, x-z\rangle < 0, \quad \forall y \in C\backslash K.$$

Lemma 2.6. ([14]) *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C\times C \to \mathbb{R}$ and $\phi_1 : C\times C \to \mathbb{R}$ be two bifunctions satisfying Assumption* 2.5*. Assume that ϕ_1 is monotone. For $r > 0$ and $x \in H_1$, define a mapping $T_r^{(F_1,\phi_1)} : H_1 \to C$ as follows:*

$$T_r^{(F_1,\phi_1)}x = \left\{ z \in C : F_1(z,y) + \phi_1(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \forall y \in C \right\}, \tag{2.2}$$

for all $x \in H_1$. Then

(i) *for each $x \in H_1, T_r^{(F_1,\phi_1)}x \neq \emptyset$,*
(ii) *$T_r^{(F_1,\phi_1)}$ is single-valued,*
(iii) *$T_r^{(F_1,\phi_1)}$ is firmly nonexpansive, that is, for any $x, y \in H_1$,*

$$\|T_r^{(F_1,\phi_1)}x - T_r^{(F_1,\phi_1)}y\|^2 \leq \langle T_r^{(F_1,\phi_1)}x - T_r^{(F_1,\phi_1)}y, x-y\rangle,$$

(iv) $F(T_r^{(F_1,\phi_1)}) = GEP(F_1, \phi_1)$,
(v) $GEP(F_1, \phi_1)$ *is compact and convex.*

Lemma 2.7. ([5]) *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $f_1 : C\times C \to \mathbb{R}$ and $\phi_1 : C\times C \to \mathbb{R}$ be two bifunctions satisfying Assumption* 2.5*, and let $T_r^{(F_1,\phi_1)}$ be defined as in Lemma* 2.6 *for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then*

$$\|T_{r_2}^{(F_1,\phi_1)}y - T_{r_1}^{(F_1,\phi_1)}x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}^{(F_1,\phi_1)}y - y\|.$$

Lemma 2.8. ([17]) *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{\infty}^{n=1} \lambda_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n + \lambda_n, \;\; n \geq 1.$$

If $\limsup\limits_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then $\lim\limits_{k\to\infty} a_n = 0$.

3. Main result

In this section, we give our main result of the sequel. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $L : H_1 \to H_2$ be a bounded linear operator with dual $L^* : H_2 \to H_1$. Assume F_1, ϕ_1 and F_2, ϕ_2 are bifunctions on $C \times C$ and $Q \times Q$, respectively satisfying Assumptions A_1-A_7. Let $S : H_1 \to H_1$ be a k-demimetric mapping where $k \in (-\infty, 1)$. We consider the problem of finding a point $x^* \in C$ such that

$$x^* \in \mathrm{Fix(S)} \cap SGEP(F_1, F_2, \phi, \phi_2). \tag{3.1}$$

We denote the solution of (3.1) by Γ. That is

$$\Gamma = \{x^* \in C : x^* \in \mathrm{Fix(S)} \cap SGEP(F_1, F_2, \phi, \phi_2)\}.$$

Assume Γ is nonempty. For approximating a point in Γ, we introduce the following iterative method:

Algorithm: **(The New Algorithm)**

Initialization: Choose $x_0, x_1 \in C$ and $\theta > 0$. Let $f : C \to C$ be a contraction mapping with a constant $\tau \in (0,1)$. Given x_n, x_{n-1}, choose θ_n such that $\theta_n \in [0, \bar{\theta}_n]$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Iterative process: Compute x_{n+1} as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \alpha_n w_n + (1-\alpha_n) T_{r_n}^{F_1,\phi_1}(w_n - \gamma_n L^*(I - T_{r_n}^{F_2,\phi_2}) L w_n), \\ z_n = \beta_n y_n + (1-\beta_n) S y_n, \\ x_{n+1} = \delta_n f(x_n) + \mu_n x_n + \lambda_n z_n, \end{cases} \tag{3.2}$$

where γ_n is chosen such that for small enough $\epsilon > 0$,

$$\gamma_n \in \left\{\epsilon, \frac{\|(I - T_{r_n}^{F_2,\phi_2}) L w_n\|^2}{\|L^*(I - T_{r_n}^{F_2,\phi_2}) L w_n\|^2} - \epsilon\right\}, \tag{3.3}$$

if $T_{r_n}^{F_2,\phi_2} L w_n \neq L w_n$ otherwise $\gamma_n = \gamma$.

First we prove the following important lemma:

Lemma 3.1. *Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ given by Algorithm (3.2) such that $x_{n_k} \rightharpoonup q \in C$. Suppose $\|w_{n_k} - x_{n_k}\| \to 0$, $\|w_{n_k} - u_{n_k}\| \to 0$ and $\|L w_{n_k} - T_{r_{n_k}}^{F_1,\phi_1} L w_{n_k}\| \to 0$ as $k \to \infty$. Then $q \in SGEP(F_1, F_2, \phi_1, \phi_2)$.*

Proof. Since $u_{n_k} = T_{r_{n_k}}^{F_1,\phi_1}(w_{n_k} - \gamma_{n_k} L^*(I - T_{r_{n_k}}^{F_2,\phi_2}) L w_{n_k})$, we get for all $u \in C$,

$$F_1(u_{n_k}, u) + \phi_1(u_{n_k}, u) + \frac{1}{r_{n_k}} \langle u - u_{n_k}, u_{n_k} - (w_{n_k} - \gamma_{n_k} L^*(I - T_{r_{n_k}}^{F_2,\phi_2}) L w_{n_k}) \rangle \geq 0.$$

This implies that

$$F_1(u_{n_k}, u) + \phi_1(u_{n_k}, u) + \frac{1}{r_{n_k}} \langle u - u_{n_k}, u_{n_k} - w_{n_k} \rangle + \frac{1}{r_{n_k}} \langle u - u_{n_k}, \gamma_{n_k} L^*(I - T_{r_{n_k}}^{F_2,\phi_2}) L w_{n_k} \rangle \geq 0, \ \forall\, u \in C$$

and

$$\frac{1}{r_{n_k}} \langle u - u_{n_k}, u_{n_k} - w_{n_k} \rangle + \frac{1}{r_{n_k}} \langle u - u_{n_k}, \gamma_{n_k} L^*(I - T_{r_{n_k}}^{F_2,\phi_2}) L w_{n_k} \rangle \geq F_1(u, u_{n_k}) + \phi_1(u, u_{n_k}). \tag{3.4}$$

Since $\|w_{n_k} - u_{n_k}\| \to 0$, then $u_{n_k} \rightharpoonup q$ as $k \to \infty$. Taking the limit of the inequality (3.4), we get

$$F_1(u, q) + \phi_1(u, q) \leq 0, \ \forall u \in C. \tag{3.5}$$

Let $u_t = tu + (1-t)q$ for any $t \in (0,1]$ and $u \in C$. Consequently, we have $u_t \in C$ and hence

$$F_1(u_t, q) + \phi_1(u_t, q) \leq 0.$$

Using assumption (A1) and (A4), we have

$$\begin{aligned}0 &\leq F_1(u_t, u_t) + \phi_1(u_t, u_t)\\ &\leq t(F_1(u_t, u) + \phi_1(u_t, u)) + (1-t)(F_1(u_t, q) + \phi_1(u_t, q))\\ &\leq F_1(u_t, u) + \phi_1(u_t, u).\end{aligned}$$

Hence, we have

$$F_1(u_t, u) + \phi_1(u_t, u) \geq 0.$$

Letting $t \to 0$ and using assumption A3, we deduce the upper semicontinuity of F_2 such that

$$F_1(q, u) + \phi_1(q, u) \geq 0, \ \forall \ u \in C.$$

This implies that $q \in GEP(F_1, \phi_1)$.

Further, since L is a bounded linear operator, then $Lu_{n_k} \rightharpoonup Lq$. Then, it follows from $\|(I - T_{r_{n_k}}^{F_2,\phi_2})Lw_{n_k}\| \to 0$ that $T_{r_{n_k}}^{F_2,\phi_2} Lw_{n_k} \rightharpoonup Lq$. By definition of $T_{r_{n_k}}^{F_2,\phi_2} Lw_{n_k}$, we have

$$\begin{aligned}&F_2(T_{r_{n_k}}^{F_2,\phi_2} Lw_{n_k}, u) + \phi_2(T_{r_{n_k}}^{F_2,\phi_2} Lw_{n_k}, u)\\ &\qquad + \frac{1}{r_{n_k}}\langle u - T_{r_{n_k}}^{F_2,\phi_2} Lw_{n_k}, T_{r_{n_k}}^{F_2,\phi_2} Lw_{n_k} - Lw_{n_k}\rangle \geq 0, \ \forall \ u \in Q.\end{aligned}$$

Since both F_2 and ϕ_2 are upper semicontinuous in the first argument, it follows from the above inequality that

$$F_2(Lq, u) + \phi_2(Lq, u) \leq 0, \ \forall u \ \in \ Q.$$

This shows that $Lq \in GEP(F_2, \phi_2)$. Hence $q \in SGEP(F_1, F_2, \phi_1, \phi_2)$. $\square$

Lemma 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm* (3.2)*. Then $\{x_n\}$ is bounded.*

Proof. Suppose $u_n = T_{r_n}^{F_1,\phi_1}(w_n - \gamma_n L^*(I - T_{r_n}^{F_2,\phi_2})Lw_n)$ in Algorithm (3.2), then $y_n = \alpha_n w_n + (1-\alpha_n)u_n$. Let $p \in \Gamma$, we have from Algorithm (3.2), that

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}^{F_1,\phi_1}(w_n - \gamma_n L^*(I - T_{r_n}^{F_2,\phi_2})Lw_n) - T_{r_n}^{F_1,\phi_1} p\|^2\\ &\leq \|w_n - \gamma_n L^*(I - T_{r_n}^{F_2,\phi_2})Lw_n - p\|^2\end{aligned}$$

$$
\begin{aligned}
&= \|w_n - p\|^2 + \gamma_n^2 \|L^*(T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n)\|^2 \\
&\quad + 2\gamma_n \langle w_n - p, L^*(T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n)\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2 \|L^*(T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n)\|^2 \\
&\quad + 2\gamma_n \langle Lw_n - Lp, T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n\rangle \\
&= \|w_n - p\|^2 + \gamma_n^2 \|L^*(T_{r_2}^{F_2,\phi_2} Lw_n - Lw_n)\|^2 + \gamma_n \|T_{r_n}^{F_2,\phi_2} Lw_n - Lp\|^2 \\
&\quad - \gamma_n \|Lw_n - Lp\|^2 - \gamma_n \|T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n\|^2 \\
&\leq \|w_n - p\|^2 + \gamma_n^2 \|L^*(T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n)\|^2 + \gamma_n \|Lw_n - Lp\|^2 \\
&\quad - \gamma_n \|Lw_n - Lp\|^2 - \gamma_n \|T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n\|^2 \\
&\leq \|w_n - p\|^2 + \gamma_n^2 \|L^*(T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n)\|^2 \\
&\quad - \gamma_n(\gamma_n + \epsilon) \|T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n\|^2 \\
&= \|w_n - p\|^2 - \gamma_n \epsilon \|L^*(T_{r_n}^{F_2,\phi_2} Lw_n - Lw_n)\|^2 \\
&\leq \|w_n - p\|^2.
\end{aligned} \tag{3.6}
$$

Thus,

$$\|u_n - p\| \leq \|w_n - p\|.$$

Also,

$$
\begin{aligned}
\|y_n - p\| &= \|\alpha_n w_n + (1-\alpha_n)u_n - p\| \\
&= \|\alpha_n (w_n - p) + (1-\alpha_n)(u_n - p)\| \\
&\leq \alpha_n \|w_n - p\| + (1-\alpha_n)\|u_n - p\| \\
&\leq \alpha_n \|w_n - p\| + (1-\alpha_n)\|w_n - p\| \\
&= \|w_n - p\|.
\end{aligned}
$$

Since $\lim_{n\to\infty} \frac{\theta_n}{\delta_n}\|x_n - x_{n-1}\| = 0$, there exists $M_1 > 0$ such that

$$\frac{\theta_n}{\delta_n}\|x_n - x_{n-1}\| \leq M_1$$

for all $n \in \mathbb{N}$. Therefore,

$$
\begin{aligned}
\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\
&\leq \|x_n - p\| + \theta_n \|x_n - x_{n-1}\| \\
&\leq \|x_n - p\| + \delta_n \frac{\theta_n}{\delta_n}\|x_n - x_{n-1}\| \\
&\leq \|x_n - p\| + \delta_n M_1.
\end{aligned} \tag{3.7}
$$

Again from Algorithm (3.2), we have

$$\begin{aligned}\|z_n - p\|^2 &= \|\beta_n y_n + (1-\beta_n)Sy_n - p\|^2\\ &= \|(1-\beta_n)(y_n - p) + \beta_n(Sy_n - p)\|^2\\ &= (1-\beta_n)\|y_n - p\|^2 + \beta_n\|Sy_n - p\|^2 - \beta_n(1-\beta_n)\|(Sy_n - y_n)\|^2\\ &\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n\|y_n - p\|^2 - \beta_n k\|(Sy_n - y_n)\|^2\\ &- \beta_n(1-\beta_n)\|(Sy_n - y_n)\|^2\\ &= \|y_n - p\|^2 - \beta_n(1-k-\beta_n)\|(Sy_n - y_n)\|^2. \end{aligned} \tag{3.8}$$

Therefore,

$$\|z_n - p\| \leq \|y_n - p\|.$$

Finally, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\delta_n f(x_n) + \mu_n x_n + \lambda_n z_n - p\|\\ &\leq \delta_n\|f(x_n) - p\| + \mu_n\|x_n - p\| + \lambda_n\|z_n - p\|\\ &\leq \delta_n\|f(x_n) - f(p)\| + \delta_n\|f(p) - p\| + \mu_n\|x_n - p\| + \lambda_n\|y_n - p\|\\ &\leq \delta_n\|f(x_n) - f(p)\| + \delta_n\|f(p) - p\| + \mu_n\|x_n - p\| + \lambda_n\|w_n - p\|\\ &\leq \delta_n\|f(x_n) - f(p)\| + \delta_n\|f(p) - p\| + \mu_n\|x_n - p\|\\ &\quad + \lambda_n(\|x_n - p\| + \delta_n M_1)\\ &\leq \delta_n\tau\|x_n - p\| + \delta_n\|f(p) - p\| + \mu_n\|x_n - p\|\\ &\quad + \lambda_n(\|x_n - p\| + \delta_n M_1)\\ &\leq \delta_n\tau\|x_n - p\| + (1-\delta_n)\|x_n - p\| + \delta_n\|f(p) - p\| + \lambda_n\delta_n M_1\\ &= (1-\delta_n+\delta_n\tau)\|x_n - p\| + \delta_n\|f(p) - p\| + \lambda_n\delta_n M_1\\ &= [1-\delta_n(1-\tau)]\|x_n - p\| + \frac{\delta_n(1-\tau)}{(1-\tau)}\|f(p) - p\| + \frac{\delta_n(1-\tau)}{(1-\tau)}\lambda_n M_1\\ &\leq \max\left\{\|x_n - p\|, \frac{\|f(p) - p\| + \lambda_n M_1}{(1-\tau)}\right\}\\ &\vdots\\ &\leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\| + \lambda_n M_1}{(1-\tau)}\right\}.\end{aligned}$$

By induction, we have,

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\| + M_1}{(1-\tau)}\right\}, \ \forall\ n \geq 1.$$

Thus, $\{x_n\}$ is bounded, so $\{u_n\}, \{y_n\}$ and $\{z_n\}$ also are bounded. □

Theorem 3.3. *Let $\{x_n\}$ be a sequence generated by Algorithm* (3.2). *Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $p = P_\Gamma f(p)$.*

Proof. Let $p \in \Gamma$. From Algorithm (3.2), we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|2\|x_n - p\| + \delta_n\frac{\theta_n}{\delta_n}\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \alpha_n M_1) \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M_2,
\end{aligned} \tag{3.9}$$

where $M_2 = \sup_n(2\|x_n - p\| + \alpha_n M_1)$.

Also from Algorithm (3.2), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n w_n + (1 - \alpha_n)u_n - p\|^2 \\
&= \|\alpha_n(w_n - p) + (1 - \alpha_n)(u_n - p)\|^2 \\
&= \alpha_n\|w_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 - \alpha_n(1 - \alpha_n)\|w_n - u_n\|^2 \\
&\le \alpha_n\|w_n - p\|^2 + (1 - \alpha_n)\|w_n - p\|^2 - \alpha_n(1 - \alpha_n)\|w_n - u_n\|^2 \\
&= \|w_n - p\|^2 - \alpha_n(1 - \alpha_n)\|w_n - u_n\|^2.
\end{aligned} \tag{3.10}$$

In addition, using Algorithm (3.2) and (3.9), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\delta_n f(x_n) + \mu_n x_n + \lambda_n z_n - p\|^2 \\
&= \|\delta_n(f(x_n) - p) + \mu_n(x_n - p) + \lambda_n(z_n - p)\|^2 \\
&\le \|\mu_n(x_n - p) + \lambda_n(z_n - p)\|^2 + 2\mu_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \mu_n^2\|x_n - p\|^2 + 2\mu_n\lambda_n\langle x_n - p, z_n - p\rangle + \lambda_n^2\|z_n - p\|^2 \\
&\quad + 2\delta_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \mu_n^2\|x_n - p\|^2 + 2\mu_n\lambda_n\|x_n - p\|\|z_n - p\| + \lambda_n^2\|z_n - p\|^2 \\
&\quad + 2\delta_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \mu_n^2\|x_n - p\|^2 + \lambda_n^2\|z_n - p\|^2 + \mu_n\lambda_n(\|x_n - p\|^2 + \|z_n - p\|^2) \\
&\quad + 2\delta_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\delta_n\langle f(p) - p, x_{n+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\|z_n-p\|^2\\
&\quad +2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\|y_n-p\|^2\\
&\quad +2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\|w_n-p\|^2\\
&\quad +2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\|x_n-p\|^2\\
&\quad +\lambda_n(\mu_n+\lambda_n)\theta_n\|x_n-x_{n-1}\|M_2\\
&\quad +2\delta_n\tau\|x_n-p\|\|x_{n+1}-p\|+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le (\mu_n+\lambda_n)^2\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\theta_n\|x_n-x_{n-1}\|M_2\\
&\quad +\delta_n\tau\|x_n-p\|^2+\delta_n\tau\|x_{n+1}-p\|^2+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&= (1-\delta_n^2)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\theta_n\|x_n-x_{n-1}\|M_2\\
&\quad +\delta_n\tau\|x_n-p\|^2+\delta_n\tau\|x_{n+1}-p\|^2+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&= (1-2\delta_n+\delta_n\tau)\|x_n-p\|^2+\delta_n^2\|x_n-p\|^2\\
&\quad +\lambda_n(1-\delta_n)\theta_n\|x_n-x_{n-1}\|M_2+\delta_n\tau\|x_n-p\|^2\\
&\quad +\delta_n\tau\|x_{n+1}-p\|^2+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle \qquad (3.11)
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \left(1-\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\right)\|x_n-p\|^2\\
&\quad +\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\lambda_n(1-\delta_n)}{2\delta_n(1-\tau)}\theta_n\|x_n-x_{n-1}\|M_2\right)\\
&\quad +\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\delta_nM_3}{2(1-\tau)}+\frac{1}{1-\tau}\langle f(p)-p,x_{n+1}-p\rangle\right)\\
&= (1-\frac{2\delta_n(1-\tau)}{1-\delta_n\tau})\|x_n-p\|^2+\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}b_n, \qquad (3.12)
\end{aligned}
$$

where $M_3=\sup_{n\in\mathbb{N}}\{\|x_n-p\|^2 : n\ge \mathbb{N}\}$ and $b_n=(\frac{\lambda_n(1-\delta_n)}{2(1-\delta_n)}\frac{\theta_n}{\delta_n}\|x_n-x_{n-1}\|M_2+\frac{\delta_nM_3}{2(1-\tau)}+\frac{1}{1-\tau}\langle f(p)-p,x_{n+1}-p\rangle)$.

According to Lemma 2.8, it is sufficient to establish that $\limsup_{k\to\infty}\ b_{n_k}\le 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying the condition

$$
\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|-\{\|x_{n_k}-p\|\}\ge 0. \qquad (3.13)
$$

To establish that $\limsup_{k\to\infty} b_{n_k} \le 0$, we assume the existence of the subsequence satisfying (3.13). Thus,

$$\begin{aligned}\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|\}^2 &= \liminf_{k\to\infty}\{(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)\\ &\quad\times(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)\}\\ &\ge 0. \end{aligned} \tag{3.14}$$

It is easy to see from (3.8) and (3.11), that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\|z_n-p\|^2\\ &\quad+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)(\|y_n-p\|^2\\ &\quad-\beta_n(1-k-\beta_n)\|(Sy_n-y_n)\|^2)\\ &\quad+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\quad+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)(\|w_n-p\|^2\\ &\quad-\beta_n(1-k-\beta_n)\|(Sy_n-y_n)\|^2)+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle\\ &\quad+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\quad+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\le \left(1-\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\right)\|x_n-p\|^2\\ &\quad+\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\lambda_n(1-\delta_n)}{2\delta_n(1-\tau)}\theta_n\|x_n-x_{n-1}\|M_2\right)\\ &\quad+\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\delta_n M_3}{2\delta_n(1-\tau)}+\frac{1}{1-\tau}\langle f(p)-p,x_{n+1}-p\rangle\right)\\ &\quad-\lambda_n(1-\delta_n)\beta_n(1-k-\beta_n)\|(Sy_n-y_n)\|^2\\ &\le \|x_n-p\|^2+\frac{\delta_n\lambda_n(1-\delta_n)}{(1-\delta_n\tau)}\frac{\theta_n}{\delta_n}\|x_n-x_{n-1}\|M_2+\frac{\delta_n M_3}{1-\delta_n\tau}\\ &\quad+\frac{2\delta_n}{1-\delta_n\tau}\langle f(p)-p,x_{n+1}-p\rangle\\ &\quad-\lambda_n(1-\delta_n)\beta_n(1-k-\beta_n)\|(Sy_n-y_n)\|^2\\ &= \|x_n-p\|^2+\delta_n M_4-\lambda_n(1-\delta_n)\beta_n(1-k-\beta_n)\|(Sy_n-y_n)\|^2, \end{aligned} \tag{3.15}$$

where $M_4 = \frac{\lambda_n(1-\delta_n)}{(1-\delta_n\tau)}\frac{\theta_n}{\delta_n}\|x_n - x_{n-1}\|M_2 + \frac{M_3}{1-\delta_n\tau} + \frac{2}{1-\delta_n\tau}\langle f(p)-p, x_{n+1}-p\rangle$. Thus we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \lambda_{n_k}(1-\delta_{n_k})\beta_{n_k}(1-k-\beta_{n_k})\|(Sy_{n_k}-y_{n_k})\|^2 \\
&\quad\le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2 - \|x_{n_k+1}-p\|^2] + M_4 \lim_{k\to\infty}\delta_{n_k} \\
&\quad\le -\liminf_{k\to\infty}(|x_{n_k+1}-p\|^2 - \|x_{n_k}-p\|^2) \\
&\quad\le 0.
\end{aligned}$$

Therefore, we obtain

$$\|Sy_{n_k} - y_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{3.16}$$

Now from (3.10) and (3.11), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2 + \frac{\delta_n M_3}{1-\delta_n\tau} \\
&\quad\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2 + \lambda_n(\mu_n+\lambda_n)\|z_n-p\|^2 \\
&\qquad + 2\delta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\delta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2 + \lambda_n(\mu_n+\lambda_n)\|y_n-p\|^2 \\
&\qquad + 2\delta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\delta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2 + \lambda_n(\mu_n+\lambda_n) \\
&\qquad \times(\|w_n-p\|^2 - \alpha_n(1-\alpha_n)\|w_n-u_n\|^2) \\
&\qquad + 2\delta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\delta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad\le \left(1-\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\right)\|x_n-p\|^2 \\
&\qquad + \frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\lambda_n(1-\delta_n)}{2\delta_n(1-\tau)}\theta_n\|x_n-x_{n-1}\|M_2\right) \\
&\qquad + \frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\delta_n M_3}{2\delta_n(1-\tau)} + \frac{1}{1-\tau}\langle f(p)-p, x_{n+1}-p\rangle\right) \\
&\qquad - \alpha_n(1-\alpha_n)\|w_n-u_n\|^2 \\
&\quad\le \|x_n-p\|^2 + \frac{\delta_n\lambda_n(1-\delta_n)}{(1-\delta_n\tau)}\frac{\theta_n}{\delta_n}\|x_n-x_{n-1}\|M_2 + \frac{\delta_n M_3}{1-\delta_n\tau} \\
&\qquad + \frac{2\delta_n}{1-\delta_n\tau}\langle f(p)-p, x_{n+1}-p\rangle - \alpha_n(1-\alpha_n)\|w_n-u_n\|^2 \\
&\quad= \|x_n-p\|^2 + \delta_n M_4 - \alpha_n(1-\alpha_n)\|w_n-u_n\|^2,
\end{aligned} \tag{3.17}$$

where M_4 is defined as before. Thus we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \alpha_{n_k}(1-\alpha_{n_k})\|w_{n_k}-u_{n_k}\|^2 \\
&\quad \le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2 - \|x_{n_k+1}-p\|^2] + M_4 \lim_{k\to\infty} \delta_{n_k} \\
&\quad \le -\liminf_{k\to\infty}(|x_{n_k+1}-p\|^2 - \|x_{n_k}-p\|^2) \\
&\quad \le 0.
\end{aligned}$$

Hence, we obtain

$$\lim_{k\to\infty} \|w_{n_k}-u_{n_k}\| = 0. \tag{3.18}$$

Again from (3.10) and (3.11), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2 + \lambda_n(\mu_n+\lambda_n)\|y_n-p\|^2 \\
&\qquad + 2\delta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\delta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad \le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2 + \lambda_n(\mu_n+\lambda_n)(\alpha_n\|w_n-p\|^2 + (1-\alpha_n)\|u_n-p\|^2) \\
&\qquad + 2\delta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\delta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad \le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2 + \lambda_n(\mu_n+\lambda_n)(\alpha_n\|w_n-p\|^2 \\
&\qquad + (1-\alpha_n)[\|w_n-p\|^2 - \gamma_n\epsilon\|L^*(T_{r_n}^{F_2,\phi_2}Lw_n - Lw_n)\|^2]) \\
&\qquad + 2\delta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\delta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad \le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2 + \lambda_n(\mu_n+\lambda_n)(\|w_n-p\|^2 \\
&\qquad - \gamma_n\epsilon(1-\alpha_n)\|L^*(T_{r_n}^{F_2,\phi_2}Lw_n - Lw_n)\|^2) \\
&\qquad + 2\delta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle + 2\delta_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\quad \le \left(1-\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\right)\|x_n-p\|^2 + \frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\lambda_n(1-\delta_n)}{2\delta_n(1-\tau)}\theta_n\|x_n-x_{n-1}\|M_2\right) \\
&\qquad + \frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\delta_n M_3}{2\delta_n(1-\tau)} + \frac{1}{1-\tau}\langle f(p)-p, x_{n+1}-p\rangle\right) \\
&\qquad - \gamma_n\epsilon(1-\alpha_n)\|L^*(T_{r_n}^{F_2,\phi_2}Lw_n - Lw_n)\|^2 \\
&\quad \le \|x_n-p\|^2 + \frac{\delta_n\lambda_n(1-\delta_n)}{(1-\delta_n\tau)}\frac{\theta_n}{\delta_n}\|x_n-x_{n-1}\|M_2 + \frac{\delta_n M_3}{1-\delta_n\tau} \\
&\qquad + \frac{2\delta_n}{1-\delta_n\tau}\langle f(p)-p, x_{n+1}-p\rangle - \gamma_n\epsilon(1-\alpha_n)\|L^*(T_{r_n}^{F_2,\phi_2}Lw_n - Lw_n)\|^2 \\
&\quad = \|x_n-p\|^2 + \delta_n M_4 - \gamma_n\epsilon(1-\alpha_n)\|L^*(T_{r_n}^{F_2,\phi_2}Lw_n - Lw_n)\|^2,
\end{aligned} \tag{3.19}$$

where M_4 is defined as before. Therefore we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \gamma_{n_k}\epsilon(1-\alpha_{n_k})\|L^*(T_{r_n}^{F_2,\phi_2}Lw_{n_k}-Lw_{n_k})\|^2\\
&\quad\le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]+M_4\lim_{k\to\infty}\delta_{n_k}\\
&\quad\le -\liminf_{k\to\infty}(|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2)\\
&\quad\le 0.
\end{aligned}$$

Hence, we have

$$\lim_{k\to\infty}\|L^*(T_{r_n}^{F_2,\phi_2}Lw_{n_k}-Lw_{n_k})\|=0. \tag{3.20}$$

Using the definition of step-size γ_n in Algorithm (3.2) and (3.20), we have

$$\lim_{k\to\infty}\|(T_{r_n}^{F_2,\phi_2}Lw_{n_k}-Lw_{n_k})\|=0. \tag{3.21}$$

From (3.2) and (3.18), we have

$$\begin{aligned}
\|y_{n_k}-u_{n_k}\| &= \|\alpha_{n_k}w_{n_k}+(1-\alpha_{n_k})u_{n_k}-u_{n_k}\|\\
&\le \alpha_{n_k}\|w_{n_k}-u_{n_k}\|+(1-\alpha_{n_k})\|u_{n_k}-u_{n_k}\|\\
&\to 0 \text{ as } k\to\infty.
\end{aligned} \tag{3.22}$$

Also from (3.2) and Lemma 3.2, let $(w_n-\gamma_n L^*(I-T_{r_n}^{F_2,\phi_2})Lw_n)=v_n$. Then we have

$$\|T_{r_n}^{F_1,\phi_1}v_n-p\|^2\le\|v_n-p\|^2-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2,$$

which implies from (3.6), that

$$\begin{aligned}
\|u_n-p\|^2 &\le \|v_n-p\|^2-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2\\
&\le \|w_n-p\|^2-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2.
\end{aligned}$$

Using this in (3.11), we deduce

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)\|y_n-p\|^2\\
&\quad+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)(\alpha_n\|w_n-p\|^2\\
&\quad\times(1-\alpha_n)\|u_n-p\|^2)+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle\\
&\quad+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)(\alpha_n\|w_n-p\|^2\\
&\quad+(1-\alpha_n)[\|w_n-p\|^2-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2])\\
&\quad+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\leq \mu_n(\mu_n+\lambda_n)\|x_n-p\|^2+\lambda_n(\mu_n+\lambda_n)(\|w_n-p\|^2-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2)\\
&\quad+2\delta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\delta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\leq \left(1-\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\right)\|x_n-p\|^2+\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\lambda_n(1-\delta_n)}{2\delta_n(1-\tau)}\theta_n\|x_n-x_{n-1}\|M_2\right)\\
&\quad+\frac{2\delta_n(1-\tau)}{1-\delta_n\tau}\left(\frac{\delta_nM_3}{2\delta_n(1-\tau)}+\frac{1}{1-\tau}\langle f(p)-p,x_{n+1}-p\rangle\right)-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2\\
&\leq \|x_n-p\|^2+\frac{\delta_n\lambda_n(1-\delta_n)}{(1-\delta_n\tau)}\frac{\theta_n}{\delta_n}\|x_n-x_{n-1}\|M_2+\frac{\delta_nM_3}{1-\delta_n\tau}\\
&\quad+\frac{2\delta_n}{1-\delta_n\tau}\langle f(p)-p,x_{n+1}-p\rangle-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2\\
&=\|x_n-p\|^2+\delta_nM_4-\|v_n-T_{r_n}^{F_1,\phi_1}v_n\|^2. \qquad (3.23)
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\limsup_{k\to\infty}\|v_{n_k}-T_{r_{n_k}}^{F_1,\phi_1}v_{n_k}\|^2 &\leq \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]+M_4\lim_{k\to\infty}\delta_{n_k}\\
&\leq -\liminf_{k\to\infty}(|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2)\\
&\leq 0.
\end{aligned}
$$

Therefore, we deduce

$$\|v_{n_k}-T_{r_{n_k}}^{F_1,\phi_1}v_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (3.24)$$

Again from (3.2), we have

$$
\begin{aligned}
\|w_{n_k}-x_{n_k}\| &= \|x_{n_k}+\theta_{n_k}(x_{n_k}-x_{n_k+1})-x_{n_k}\|\\
&=\theta_{n_k}\|x_{n_k}-x_{n_k+1}\|\\
&=\delta_{n_k}\frac{\theta_{n_k}}{\delta_{n_k}}\|x_{n_k}-x_{n_k+1}\|\\
&\to 0 \text{ as } k\to\infty. \qquad (3.25)
\end{aligned}
$$

It is easy to see from (3.18), (3.22) and (3.25) that

$$\|y_{n_k}-x_{n_k}\|\leq\|y_{n_k}-u_{n_k}\|+\|w_{n_k}-u_{n_k}\|+\|w_{n_k}-x_{n_k}\|,$$

which implies

$$\lim_{k\to\infty}\|y_{n_k}-x_{n_k}\|=0. \qquad (3.26)$$

From (3.2), (3.16) and (3.26), we deduce that

$$\begin{aligned}\|z_{n_k}-x_{n_k}\| &= \|(\beta_{n_k}y_{n_k}+(1-\beta_{n_k})Sy_{n_k}-x_{n_k}\| \\ &\le \beta_{n_k}\|y_{n_k}-x_{n_k}\|+(1-\beta_{n_k})\|Sy_{n_k}-x_{n_k}\| \\ &\le \beta_{n_k}\|y_{n_k}-x_{n_k}\|+(1-\beta_{n_k})\|Sy_{n_k}-y_{n_k}+y_{n_k}-x_{n_k}\| \\ &= \|y_{n_k}-x_{n_k}\|+(1-\beta_{n_k})\|Sy_{n_k}-y_{n_k}\|.\end{aligned}$$

Then, we have

$$\|z_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \tag{3.27}$$

Again from (3.2) and (3.27), we have

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &= \|(f(x_{n_k})+\mu_{n_k}x_{n_k}+\lambda_{n_k}z_{n_k})-x_{n_k}\| \\ &\le \delta_{n_k}\|f(x_{n_k})-x_{n_k}\|+\mu_{n_k}\|x_{n_k}-x_{n_k}\|+\lambda_{n_k}\|z_{n_k}-x_{n_k}\|.\end{aligned}$$

Thus,

$$\|x_{n_k+1}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \tag{3.28}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $q\in H_1$. It follows from demiclosedness of S, (3.16) and (3.26) that $q\in Fix(S)$. Also from (3.24), we have $v_{n_k}\to q$. Using Lemma 3.1, we have $q\in SGEP(F_1,F_2,\phi_1,\phi_2)$. Thus, $q\in\Gamma$. Moreover, since $\{x_{n_{k_i}}\}$ converges weakly to q, we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle &= \lim_{i\to\infty}\langle f(p)-p,x_{n_{k_i}}-p\rangle \\ &= \langle f(p)-p,q-p\rangle.\end{aligned}$$

Hence, p is the unique solution of Γ, it follows that

$$\limsup_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle\le 0. \tag{3.29}$$

We deduce this from (3.28) and (3.29),

$$\limsup_{k\to\infty}\langle f(p)-p,x_{n_k+1}-p\rangle\le 0. \tag{3.30}$$

By assumption and (3.29), we have

$$\begin{aligned}\lim_{k\to\infty}M_4 = \lim\Big(&\frac{\lambda_n(1-\delta_n)}{(1-\delta_n\tau)}\frac{\theta_n}{\delta_n}\|x_n-x_{n-1}\|M_2 \\ &+\frac{M_3}{1-\delta_n\tau}+\frac{2}{1-\delta_n\tau}\langle f(p)-p,x_{n+1}-p\rangle\Big) \\ \le 0.&\end{aligned}$$

Thus, from Lemma 2.8, we get that $\lim_{n\to\infty}\|x_n-p\|=0$. □

4. Numerical example

In this section, we provide a numerical example to illustrate the efficiency of our Algorithm.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^3$ and $C = Q = [0,10] \times [0,10] \times [0,10]$. Let $L : H_1 \to H_2$ be defined by $Lx = \frac{x}{2}$ for each $x \in H_1$. For $x \in C$, define the mapping $S : C \to C$ by

$$Sx = \left[0, \frac{x}{10}\right].$$

Then S is 0-demimetric mapping for all $Fix(T) = \{0\}$. Define the bifunctions $F_1, \phi_1 : C \times C \to \mathbb{R}$ by $F_1(x,y) = x^2 + xy - 2y^2$ and $\phi_1(x,y) = x - y$ for each $x, y \in C$. Also define $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ by $F_2(u,z) = 3u^2 + uz - 4z^2$ and $\phi_2(u,z) = z^2 - u^2$ for each $u, z \in Q$. It is easy to check that

$$T_{r_n}^{(F_1,\phi_1)} z = \frac{z + r_n}{1 - 3r_n},$$

$$T_{r_n}^{(F_2,\phi_2)} v = \frac{v}{1 - 7r_n}.$$

We set $F(x) = \frac{x}{7}$ and $Ax = x$, $\theta = 0.5$, $\tau = 0.5$, $r_n = \frac{n}{n+1}$ and $\alpha_n = \frac{\theta}{\|L\|}$ from Algorithm. It can easily deduced that $F_1 F_2, \phi_1\phi_2$ and $\{r_n\}$ satisfy all conditions in Theorem 3.3. Let $\epsilon > 0$, the Algorithm stops if $\|x_n - x^*\| \leq \epsilon$.

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