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VARIANTS OF F-CONTRACTIONS AND EXISTENCE RESULTS OF INTEGRAL EQUATIONS

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Abstract. In this paper, the concept of admissible hybrid F-(G-α-ϕ)-contraction is initiated and new conditions for the existence of fixed points for such family of contractions are examined in the setting of generalized metric spaces. It is noted that the principal ideas investigated herein improves and harmonizes some existing fixed point results in the related literature. A few consequences of the presented main notion are highlighted and analyzed with respect to the available brochure. To substantiate the assumptions and demonstrate the importance of the obtained results, a nontrivial and comparative example is constructed. It is observed that the ideas established in this work cannot be inferred from their analogues in either metric or quasi-metric spaces. From application standpoint, one of the deduced corollaries is applied to discuss and ascertain new conditions for the existence and uniqueness of solutions to certain class of nonlinear integral equations.

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1. Introduction

In the progression from classical to modern analysis, the role of fixed point theory cannot be over-rated. Due to its versatility, the study of metric fixed point theory has a wide variety of applications in various branches of quantitative sciences ranging from approximation theory, optimization and so on. In the field of metric fixed point theory, the Banach Contraction Principle [5] is a distinguished result which is a key tool for both theoretical and computational aspects of mathematics. Owing to its simplicity of approach, this theorem has undergone numerous generalizations as many researchers have extended the result by applying different contractive constraints on various types of spaces. For recent advancements on metric fixed point results, refer to [7].

The idea of generalization of the metric space has drawn the interest of many researchers over the past few decades. In 1992, Dhage [8] presented the concept of a generalized metric space under the name D-metric space. In line with that, Mustafa and Sims [21] introduced a more appropriate concept of generalized metric space called G-metric space and they demonstrated that most of the results established by Dhage were flawed. Mustafa et al. [20] established some fixed point results for mappings satisfying different contractive conditions in the framework of G-metric space. Based on the notion of G-metric space, Jleli et al. [13] and Samet et al. [25] remarked that some fixed point results in the context of G-MS can be deduced directly by some existing results in the setting of symmetric or asymmetric metric spaces. Some authors [6, 15] noticed that the approaches given in [13, 25] can only be applicable if the contractive constraints in the theorem can be reduced to two variables. For a recent survey in the developments of fixed point results in G-MS, the reader can refer to Jiddah et al. [11].

The notion of α-admissibility and α-ψ-contraction was presented by Samet et al [24] and some fixed point results were established. The idea of triangular α-admissibility was birthed as a generalization of α-admissibility by Karapinar et al [17]. In 2012, Wardowski [29] initiated an interesting generalization of the Banach Contraction theorem called F-contraction and established a fixed point result. In 2014, Piri and Kumam [22] extended the work of Wardowski by imposing weaker auxiliary conditions on the self-map of a complete metric space on the mapping F. Minak et al. [18] presented some fixed point results for generalized F-contractions including Ćirić type generalized F-contraction on a complete metric space. In 2016, Singh et al. [27] studied a new form of Hardy-Roger-type contraction in G-metric spaces and improved the main results of [22]. Aydi et al. [4] proposed the notion of a modified F-contraction via α-admissible mappings and some theorems that guarantees the existence and uniqueness of fixed point for such mappings were investigated. Furthermore, Vujaković et al. [28] initiated the idea of (ϕ -F)-weak contraction and proved the corresponding fixed point results. For some important trends in F-contraction type fixed point results, we refer to Fabiano et al. [9], Joshi and Jain [14].

The concept of admissible hybrid contraction was introduced by Karapinar and Fulga [16] as a contraction that unifies several nonlinear and linear contractions in the framework of a complete metric space. Jiddah et al. [12] extended the work of [16] in the set-up of G-metric space and established some fixed point results via (G-α-ϕ)-contraction. Going in the same direction, we observe that hybrid fixed point results via F-contractive-type mappings in G-metric space are not adequately studied. However, the work of [12] did not consider other conditions to study the existence of fixed point apart from the continuity of the mapping. Motivated by this, we introduce a new concept called admissible hybrid F-(G-α-ϕ)-contraction. With the help of new auxiliary functions, some fixed point results are discussed for this class of contractions in the set-up of a G-metric space. A comprehensive, non-trivial example is constructed to demonstrate the validity of our result and its improvement over previous findings. It is worthy of note that the key ideas established herein cannot be reduced to any existing result. With the help of some consequences presented, it has been established that the idea proposed herein is a generalization of some well-known fixed point results in the domain of F-contractive operators in G-metric space. Finally, one of our obtained corollaries is applied to prove the existence and uniqueness of a solution to a class of nonlinear integral equations.

The paper is settled in the following form: In Section 1, the introduction and overview of related literature are presented. The basic concepts needed in this work are compiled in Section 2. In Section 3, the principal results and some corollaries of the obtained fixed point results are discussed. With the aid of one of the obtained results herein, the existence and uniqueness of a solution to a nonlinear integral equation is explored in Section 4. In Section 5, deductions, recommendation and conclusion are given.

2. Preliminaries

In this section, we will present some basic notations and results that will be used subsequently. Throughout this paper, every set X is considered non-empty. We denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$, the set of real numbers, the set of non-negative real numbers and the set of natural numbers, respectively.

Definition 2.1. ([21]) Let X be a non-empty set and let $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying:

(G_1) $G(x, y, z) = 0$ if $x = y = z$;
(G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G_3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
(G_4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$ (symmetry in all three variables);
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Example 2.2. ([20]) Let (X, d) be a metric space, then (X, G_s) and (X, G_m) are G-metric space, where

$$G_m(x, y, z) = d(x, y) + d(y, z) + d(x, z), \quad \forall x, y, z \in X, \tag{2.1}$$

$$G_n(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \quad \forall x, y, z \in X. \tag{2.2}$$

Definition 2.3. ([20]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to x if

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq n_0$. We refer to x as the limit of the sequence $\{x_n\}$.

Proposition 2.4. ([20]) *Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) *$\{x_n\}$ is G-convergent to x.*
(ii) *$G(x, x_n, x_m) \longrightarrow 0$, as $n, m \to \infty$.*
(iii) *$G(x_n, x, x) \longrightarrow 0$, as $n \to \infty$.*
(iv) *$G(x_n, x_n, x) \longrightarrow 0$, as $n \to \infty$.*

Definition 2.5. ([20]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$G(x_n, x_m, x_l) < \epsilon$$

for all $l, n, m, \geq n_0$. That is, $G(x_n, x_m, x_l) \longrightarrow 0$, as $n, m, l \to \infty$.

Proposition 2.6. ([20]) *In a G-metric space (X, G), the following are equivalent:*

(i) *The sequence $\{x_n\}$ is G-Cauchy.*
(ii) *For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq n_0$.*

Definition 2.7. ([20]) Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \longrightarrow (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$ for all $x, y \in X$. The function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 2.8. ([20]) *Let (X, G) and (X', G') be two G-metric spaces. Then a function $f : (X, G) \longrightarrow (X', G')$ is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x. That is, whenever $\{x_n\}$ is G-convergent to x, $\{fx_n\}$ is G-convergent to fx.*

Definition 2.9. ([20]) A G-metric space (X, G) is called symmetric G-metric spaces if

$$G(x, x, y) = G(y, x, x), \quad \forall x, y \in X.$$

Definition 2.10. ([20]) A G-metric space (X, G) is said to be G-complete (or complete G-metric), if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Mustafa [19] proved the following result in the framework of G-metric space.

Theorem 2.11. ([19]) *Let (X, G) be a complete G-metric space and $T : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$G(Tx, Ty, Tz) \leq kG(x, y, z) \tag{2.3}$$

for all $x, y, z \in X$, where $0 \leq k < 1$. Then T has a unique fixed point (say u, that is, $Tu = u$), and T is G-continuous at u.

Following the direction of [29], the idea of F-contraction is defined as follows:

Definition 2.12. ([29]) Let Δ_f denote the family of functions $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following auxiliary conditions:

- $(F1)$ F is strictly increasing; that is, for all a, b $\in \mathbb{R}_+$, if $a < b$ then $F(a) < F(b)$;
- $(F2)$ for every sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
- $(F3)$ there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.13. ([29]) Let (X, d) be a metric space. A self-mapping T on X is called an F-contraction, if there exists $\tau > 0$ and $F \in \Delta_f$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \quad \Longrightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{2.4}$$

Remark 2.14. From (F1) and (2.4), it is clear that if T is an F-contraction, then $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$ such that $Tx \neq Ty$. That is, T is a contractive mapping and hence, every F-contraction is a continuous mapping.

Wardowski [29] presented a variant of the Banach fixed point theorem as follows:

Theorem 2.15. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be an F-contraction. Then, T has a unique fixed point $x_0 \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x_0.*

In line with [26], let Φ be the set of all functions $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that ϕ is a non-decreasing function with $\lim_{n\to+\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called a Φ-map.

Let $\phi \in \Phi$ be a Φ-map such that there exist $n_0 \in \mathbb{N}$, $k \in (0, 1)$ and a convergent series of non-negative terms $\sum_{n=1}^{\infty} v_n$ satisfying

$$\phi^{n+1}(t) \leq k\phi^n(t) + v_n$$

for $n \geq n_0$ and any $t > 0$. Then ϕ is called a (c)-comparison function [2].

Lemma 2.16. ([2]) *If $\phi \in \Phi$, then the following hold:*

(i) *$\{\phi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for $t \geq 0$;*
(ii) *$\phi(t) < t$ for all $t \in \mathbb{R}_+$;*
(iii) *ϕ is continuous;*
(iv) *$\phi(t) = 0$ if and only if $t = 0$;*
(v) *the series $\sum_{i=1}^{\infty} \phi^i(t)$ is convergent for $t \geq 0$.*

Popescu [23] presented the following definitions in the framework of metric spaces.

Definition 2.17. ([23]) Let $\alpha : X \times X \longrightarrow \mathbb{R}_+$ be a function. A mapping $T : X \longrightarrow X$ is said to be α-orbital admissible, if for all $x \in X$, $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.

Definition 2.18. ([23]) Let $\alpha : X \times X \longrightarrow \mathbb{R}_+$ be a function. A mapping $T : X \longrightarrow X$ is said to be triangular α-orbital admissible, if for all $x \in X$, T is α-orbital admissible, $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ implies $\alpha(x, Ty) \geq 1$.

The above definitions were modified and presented in the setting of G-metric space by Jiddah et al [12] as follows:

Definition 2.19. ([12]) Let $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function. A mapping $T : X \longrightarrow X$ is said to be (G$-\alpha$)-orbital admissible, if for all $x \in X$, $\alpha(x, Tx, T^2x) \geq 1$ implies $\alpha(Tx, T^2x, T^3x) \geq 1$.

Definition 2.20. ([12]) Let $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function. A mapping $T : X \longrightarrow X$ is said to be triangular (G$-\alpha$)-orbital admissible, if for all $x \in X$, T is (G$-\alpha$)-orbital admissible, $\alpha(x, y, Ty) \geq 1$ and $\alpha(y, Ty, T^2y) \geq 1$ implies $\alpha(x, Ty, T^2y) \geq 1$.

Lemma 2.21. ([12]) *Let $T : X \longrightarrow X$ be a triangular $(G\text{–}\alpha)$-orbital admissible mapping. If we can find $x_0 \in X$ such that $\alpha(x_0, Tx_0, T^2x_0) \geq 1$, then*

$$\alpha(x_n, x_m, x_l) \geq 1, \quad \forall\ n, m, l \in \mathbb{N}, \tag{2.5}$$

where the sequence $\{x_n\}_{n\in\mathbb{N}}$ is defined by $x_{n+1} = Tx_n,\ n \in \mathbb{N}$.

Definition 2.22. ([3]) Let $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a mapping. The set X is called regular with respect to α if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, we have $\alpha(x_n, x, x) \geq 1$ for all n.

Jiddah et al. [12] presented the following definition of admissible hybrid (G-α-ϕ)-contraction in G-metric space.

Definition 2.23. ([12]) Let (X, G) be a G-metric space. A mapping $T : X \longrightarrow X$ is called an admissible hybrid (G-α-ϕ)-contraction, if there exists $\phi \in \Phi$ and a mapping $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ such that

$$\alpha(x, y, Ty)G(Tx, Ty, T^2y) \leq \phi(M(x, y, Ty)) \tag{2.6}$$

for all $x, y \in X\backslash Fix(T)$, where

$$M(x,y,Ty)=\begin{cases} \Bigg[\lambda_1 G(x,y,Ty)^q + \lambda_2 G(x,Tx,T^2x)^q + \lambda_3 G(y,Ty,T^2y)^q \\ +\lambda_4\left(\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q + \lambda_5\left(\frac{G(x,y,Ty)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q\Bigg]^{\frac{1}{q}}, \\ \qquad \text{for} \quad \text{some } q>0,\ x,y\in X; \\ \\ G(x,y,Ty)^{\lambda_1}\cdot G(x,Tx,T^2x)^{\lambda_2}\cdot G(y,Ty,T^2y)^{\lambda_3} \\ \cdot\left[\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right]^{\lambda_4}\cdot\left[\frac{G(x,y,Ty)+G(y,Ty,T^2y)}{2}\right]^{\lambda_5}, \\ \qquad \text{for} \quad q=0,\ x,y\in X\backslash Fix(T) \end{cases} \tag{2.7}$$

$q \geq 0,\ \lambda_i \geq 0;\ i = 1, 2, \cdots, 5$ such that $\sum_{i=1}^{5} \lambda_i = 1$ and $Fix(T) = \{x \in X : Tx = x\}$.

3. Main results

In this section, the idea of admissible hybrid F-(G-α-ϕ)-contraction is introduced in the setting of G-metric spaces and the conditions for the existence of fixed points for such operators are investigated.

Definition 3.1. Let (X, G) be a G-metric space. A mapping $T : X \longrightarrow X$ is called an admissible hybrid F-(G-α-ϕ)-contraction if there exist $\tau > 0$, $\phi \in \Phi$, $F \in \Delta_f$ and a mapping $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ such that $G(Tx, Ty, T^2y) > 0$ implies

$$\tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2y)) \leq F(\phi(M(x, y, Ty))) \tag{3.1}$$

for all $x, y \in X \backslash Fix(T)$, where

$$M(x,y,Ty)=\begin{cases} \Big[\lambda_1 G(x,y,Ty)^q + \lambda_2 G(x,Tx,T^2x)^q + \lambda_3 G(y,Ty,T^2y)^q \\ +\lambda_4\left(\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q + \lambda_5\left(\frac{G(x,y,Ty)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q\Big]^{\frac{1}{q}}, \\ \qquad \text{for} \quad \text{some } q > 0,\ x, y \in X; \\ \\ G(x,y,Ty)^{\lambda_1} \cdot G(x,Tx,T^2x)^{\lambda_2} \cdot G(y,Ty,T^2y)^{\lambda_3} \\ \cdot \left[\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right]^{\lambda_4} \cdot \left[\frac{G(x,y,Ty)+G(y,Ty,T^2y)}{2}\right]^{\lambda_5}, \\ \qquad \text{for} \quad q = 0,\ x, y \in X\backslash Fix(T) \end{cases} \tag{3.2}$$

$q \geq 0,\ \lambda_i \geq 0;\ i = 1, 2, \cdots, 5$ such that $\sum_{i=1}^{5} \lambda_i = 1$ and $Fix(T) = \{x \in X : Tx = x\}$.

Example 3.2. Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be defined by $F(t) = \ln(t)$, $t > 0$. It is easy to see that $F \in \Delta_f$. Each mapping $T : X \longrightarrow X$ satisfying (3.1) is an admissible hybrid F-(G-α-ϕ)-contraction such that

$$\alpha(x, y, Ty)G(Tx, Ty, T^2y) \leq e^{-\tau}(\phi(M(x, y, Ty))) \tag{3.3}$$

for all $x, y \in X$. Note that for $x, y \in X$ such that $Tx = Ty = T^2y$, the inequality (3.1) is still valid. That is, T is an admissible hybrid F-(G-α-ϕ)-contraction.

Our first main result is presented as follows:

Theorem 3.3. *Let (X,G) be a complete G-metric space and let $T : X \longrightarrow X$ be an admissible hybrid F-(G-α-ϕ)-contraction. Assume further that:*

(i) *T is triangular (G-α)-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, T^2x_0) \geq 1$;*
(iii) *either T is continuous or;*
(iv) *T^3 is continuous and $\alpha(x, Tx, T^2x) \geq 1$ for any $x \in Fix(T^3)$.*

Then T has a fixed point in X.

Proof. By hypothesis (ii), we have $\alpha(x_0, Tx_0, T^2x_0) \geq 1$ for some $x_0 \in X$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Suppose that we can find some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. Then, x_{n_0} is a fixed point of T and hence, the proof. Assume on the contrary that $x_n \neq x_{n-1}$, for all $n \in \mathbb{N}$.

Since $\alpha(x_0, x_1, x_2) = \alpha(x_0, Tx_0, T^2x_0) \geq 1$ and T is triangular (G-α)-orbital admissible, then

$$\alpha(x_{n-1}, x_n, x_{n+1}) \geq 1, \quad \forall\ n = 0, 1, \cdots . \tag{3.4}$$

Given the fact that T is an admissible hybrid F-(G-α-ϕ)-contraction, then we have that for $0 < G(x_n, x_{n+1}, x_{n+2})$,

$$\tau + F(\alpha(x_{n-1}, x_n, Tx_n)G(Tx_{n-1}, Tx_n, T^2x_n)) \leq F(\phi(M(x_{n-1}, x_n, Tx_n))). \tag{3.5}$$

Together with (3.4) and (3.5), we have

$$\begin{aligned}
\tau + F(G(x_n, x_{n+1}, Tx_{n+1})) &= \tau + F(G(x_n, x_{n+1}, x_{n+2})) \\
&\leq \tau + F(\alpha(x_{n-1}, x_n, Tx_n)G(x_n, x_{n+1}, x_{n+2})) \\
&\leq F(\phi(M(x_{n-1}, x_n, x_{n+1}))).
\end{aligned}$$

On account of ($F1$) and (3.4), we get

$$\tau + F(G(x_n, x_{n+1}, x_{n+2})) \leq F(\phi(M(x_{n-1}, x_n, x_{n+1}))). \tag{3.6}$$

We now consider the following cases of (3.2).

Case 1**:** For $q > 0$, taking $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned} M(x_{n-1}, x_n, Tx_n) &= \Big[\lambda_1 G(x_{n-1}, x_n, Tx_n)^q + \lambda_2 G(x_{n-1}, Tx_{n-1}, T^2x_{n-1})^q \\ &\quad + \lambda_3 G(x_n, Tx_n, T^2x_n)^q \\ &\quad + \lambda_4 \Big(\frac{G(x_n, Tx_n, T^2x_n)(1 + G(x_{n-1}, Tx_{n-1}, T^2x_{n-1}))}{1 + G(x_{n-1}, x_n, Tx_n)}\Big)^q \\ &\quad + \lambda_5 \Big(\frac{G(x_{n-1}, x_n, Tx_n)(1 + G(x_{n-1}, Tx_{n-1}, T^2x_{n-1}))}{1+G(x_{n-1}, x_n, Tx_n)}\Big)^q\Big]^{\frac{1}{q}} \\ &= \Big[\lambda_1 G(x_{n-1}, x_n, x_{n+1})^q + \lambda_2 G(x_{n-1}, x_n, x_{n+1})^q \\ &\quad + \lambda_3 G(x_n, x_{n+1}, x_{n+2})^q \\ &\quad + \lambda_4 \left(\frac{G(x_n, x_{n+1}, x_{n+2})(1 + G(x_{n-1}, x_n, x_{n+1}))}{1 + G(x_{n-1}, x_n, x_{n+1})}\right)^q \\ &\quad + \lambda_5 \left(\frac{G(x_{n-1}, x_n, x_{n+1})(1 + G(x_{n-1}, x_n, x_{n+1}))}{1 + G(x_{n-1}, x_n, x_{n+1})}\right)^q\Big]^{\frac{1}{q}} \\ &= \Big[\lambda_1 G(x_{n-1}, x_n, x_{n+1})^q + \lambda_2 G(x_{n-1}, x_n, x_{n+1})^q \\ &\quad + \lambda_3 G(x_n, x_{n+1}, x_{n+2})^q \\ &\quad + \lambda_4 G(x_n, x_{n+1}, x_{n+2})^q + \lambda_5 G(x_{n-1}, x_n, x_{n+1})^q\Big]^{\frac{1}{q}} \\ &= \Big[(\lambda_1 + \lambda_2 + \lambda_5) G(x_{n-1}, x_n, x_{n+1})^q \\ &\quad + (\lambda_3 + \lambda_4) G(x_n, x_{n+1}, x_{n+2})^q\Big]^{\frac{1}{q}}. \end{aligned} \tag{3.7}$$

Suppose that

$$G(x_{n-1}, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}).$$

Then, (3.6) becomes

$$\begin{aligned} F(G(x_n, x_{n+1}, x_{n+2})) &\leq F(\phi(M(x_{n-1}, x_n, Tx_n))) - \tau \\ &= F(\phi\big[(\lambda_1 + \lambda_2 + \lambda_5) G(x_{n-1}, x_n, x_{n+1})^q \\ &\quad + (\lambda_3 + \lambda_4) G(x_n, x_{n+1}, x_{n+2})^q\big]^{\frac{1}{q}}) - \tau \end{aligned}$$

$$
\begin{aligned}
&\leq F(\phi\big[(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)G(x_n,x_{n+1},x_{n+2})^q\big]^{\frac{1}{q}})-\tau\\
&= F(\phi((\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{\frac{1}{q}}G(x_n,x_{n+1},x_{n+2})))-\tau\\
&\leq F(\phi(G(x_n,x_{n+1},x_{n+2})))-\tau\\
&< F(G(x_n,x_{n+1},x_{n+2})-\tau,
\end{aligned}
\tag{3.8}
$$

which is a contradiction. Therefore, for every $n \in \mathbb{N}$, we have

$$G(x_n,x_{n+1},x_{n+2}) < G(x_{n-1},x_n,x_{n+1}),$$

so that (3.6) becomes

$$
\begin{aligned}
F(G(x_n,x_{n+1},x_{n+2})) &\leq F(\phi\big[(\lambda_1+\lambda_2+\lambda_5)G(x_{n-1},x_n,x_{n+1})^q\\
&\qquad + (\lambda_3+\lambda_4)G(x_n,x_{n+1},x_{n+2})^q\big]^{\frac{1}{q}})-\tau\\
&\leq F(\phi\big[(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)G(x_{n-1},x_n,x_{n+1})^q\big]^{\frac{1}{q}})-\tau\\
&= F(\phi((\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{\frac{1}{q}}G(x_{n-1},x_n,x_{n+1})))-\tau\\
&\leq F(\phi(G(x_{n-1},x_n,x_{n+1})))-\tau.
\end{aligned}
\tag{3.9}
$$

By letting $\gamma_n = G(x_n,x_{n+1},x_{n+2})$, we deduce from (3.9) that

$$F(\gamma_n) \leq F(\phi(\gamma_{n-1}))-\tau \leq F(\phi^2(\gamma_{n-2}))-2\tau \leq \cdots \leq F(\phi^n(\gamma_0))-n\tau \tag{3.10}$$

for all $n \geq 1$ with $x_{n+1} \neq x_{n+2}$. Letting $n \to \infty$ in (3.10), yields

$$
\begin{aligned}
\lim_{n\to\infty} F(\gamma_n) &\leq \lim_{n\to\infty} F(\phi^n(\gamma_0)) - \lim_{n\to\infty} n\tau\\
&= F\lim_{n\to\infty}(\phi^n(\gamma_0)) - \lim_{n\to\infty} n\tau\\
&= -\infty.
\end{aligned}
$$

And by $(F2)$, we obtain

$$\lim_{n\to\infty} \gamma_n = 0. \tag{3.11}$$

On account of $(F3)$,

$$\lim_{n\to\infty} {\gamma_n}^k F(\gamma_n) = 0 \quad \text{for } k \in (0,1).$$

By (3.11), the following is true for all $n \geq 0$:

$$
\begin{aligned}
0 &\leq {\gamma_n}^k F(\gamma_n) - {\gamma_n}^k F(\phi^n(\gamma_0))\\
&\leq {\gamma_n}^k\big[F(\phi^n(\gamma_0)) - n\tau\big] - {\gamma_n}^k F(\phi^n(\gamma_0))\\
&= -{\gamma_n}^k n\tau\\
&\leq 0.
\end{aligned}
\tag{3.12}
$$

Letting $n \to \infty$ in (3.12),

$$\lim_{n\to\infty} n{\gamma_n}^k = 0. \tag{3.13}$$

From (3.13), we can find $n_1 \in \mathbb{N}$ such that $n{\gamma_n}^k \leq 1$ for all $n \geq n_1$. Thus, we have $\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_1$. For $m, n \in \mathbb{N}$ with $m > n \geq n_1$, we have

$$\begin{aligned} G(x_n, x_n, x_m) &\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) \\ &\quad + \cdots + G(x_{m-1}, x_{m-1}, x_m) \\ &= \gamma_n + \gamma_{n-1} + \gamma_{n-2} + \cdots + \gamma_{m-1} \\ &= \sum_{i=n}^{m-1} \gamma_i \leq \sum_{i=n}^{\infty} \gamma_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} . \end{aligned}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ converges, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is G-Cauchy in (X, G). From the completeness of (X, G), there exists $u \in X$ such that $\{x_n\}$ converges to u. That is, $\lim_{n\to\infty} G(x_n, x_n, u) = 0$.

We now show that u is a fixed point of T. By assumption (iii), we obtain

$$\begin{aligned} \lim_{n\to\infty} G(u, u, Tu) &= \lim_{n\to\infty} G(x_{n+1}, x_{n+1}, Tu) \\ &= \lim_{n\to\infty} G(Tx_n, Tx_n, Tu) \\ &= \lim_{n\to\infty} G(Tx_n, Tx_n, Tx_n) \\ &= 0. \end{aligned}$$

So we get $Tu = u$, that is, u is a fixed point of T.

Also, using assumption (iv), $T^3u = \lim_{n\to\infty} T^3x_n = u$. To illustrate that $Tu = u$, assume on the contrary that $Tu \neq u$. By (3.5), we obtain

$$\begin{aligned} \tau + F(G(u, Tu, T^2u)) &= \tau + F(G(T^3u, Tu, T^2u)) \\ &\leq \tau + F(\alpha(u, Tu, T^2u)G(Tu, T^2u, T^3u)) \\ &= \tau + F(\alpha(u, Tu, T^2u)G(Tu, T^2u, u)) \\ &\leq F(\phi(M(u, Tu, T^2u))) \\ &< F(M(u, Tu, T^2u)), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
M(u,Tu,T^2u) &= \Bigg[\lambda_1 G(u,Tu,T^2u)^q + \lambda_2 G(u,Tu,T^2u)^q + \lambda_3 G(Tu,T^2u,T^3u)^q \\
&\quad + \lambda_4\left(\frac{G(Tu,T^2u,T^3u)(1+G(u,Tu,T^2u))}{1+G(u,Tu,T^2u)}\right)^q \\
&\quad + \lambda_5\left(\frac{G(u,Tu,T^2u)(1+G(u,Tu,T^2u))}{1+G(u,Tu,T^2u)}\right)^q\Bigg]^{\frac{1}{q}} \\
&= \Bigg[\lambda_1 G(u,Tu,T^2u)^q + \lambda_2 G(u,Tu,T^2u)^q + \lambda_3 G(Tu,T^2u,T^3u)^q \\
&\quad + \lambda_4 G(Tu,T^2u,T^3u)^q + \lambda_5 G(u,Tu,T^2u)^q\Bigg]^{\frac{1}{q}} \\
&= \Bigg[\lambda_1 G(u,Tu,T^2u)^q + \lambda_2 G(u,Tu,T^2u)^q + \lambda_3 G(Tu,T^2u,u)^q \\
&\quad + \lambda_4 G(u,Tu,T^2u)^q + \lambda_5 G(u,Tu,T^2u)^q\Bigg]^{\frac{1}{q}} \\
&= \Bigg[(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5) G(u,Tu,T^2u)^q\Bigg]^{\frac{1}{q}} \\
&= (\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{\frac{1}{q}} G(u,Tu,T^2u) \\
&= G(u,Tu,T^2u).
\end{aligned}$$

Hence, (3.14) becomes

$$\tau + F(G(u,Tu,T^2u)) < F(G(u,Tu,T^2u)),$$

which is a contradiction. Hence, $Tu = u$.

Case 2: For $q = 0$, we have

$$\begin{aligned}
M(x_{n-1},x_n,Tx_n) &= G(x_{n-1},x_n,Tx_n)^{\lambda_1} \cdot G(x_{n-1},Tx_{n-1},T^2x_{n-1})^{\lambda_2} \\
&\quad \cdot G(x_n,Tx_n,T^2x_n)^{\lambda_3} \\
&\quad \cdot \left[\frac{G(x_n,Tx_n,T^2x_n)(1+G(x_{n-1},Tx_{n-1},T^2x_{n-1}))}{1+G(x_{n-1},x_n,Tx_n)}\right]^{\lambda_4} \\
&\quad \cdot \left[\frac{G(x_{n-1},x_n,Tx_n)+G(x_n,Tx_n,T^2x_n)}{2}\right]^{\lambda_5}
\end{aligned}$$

$$
\begin{aligned}
&= G(x_{n-1}, x_n, x_{n+1})^{\lambda_1} \cdot G(x_{n-1}, x_n, x_{n+1})^{\lambda_2} \cdot G(x_n, x_{n+1}, x_{n+2})^{\lambda_3} \\
&\quad \cdot G(x_n, x_{n+1}, x_{n+2})^{\lambda_4} \cdot \left[\frac{G(x_{n-1}, x_n, x_{n+1}) + G(x_n, x_{n+1}, x_{n+2})}{2}\right]^{\lambda_5} \\
&= [G(x_{n-1}, x_n, x_{n+1})]^{(\lambda_1+\lambda_2)} \cdot [G(x_n, x_{n+1}, x_{n+2})]^{(\lambda_3+\lambda_4)} \\
&\quad \cdot \left[\frac{G(x_{n-1}, x_n, x_{n+1}) + G(x_n, x_{n+1}, x_{n+2})}{2}\right]^{\lambda_5}.
\end{aligned}
$$

If $G(x_{n-1}, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2})$, then

$$
\begin{aligned}
M(x_{n-1}, x_n, Tx_n) &= [G(x_n, x_{n+1}, x_{n+2})]^{(\lambda_1+\lambda_2)} \cdot [G(x_n, x_{n+1}, x_{n+2})]^{(\lambda_3+\lambda_4)} \\
&\quad \cdot [G(x_n, x_{n+1}, x_{n+2})]^{\lambda_5} \\
&= [G(x_n, x_{n+1}, x_{n+2})]^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \\
&= G(x_n, x_{n+1}, x_{n+2}).
\end{aligned}
$$

Hence, (3.6) becomes

$$
\begin{aligned}
F(G(Tx_{n-1}, Tx_n, T^2x_n)) &\leq F(\phi(M(x_{n-1}, x_n, Tx_n))) - \tau \\
&\leq F(\phi[G(x_n, x_{n+1}, x_{n+2})^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)}]) - \tau \\
&= F(\phi[G(x_n, x_{n+1}, x_{n+2})]) - \tau \\
&< F(G(x_n, x_{n+1}, x_{n+2})) - \tau.
\end{aligned}
$$

That is,

$$
F(G(x_n, x_{n+1}, x_{n+2})) < F(G(x_n, x_{n+1}, x_{n+2})) - \tau,
$$

which is a contradiction. Hence, we have

$$
G(x_n, x_{n+1}, x_{n+2}) < G(x_{n-1}, x_n, x_{n+1}), \ \forall n.
$$

Therefore, by (3.6) we have

$$
\begin{aligned}
F(G(x_n, x_{n+1}, x_{n+2})) &< F(\phi(G(x_{n-1}, x_n, x_{n+1}))) - \tau \\
&< F(\phi^2(G(x_{n-2}, x_{n-1}, x_n))) - 2\tau \\
&\vdots \\
&< F(\phi^n(G(x_0, x_1, x_2))) - n\tau.
\end{aligned}
$$

By similar argument as in the case of $q > 0$, we can show that $\{x_n\}_{n\in\mathbb{N}}$ in (X, G) is G-Cauchy and therefore (X, G) being complete, there exists a point $u \in X$ such that $\lim_{n\to\infty} x_n = u$. To see that u is a fixed point of T, under the hypothesis that T is continuous and by the uniqueness of limit, we have $Tu = u$. That is, u is a fixed point of T.

In a similar manner, if T^3 is continuous as in Case 1, we have $T^3u = u$. Suppose on the contrary that $Tu \neq u$. Then

$$\begin{aligned}\tau + F(G(u,Tu,T^2u)) &\leq \tau + F(\alpha(u,Tu,T^2u)G(Tu,T^2u,T^3u))\\ &= \tau + F(\alpha(u,Tu,T^2u)G(u,Tu,T^2u))\\ &\leq F(\phi(M(u,Tu,T^2u)))\\ &< F(M(u,Tu,T^2u)),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}M(u,Tu,T^2u) &= G(u,Tu,T^2u)^{\lambda_1}\cdot G(u,Tu,T^2u)^{\lambda_2}\cdot G(Tu,T^2u,T^3u)^{\lambda_3}\\ &\quad\cdot\left[\frac{G(Tu,T^2u,T^3u)(1+G(u,Tu,T^2u))}{1+G(u,Tu,T^2u)}\right]^{\lambda_4}\\ &\quad\cdot\left[\frac{G(u,Tu,T^2u)+G(Tu,T^2u,T^3u)}{2}\right]^{\lambda_5}\\ &= G(u,Tu,T^2u)^{\lambda_1}\cdot G(u,Tu,T^2)^{\lambda_2}\cdot G(u,Tu,T^2u)^{\lambda_3}\\ &\quad\cdot G(u,Tu,T^2u)^{\lambda_4}\cdot G(u,Tu,T^2u)^{\lambda_5}\\ &= G(u,Tu,T^2u)^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)}\\ &= G(u,Tu,T^2u).\end{aligned}$$

Hence, (3.15) becomes

$$\tau + F(G(u,Tu,T^2u)) \leq F(G(u,Tu,T^2u)),$$

which is a contradiction. Therefore, $Tu = u$. □

Theorem 3.4. *If in Theorem* 3.3*, in the case of* $q > 0$*, we assume an additional condition that* $\alpha(x,y,Ty) \geq 1$ *for all* $x,y \in Fix(T)$*, then the fixed point of* T *is unique.*

Proof. Let u,w be two fixed points of T such that $u \neq w$. Taking into account the additional hypothesis and by (3.5), we have

$$G(u,w,Tw) \leq \alpha(u,w,Tw)G(Tu,Tw,T^2w).$$

This yields $F(G(u,w,Tw)) \leq F(\alpha(u,w,Tw)G(Tu,Tw,T^2w))$, and so,

$$\begin{aligned}\tau + F(G(u,w,Tw)) &\leq \tau + F(\alpha(u,w,Tw)G(Tu,Tw,T^2w))\\ &\leq F(\phi(M(u,w,Tw)))\\ &< F(M(u,w,Tw)),\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
M(u,w,Tw) &= \Bigg[\lambda_1 G(u,w,Tw)^q + \lambda_2 G(u,Tu,T^2u)^q + \lambda_3 G(w,Tw,T^2w)^q \\
&\quad + \lambda_4\left(\frac{G(w,Tw,T^2w)(1+G(u,Tu,T^2u))}{1+G(u,w,Tw)}\right)^q \\
&\quad + \lambda_5\left(\frac{G(u,w,Tw)(1+G(u,Tu,T^2u))}{1+G(u,w,Tw)}\right)^q\Bigg]^{\frac{1}{q}} \\
&= \left[\lambda_1 G(u,w,Tw)^q + \lambda_5\left(\frac{G(u,w,Tw)}{1+G(u,w,Tw)}\right)^q\right]^{\frac{1}{q}} \\
&\leq \left[\lambda_1 G(u,w,Tw)^q + \lambda_5 G(u,w,Tw)^q\right]^{\frac{1}{q}} \\
&= \left[(\lambda_1+\lambda_5)G(u,w,Tw)^q\right]^{\frac{1}{q}} \\
&= (\lambda_1+\lambda_5)^{\frac{1}{q}} G(u,w,Tw) \\
&\leq G(u,w,Tw).
\end{aligned}$$

Therefore, (3.16) becomes

$$\tau + F(G(u,w,Tw)) < F(G(u,w,Tw)),$$

which is a contradiction. Thus, $u = w$ and so it follows that T has exactly unique fixed point. $\square$

The following example is constructed to verify the hypotheses of Theorems (3.3) and (3.4).

Example 3.5. Let $X = [0,\infty)$ and $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be defined by $G(x,y,z) = |x-y| + |x-z| + |y-z|$ for all $x,y \in X$. Then (X,G) is a complete G-metric space. Take $\tau > 0$, and consider the mapping $T : X \longrightarrow X$ defined by

$$Tx = \begin{cases} \frac{3}{7}xe^{-\tau}, & \text{if} \quad x \in [0,1]; \\ \frac{3}{7}e^{-\tau}, & \text{if} \quad x > 1. \end{cases}$$

Define the mapping $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ by

$$\alpha(x,y,Ty) = \begin{cases} 1, & \text{if} \quad x,y \in [0,1]; \\ 0, & \text{otherwise,} \end{cases}$$

for all $x,y \in X$. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined by $\phi(t) = \frac{3}{7}t$ for all $t > 0$. Clearly, ϕ is a (c)-comparison function. Let $F(t) = \ln(t^2 + t)$, $t > 0$ and therefore $F \in \Delta_f$. It is obvious that T is triangular (G$-\alpha$)-orbital admissible and there exists $x_0 = 0 \in X$ such that $\alpha(0, T0, T^2 0) = \alpha(0, 0, 0) \geq 1$. Also, T is continuous for all $x \in X$ and likewise, T^3 is continuous for any $x \in Fix(T^3)$.

In the case where x or $y \notin [0, 1]$, then $\alpha(x, y, Ty) = 0$ and $G(Tx, Ty, T^2 y) = 0$ for all $x > 1$. Hence, the inequality (3.1) holds for all $x, y \notin [0, 1]$.

Now for all $x, y \in [0, 1]$, take $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. To show that the mapping T is an admissible hybrid F-(G-α-ϕ)-contraction, we examine the following two cases:

Case 1: For $q > 0$, consider $q = 1$. Then,

$$\begin{aligned} G(Tx, Ty, T^2 y) &= |Tx - Ty| + \left|Tx - T^2 y\right| + \left|Ty - T^2 y\right| \\ &= \left|\frac{3}{7}xe^{-\tau} - \frac{3}{7}ye^{-\tau}\right| + \left|\frac{3}{7}xe^{-\tau} - \frac{9}{49}ye^{-2\tau}\right| + \left|\frac{3}{7}ye^{-\tau} - \frac{9}{49}ye^{-2\tau}\right| \\ &= \frac{3}{7}e^{-\tau}\left[\left|x - y\right| + \left|x - \frac{3}{7}ye^{-\tau}\right| + \left|y - \frac{3}{7}ye^{-\tau}\right|\right] \\ &= e^{-\tau}\phi(G(x, y, Ty)) \\ &\leq e^{-\tau}\phi(M(x, y, Ty)). \end{aligned} \tag{3.17}$$

Now,

$$G(Tx, Ty, T^2 y)\left[G(Tx, Ty, T^2 y) + 1\right] = \left[G(Tx, Ty, T^2 y)\right]^2 + G(Tx, Ty, T^2 y).$$

This implies that

$$\begin{aligned} F(\alpha(x, y, Ty)G(Tx, Ty, T^2 y)) &= F(G(Tx, Ty, T^2 y)) \\ &= \ln\left[(G(Tx, Ty, T^2 y))^2 + G(Tx, Ty, T^2 y)\right] \\ &\leq \ln\left[(e^{-\tau}(\phi(M(x, y, Ty)))^2 + \phi(M(x, y, Ty))\right] \\ &\leq \ln\left[e^{-\tau}((\phi(M(x, y, Ty)))^2 + \phi(M(x, y, Ty)))\right] \\ &= \ln e^{-\tau} + \ln\left[(\phi(M(x, y, Ty)))^2 + \phi(M(x, y, Ty))\right] \\ &= -\tau + F(\phi(M(x, y, Ty))). \end{aligned}$$

Therefore, we have

$$\tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2 y)) \leq F(\phi(M(x, y, Ty))).$$

Case 2: Similarly, for $q = 0$, we obtain

$$G(Tx, Ty, T^2 y) \leq e^{-\tau}\phi(M(x, y, Ty)). \tag{3.18}$$

In similar manner as in Case 1, the inequality (3.18) gives

$$\tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2 y)) \leq F(\phi(M(x, y, Ty))).$$

In the following Figure 1, we demonstrate the authenticity of contractive inequality (3.1) using Example 3.5.

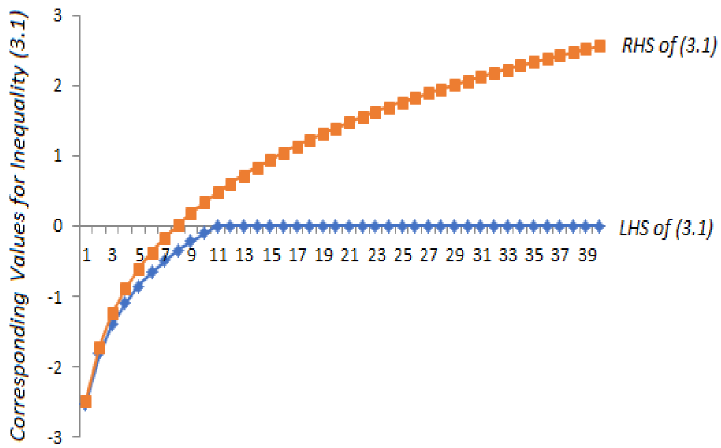

FIGURE 1. Illustration of contractive inequality (3.1) using Example 3.5

Figure 1 above illustrates that the right-hand side (RHS) of contractive inequality (3.1) predominates the left-hand side (LHS) as defined in Example 3.5. Hence, all the assumptions of Theorems 3.3 and 3.4 are satisfied. Consequently, we see that $x = 0$ is the unique fixed point of T.

We now show that our principal idea in this paper refines the corresponding ones in [12]. However, it is easy to verify that the main result of Jiddah et al. [12] is not applicable to this example. In fact, suppose that the mapping T is an admissible hybrid (G-α-ϕ)-contraction; that is, for all $x, y \in X/Fix(T)$,

$$\alpha(x, y, Ty)G(Tx, Ty, T^2y) \leq \phi(M(x, y, Ty)), \tag{3.19}$$

where

$$M(x,y,Ty) = \begin{cases} \Bigg[\lambda_1 G(x,y,Ty)^q + \lambda_2 G(x,Tx,T^2x)^q + \lambda_3 G(y,Ty,T^2y)^q \\ +\lambda_4 \left(\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q + \lambda_5 \left(\frac{G(x,y,Ty)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q\Bigg]^{\frac{1}{q}}, \\ \qquad \text{for} \quad q>0,\ x,y \in X; \\ \\ G(x,y,Ty)^{\lambda_1} \cdot G(x,Tx,T^2x)^{\lambda_2} \cdot G(y,Ty,T^2y)^{\lambda_3} \\ \cdot \left[\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right]^{\lambda_4} \cdot \left[\frac{G(x,y,Ty)+G(y,Ty,T^2y)}{2}\right]^{\lambda_5}, \\ \qquad \text{for} \quad q=0,\ x,y \in X \setminus Fix(T) \end{cases}$$

$q \geq 0,\ \lambda_i \geq 0;\ i = 1, 2, \cdots, 5$ such that $\sum_{i=1}^{5} \lambda_i = 1$ and $Fix(T) = \{x \in X : Tx = x\}$. Then, for the chosen parameters $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $q = 1$, take $x = \frac{3}{7}e^{-\tau}$ and $y = \frac{5}{7}e^{-\tau}$ for $\tau > 0$. Clearly, $x, y \in [0, 1]$. Then, by direct calculation, we have

$$\begin{aligned} G(Tx, Ty, T^2y) &= |Tx - Ty| + \left|Tx - T^2y\right| + \left|Ty - T^2y\right| \\ &= \left|\frac{9}{49}e^{-2\tau} - \frac{15}{49}e^{-2\tau}\right| \\ &\quad + \left|\frac{9}{49}e^{-2\tau} - \frac{45}{343}e^{-3\tau}\right| + \left|\frac{15}{49}e^{-2\tau} - \frac{45}{343}e^{-3\tau}\right| \\ &= \frac{30}{49}e^{-2\tau} - \frac{90}{343}e^{-3\tau} \\ &= \frac{210e^{-2\tau} - 90e^{-3\tau}}{343}. \end{aligned} \tag{3.20}$$

Similarly,

$$\begin{aligned} M(x, y, Ty) &= G(x, y, Ty) \\ &= |x - y| + |x - Ty| + |y - Ty| \\ &= \left|\frac{3}{7}e^{-\tau} - \frac{5}{7}e^{-\tau}\right| + \left|\frac{3}{7}e^{-\tau} - \frac{15}{49}e^{-2\tau}\right| + \left|\frac{5}{7}e^{-\tau} - \frac{15}{49}e^{-2\tau}\right| \\ &= \frac{10}{7}e^{-\tau} - \frac{30}{49}e^{-2\tau} \\ &= \frac{70e^{-\tau} - 30e^{-2\tau}}{49}, \end{aligned} \tag{3.21}$$

$$\phi(M(x, y, Ty)) = \frac{30}{343}\left[7e^{-\tau} - 3e^{-2\tau}\right]. \tag{3.22}$$

By (3.19),

$$\frac{\alpha(x, y, Ty)G(Tx, Ty, T^2y)}{\phi(M(x, y, Ty))} \leq 1.$$

Since $x, y \in [0, 1]$, then $\alpha(x, y, Ty) = 1$. Hence, from (3.20), (3.21) and (3.22), we have

$$\frac{210e^{-2\tau} - 90e^{-3\tau}}{30\left[7e^{-\tau} - 3e^{-2\tau}\right]} \leq 1. \tag{3.23}$$

Letting $\tau \longrightarrow \infty$ in (3.23) gives $\infty \leq 1$, which is a contradiction.

The following are some immediate consequences of our results.

Corollary 3.6. *Given a complete G-metric space (X, G) and a continuous mapping $T : X \longrightarrow X$. Suppose that there exists $\tau > 0$ such that*

$$G(Tx, Ty, T^2y) > 0 \implies \tau + F(G(Tx, Ty, T^2y)) \leq F(\phi(M(x, y, Ty)))$$

for all $x, y \in X$, where F satisfies $(F1) - (F3)$, $\phi \in \Phi$ and $M(x, y, Ty)$ is as given in (3.2). Then T possesses a unique fixed point.

Proof. It is sufficient to take $\alpha(x, y, Ty) = 1$ in Theorem 3.4. □

Corollary 3.7. *Let (X, G) be a complete G-metric space and $T : X \longrightarrow X$ be a continuous mapping satisfying the following*

$$G(Tx, Ty, T^2y) > 0 \implies \tau + F(G(Tx, Ty, T^2y)) \leq F(\eta(M(x, y, Ty)))$$

for all $x, y \in X$, $\tau > 0$ where $F \in \Delta_f, \phi \in \Phi$ and $\eta \in (0, 1)$. Then the fixed point of T in X is unique.

Proof. It follows from Corollary 3.6 with $\phi(t) = \eta t$ for all $t \geq 0$. □

Corollary 3.8. *Given a mapping $T : X \longrightarrow X$ defined on a complete G-metric space (X, G) and suppose there exists $\tau > 0$ such that for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) > 0 \implies \tau + F(G(Tx, Ty, Tz)) \leq F(\phi(G(x, y, z))),$$

where $F \in \Delta_f$ and $\phi \in \Phi$. Then T has a unique fixed point in X.

Proof. Consider Definition 3.1 and let $\alpha(x, y, Ty) = 1$ for all $x, y \in X$. Take $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $Ty = z$. We have

$$M(x, y, z) = G(x, y, z)$$

for all $x, y, z \in X$ and $q \geq 0$. The proof follows from Theorem 3.4. □

Corollary 3.9. (Jiddah et al. [12], Theorem 3.3) *Let (X, G) be a complete G-metric space and let $T : X \longrightarrow X$ be a continuous mapping satisfying the following condition*

$$\alpha(x, y, Ty)G(Tx, Ty, T^2y) \leq \phi(M(x, y, Ty)),$$

where $\phi \in \Phi$. Then T has a unique fixed point in X.

Proof. It is enough to take $F(t) = \ln(t)$, $t > 0$ in Theorem 3.4. □

Corollary 3.10. (Jiddah et al. [12], Theorem 3.3) *Given a mapping $T : X \longrightarrow X$ defined on a complete G-metric space (X, G) satisfying the following constraint*

$$\alpha(x, y, z)G(Tx, Ty, Tz) \leq \phi(G(x, y, z))$$

for all $x, y, z \in X$, where $\phi \in \Phi$. Then T possesses a unique fixed point in X.

Proof. Take $F(t) = \ln(t), t > 0$ in Corollary 3.8. □

Definition 3.11. ([2]) Let $T : X \longrightarrow X$ and $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be two mappings. Then T is said to be α-admissible, if for all $x, y, z \in X$,

$$\alpha(x, y, z) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty, Tz) \geq 1.$$

Definition 3.12. Let (X, G) be a G-metric space and $T : X \longrightarrow X$ be a given mapping. Then T is said to be an F-(G-α-ϕ)-contraction of type I, if there exist functions $\alpha : X \times X \times X \longrightarrow [0, \infty)$, $F \in \Delta_f$ and $\phi \in \Phi$ such that for all $x, y, z \in X$,

$$G(Tx, Ty, Tz) > 0 \implies \tau + F(\alpha(x, y, z)G(Tx, Ty, Tz)) \leq F(\phi(G(x, y, z))).$$

Note that if $F(t) = \ln(t)$, $t > 0$, then Definition 3.12 coincides with (G-α-ϕ)-contraction mapping of type I in the sense of Jiddah et al. [12].

Corollary 3.13. *Let (X, G) be a complete G-metric space. Suppose that $T : X \longrightarrow X$ is an F-(G-α-ϕ)-contraction of type I satisfying the following conditions*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, T^2x_0) \geq 1$;*
(iii) *T is G-continuous.*

Then T possesses a fixed point.

Proof. Consider Definition 3.1 and let $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a given mapping. Suppose that $T : X \longrightarrow X$ is an α-admissible mapping and take $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Then T is an F-(G-α-ϕ)-contraction of type I and so from Corollary 3.8, for all $x, y, z \in X$, $F \in \Delta_f$ and $\phi \in \Phi$ and the proof follows. □

4. Applications to an integral equation

The importance of fixed point theory to the solution of differential and integral equations can be considered as having significant value given that practically almost all real-life problems can be transformed into differential and integral equations. Huang et al. [10] investigated the conditions for the existence of a solution to a class of differential equations and whether such solution is unique using their obtained main result. In this section, we present an application to an integral equation using one of our obtained results.

Let X be a Banach space, μ an open set of $\mathbb{R} \times \mathbb{R} \times X$, $\mu_o = (t_0, s_0, x_0) \in \mu$, $f : \mu \to X$ a continuous function. We need to obtain a closed interval I such that $t_0 \in I$ and a differentiable function $x : I \to X$ satisfying

$$\begin{cases} x'(t) = f(t, s, x(t)), & t \in I; \\ x(t_0) = x_0. \end{cases} \tag{4.1}$$

It is obvious that (4.1) satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(t, s, x(s))ds, \qquad t \in I.$$

Theorem 4.1. *Suppose that the following conditions are satisfied:*

(i) *there exists a function* $\zeta(t) \in L^1(t_0 - \nu, t_0 + \nu) \cap \mathbb{R}_+$ *for some* $\nu > 0$ *such that*

$$\begin{aligned} &\|f(t,s,x) - f(t,s,y)\|_X + \|f(t,s,x) - f(t,s,z)\|_X \\ &\quad + \|f(t,s,y) - f(t,s,z)\|_X \\ &\leq \zeta(t)\Big[\|x - y\|_X + \|x - z\|_X + \|y - z\|_X\Big] \end{aligned}$$

holds for all $(t, s, x), (t, s, y), (t, s, z) \in \mu$, *where* $\|.\|_X$ *is a norm defined on* X;

(ii) *there exists a constant* $\delta > 0$ *and a* G*-closed ball* $\overline{B_G}(\mu_0, s)$ *of* μ *such that* $\|f(t, s, x)\|_X \leq \delta$ *for any* $(t, s, x) \in \overline{B_G}(\mu_0, s)$.

Then, there exists $\tau_0 > 0$ *such that for each* $\tau < \tau_0$, (4.1) *has a unique solution* $x \in C^1(I_\tau, X)$, *where* $I_\tau = [t_0 - \tau, t_0 + \tau]$.

Proof. Take $r = \min\{\nu, s\}, \quad \tau_0 = \min\{r, \frac{r}{\delta}\}$. Let $\tau < \tau_0$ and ϑ be the G-closed ball in X. Then, ϑ endowed with the Tehebyshev norm is a complete G-metric space. By virtue of $\tau < r$, if $y, z \in \vartheta$, then

$$(t, s, y(t)), (t, s, z(t)) \in \overline{B_G}(\mu_0, r) \subset \mu$$

for all $t \in I_\tau$. Therefore, for $y, z \in \vartheta$, define

$$\begin{aligned} Ty(t) &= x_0 + \int_{t_0}^{t} f(t, s, y(s))ds, \\ Tz(t) &= x_0 + \int_{t_0}^{t} f(t, s, z(s))ds, \qquad t \in I \end{aligned}$$

and $F(t) = \ln(t), \quad t \in (0, \infty)$.

We can show that

$$\ln\left(\frac{3}{2}\right) + F(\alpha(x,y,z)G(T^{n+1}x, T^{n+1}y, T^{n+1}z)) \leq F(\phi(M(x,y,z))) \tag{4.2}$$

for any $n \in \mathbb{N}$, where

$$M(x,y,z) = \frac{3}{(n+1)!}\|\zeta\|^{n+1}_{L^1(I_\tau)} G(x,y,z),$$

and $\phi(t) = \frac{t}{2}$ for all $t > 0$. Clearly, (4.2) is equivalent to

$$\begin{aligned} \alpha(x,y,z)G(T^{n+1}x, T^{n+1}y, T^{n+1}z) &\leq \frac{2}{3}(\phi(M(x,y,z))) \\ &= \frac{1}{(n+1)!}\|\zeta\|^{n+1}_{L^1(I_\tau)} G(x,y,z). \end{aligned}$$

Taking $\alpha(x,y,z) = 1$, we have

$$G(T^{n+1}x, T^{n+1}y, T^{n+1}z) \leq \frac{1}{(n+1)!}\|\zeta\|^{n+1}_{L^1(I_\tau)} G(x,y,z),$$

that is,

$$\begin{aligned} &\|T^{n+1}x - T^{n+1}y\| + \|T^{n+1}x - T^{n+1}z\| + \|T^{n+1}y - T^{n+1}z\| \\ &\leq \frac{1}{(n+1)!}\|\zeta\|^{n+1}_{L^1(I_\tau)} \Big[\|x-y\| + \|x-z\| + \|y-z\|\Big]. \end{aligned} \tag{4.3}$$

Note that

$$\begin{aligned} \sup_{t\in I_\tau}\|Ty(t) - x_0\|_X &\leq \sup_{t\in I_\tau}\left|\int_{t_0}^{t}\|f(t,s,y(s))\|_X ds\right| \\ &\leq \delta\tau \\ &\leq r \end{aligned}$$

and

$$\begin{aligned} \sup_{t\in I_\tau}\|Tz(t) - x_0\|_X &\leq \sup_{t\in I_\tau}\left|\int_{t_0}^{t}\|f(t,s,z(s))\|_X ds\right| \\ &\leq \delta\tau \\ &\leq r. \end{aligned}$$

Hence, T maps ϑ into ϑ.

To complete the proof of (4.3), we need to prove by induction on $n+1$, for every $t \in I_\tau$,

$$\begin{aligned} &\|T^{n+1}x(t) - T^{n+1}y(t)\|_X + \|T^{n+1}x(t) - T^{n+1}z(t)\| \\ &\quad + \|T^{n+1}y(t) - T^{n+1}z(t)\| \\ &\leq \frac{1}{(n+1)!}\left(\int_{t_0}^{t}\zeta(s)ds\right)^{n+1}\Big[\|x-y\| + \|x-z\| + \|y-z\|\Big]. \end{aligned} \tag{4.4}$$

For $n = 1$, it is easy to see that (4.4) holds. Suppose that (4.4) is true for $n,\ n \geq 2$. Then, taking $t > t_0$ (note that it is similar for $t < t_0$), we have

$$\begin{aligned}
&\left\|T^{n+1}x(t) - T^{n+1}y(t)\right\|_X + \left\|T^{n+1}x(t) - T^{n+1}z(t)\right\|_X \\
&\quad + \left\|T^{n+1}y(t) - T^{n+1}z(t)\right\|_X \\
&\quad = \left\|T(T^n x(t)) - T(T^n y(t))\right\|_X + \left\|T(T^n x(t)) - T(T^n z(t))\right\|_X \\
&\quad + \left\|T(T^n y(t)) - T(T^n z(t))\right\|_X \\
&\quad \leq \int_{t_0}^{t} \Big[\left\|f(t,s,T^n x(s)) - f(t,s,T^n y(s))\right\|_X \\
&\quad + \left\|f(t,s,T^n x(s)) - f(t,s,T^n z(s))\right\|_X \\
&\quad + \left\|f(t,s,T^n y(s)) - f(t,s,T^n z(s))\right\|_X \Big] ds \\
&\quad \leq \int_{t_0}^{t} \zeta(s) \Big[\left\|T^n x(s) - T^n y(s)\right\|_X + \left\|T^n x(s) - T^n z(s)\right\|_X \\
&\quad + \left\|T^n y(s) - T^n z(s)\right\|_X \Big] ds \\
&\quad \leq \frac{1}{n!} \left[\int_{t_0}^{t} \zeta(s) \left(\int_{t_0}^{s} \zeta(w) dw \right)^n ds \left(\|x-y\| + \|x-z\| + \|y-z\| \right) \right. \\
&\quad = \frac{1}{(n+1)!} \left(\int_{t_0}^{t} \zeta(s) ds \right)^{n+1} \Big[\|x-y\| + \|x-z\| + \|y-z\| \Big].
\end{aligned}$$

(4.4) leads to (4.3), where

$$\sup_{t \in I_\tau} \left(\int_{t_0}^{t} \zeta(s) ds \right)^{n+1} = \|\zeta\|^{n+1}, \quad \forall n.$$

Hence, all the assumptions of Corollary 3.8 are satisfied. Hence, T has a fixed point which corresponds to the solution of (4.1). □

Remark 4.2.

(i) We can obtain further special cases of Theorems 3.3 and 3.4 by fixing the parameters $\lambda_i (i = 1, 2, \cdots, 5)$ and q.

(ii) None of the proposed results in this work can be written in the form of $G(x, y, y)$ or $G(x, x, y)$. Hence, they cannot be inferred from their equivalents in metric space.

5. Conclusion

An intriguing generalization of the Banach contraction principle regarding the existence of fixed points in complete metric space was presented by Wardowski [29]. In this paper, some new fixed point theorems were established in the framework of complete G-metric space via a new type of contractive mapping called admissible hybrid F-(G-α-ϕ)-contraction. Some immediate consequences of the principal ideas were presented. An example which support the assumptions and effectiveness of the proposed results was constructed. It is also observed that the established result is an extension of F-contraction in metric spaces and some related findings in generalized metric spaces. Hence, the derived fixed point results cannot be reduced to their corresponding ones in the literature. From the perspective of application, one of the obtained corollaries was applied to guarantee the conditions for the existence of solutions to a nonlinear integral equation. This work is limited in scope by the fact that the mathematical formulation, analysis and results presented are purely abstract. The application to the integral equation has been developed analytically and conclusion is deduced based on the theoretical formulations of our theorems. The results obtained herein can be studied and advanced via other contractions and it will be interesting to apply these concepts in the setting of various spaces and the concerned mapping can also be extended to set-valued mappings.

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