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NEW TYPE OF GENERALIZED DIFFERENCE SEQUENCES OF FUZZY NUMBERS

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Abstract. In this paper, we introduce the notion of generalized difference sequence space $\triangle_m^n(u)$ and study $l_\infty^F(\triangle_m^n, u)$, $c^F(\triangle_m^n, u)$ and $c_0^F(\triangle_m^n, u)$. We give some of their properties like completeness, solidity, symmetricity.

1. Preliminaries, Background and Notation

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers respectively. Let ω denote the space of all sequences (real or complex), l_∞ and c respectively, denotes the space of all bounded sequences, the space of convergent sequences.

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [29] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [18] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Matloka [18] also has shown that every convergent sequence of fuzzy numbers is bounded.

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Later on sequences of fuzzy numbers have been discussed by many others(see, [1]-[16], [19]-[28]).

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on the real line $\mathbb{R}$. For X, $Y \in D$ we define

$$d(X : Y) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where $X = [a_1, a_2]$, $Y = [b_1, b_2]$. It is known that (D, d) is a complete metric space.

Let $I = [0, 1]$. A fuzzy real number X is a fuzzy set on $\mathbb{R}$ and is a mapping $X : \mathbb{R} \to I$ associating each real number t with its grade membership $X(t)$. A fuzzy real number X is called *convex* if

$$X(t) \geq X(s) \wedge X(r) = \min(X(s), X(s)), \text{ where } s < t < r.$$

A fuzzy real number X is called if *normal* if there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$.

A fuzzy real number X is called if *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$ for all $a \in I$ and given $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$ is open in the usual topology of $\mathbb{R}$.

The set of all *upper semi-continuous, normal, convex* fuzzy numbers is denoted by $R(I)$. The α-level set of a fuzzy real number X for $0 < \alpha \leq 1$ denoted by X^α is defined by $X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$. The 0-level set is the closure of strong 0-cut.

For each $r \in \mathbb{R}$, $\bar{r} \in \mathbb{R}(I)$ is defined by

$$\bar{r} = \begin{cases} \bar{r}, & \text{if } t = r, \\ 0, & \text{if } t \neq r. \end{cases}$$

The absolute value of $|X|$ of $X \in \mathbb{R}(I)$ is defined by(see for instance Kelava and Seikkala [16])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Let $\bar{d} : \mathbb{R}(I) \times \mathbb{R}(I) \to \mathbb{R}$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $\bar{d}$ defines a metric on $\mathbb{R}(I)$ (Matloka [18]). The additive identity and multiplicative identity in $\mathbb{R}(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

In this paper we introduce $l_\infty^F(\triangle_m^n, u)$, $c^F(\triangle_m^n, u)$ and $c_0^F(\triangle_m^n, u)$. We study some of their properties like completeness, solidity, symmetrically.

Throughout the article ω^F, c^F, c_0^F and l_∞^F denote the classes of all, convergent, null, bounded sequence spaces of fuzzy real numbers.

A fuzzy real valued sequence $\{X_n\}$ is said to be convergent to fuzzy real number X, if for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\bar{d}(X, Y) < \varepsilon$ for all $k \geq n_0$ (see, [15]).

A fuzzy real valued sequence $\{X_n\}$ is said to be solid (normal) if $(X_k) \in E^F$ implies that $(\alpha_k X_k) \in E^F$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E^F be a sequence space. A k-step space of E^F is a sequence space $\lambda_K^{E^F} = \{(X_{k_n}) \in \omega^F : (X_n) \in E^F\}$.

A canonical preimage of a sequence $\{X_k\} \in \lambda_K^{E^F}$ is a sequence $\{Y_n\} \in \omega^F$ defined as

$$Y_n = \begin{cases} X_n, & \text{if } \ n \in K, \\ \bar{0}, & \quad \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\lambda_K^{E^F}$ is a set of all elements in $\lambda_K^{E^F}$, i.e., Y is in canonical preimage of $\lambda_K^{E^F}$ if and only if Y is canonical preimage of some $X \in \lambda_K^{E^F}$.

A sequence space E^F is said to be monotone if it contains the canonical preimages of its step spaces.

A sequence space E^F is said convergence free if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $Y_k = \bar{0}$ whenever $X_k = \bar{0}$.

The difference sequence spaces, $H(\Delta) = \{x = (x_k) : \Delta x \in H\}$,where $H = l_\infty, c$ and c_0, were studied by Kizmaz [17].

It was further generalized by Tripathy and Esi [25], as follows. Let $m \geq 0$ be an integer then $H(\Delta_m) = \{x = (x_k) : \Delta_m x \in H\}$, for $H = l_\infty, c$ and c_0, where $\Delta_m x_k = x_k - x_{k+m}$. The idea of Kizmaz [17] was applied by Savas [22] for introducing the notion of difference sequences for fuzzy real numbers and study their different properties. The difference sequence space were further studies by Çolak e*t al.* [6-8], Mursaleen and Başarir [19], Tripathy and Esi [25] and etc.

Further, in Tripathy *et al.* [28] generalized the above notions and unified these as follows:

$$\Delta_m^n x_k = \{x \in \omega : (\Delta_n^m x_k) \in Z\},$$

where

$$\Delta_m^n x_k = \sum_{\mu=0}^{n} (-1)^{\mu} \begin{pmatrix} n \\ r \end{pmatrix} x_{k+m\mu}$$

and

$$\Delta_0^n x_k = x_k \forall \ k \in \mathbb{N}.$$

Following [4-29], we introduce the difference sequences of fuzzy real numbers type as follows:

$$H(\Delta_n^m, u) \ = \ \{ \, (X_k) \ : \ (u\Delta_n^m X) \ \in \ H \} \, ,$$

for $H = l_{\infty}^F \, , c^F$ and c_0^F, where $u = (u_k)$ is such that $u_k \neq 0$ for all $k \in \mathbb{N}$.

We note that if $n = 1$, we get results obtained by Sheikh and Ganie(see, [24]).

2. Main Results

Theorem 2.1. *The sequence spaces* $l_{\infty}^F(\triangle_m^n, u)$, $c^F(\triangle_m^n, u)$ *and* $c_0^F(\triangle_m^n, u)$ *are complete metric spaces by the metric*

$$\varrho(X, Y) = \sum_{k=0}^{m} \bar{d}(u_k X_k, u_k Y_k) + \sup_k \bar{d}(u_k \Delta_m^n X_k, u_k \Delta_m^n Y_k). \tag{2.1}$$

Proof. The proof is left as an easy exercise for the reader. □

Theorem 2.2. *The sequence spaces* $l_{\infty}^F(\triangle_m^n, u)$, $c^F(\triangle_m^n, u)$ *and* $c_0^F(\triangle_m^n, u)$ *are not solid in general.*

Proof. We consider only $c^F(\triangle_m^n, u)$. Thus to prove the result we consider the following examples:

Let

$$X_l(t) = \begin{cases} \frac{lt+l+1}{l+1}, & \text{if} \quad -1-\frac{1}{l} \leq t \leq 0, \\ \frac{l+1-lt}{l+1}, & \text{if} \ \ 0 \leq t \leq 1+\frac{1}{l}, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Now taking $l = 3$ and $u_k = 1 = n$ for all $k \in \mathbb{N}$, we have that

$$u\triangle_m^n X_l(t) = \begin{cases} \frac{tl^2+2l^2+3lt+8l+3}{2l^2+3lt+8l+3}, & \text{if } \; -2-\frac{1}{l}-\frac{1}{l+3} \le t \le 0, \\ \\ \frac{-tl^2-3lt+2l^2+8l+3}{2l^2+3lt+8l+3}, & \text{if } \; 0 \le t \le 2+\frac{1}{l}+\frac{1}{l+3}, \\ \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Now, $\lim\limits_{l} \triangle_1^3 X_l(t) = X$, where

$$X(t) = \begin{cases} \frac{t+2}{2}, & \text{if } \; -2 \le t \le 0, \\ \\ \frac{2-t}{2}, & \text{if } \; 0 \le t \le 2, \\ \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Thus, $X_l \in c^F(\triangle_1^3)$. Now consider the sequence of scalars (α_l) defined by

$$(\alpha_l) = \begin{cases} 1, & \text{if } \; l = 3k-2, \text{ for } k \in \mathbb{N}, \\ \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Then, $(\alpha_l X_l) = \{X_1, \bar{0}, \bar{0}, X_4, \bar{0}, \bar{0}, X_7, \bar{0}, \bar{0}, X_{10}, ...\}$. But

$$(\triangle_1^3 \alpha_n X_n) = \{X_1 - X_4, \bar{0}, \bar{0}, X_4 - X_7, \bar{0}, \bar{0}, ...\} \notin c^F(\triangle_1^3).$$

Hence, $c^F(u\triangle_m^n)$ is not solid. □

Theorem 2.3. *The sequence spaces $l_\infty^F(u\triangle_m^n)$, $c^F(u\triangle_n^n)$ and $c_0^F(u\triangle_n^n)$ are not symmetric.*

Proof. We consider only $c^F(u\triangle_m^n)$. Thus to prove the result we consider the following examples with $m = 1 = n$ and $u_k = 1 \; \forall \; k \in \mathbb{N}$:

Consider the sequence $X = \{N, H, N, H, N, H, ...\}$, where

$$N = \begin{cases} \frac{t+4}{4}, & \text{if } \; -4 \le t \le 0, \\ \\ \frac{4-t}{4}, & \text{if } \; 0 \le t \le 4, \\ \\ \bar{0}, & \text{otherwise,} \end{cases}$$

and

$$H = \begin{cases} \frac{t+5}{5}, & \text{if } -5 \leq t \leq 0, \\ \frac{5-t}{5}, & \text{if } 0 \leq t \leq 5, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Now, consider the re-arrangement (Y_n) of the sequence (X_n) as

$$(Y_n) = (N, N, H, H, N, N, ...) \notin C^F(\triangle_1^1), \quad \text{but} \quad (X_n) \in (\triangle_1^1).$$

Hence, $c^F(u\triangle_m^n)$ is not symmetric for any $m \in \mathbb{N}$. □

Theorem 2.4. *The sequence spaces $l_\infty^F(u\triangle_m^n)$, $c^F(u\triangle_m^n)$ and $c_0^F(u\triangle_m^n)$ are not convergence free.*

Proof. We consider only $c^F(u\triangle_m^n)$. Thus to prove the result we consider the following examples with $m = 3$ and $u = 1 = n$.

Consider the sequence (X_l) defined as follows :

$$X_k(t) = \begin{cases} \frac{kt+k+1}{k+1}, & \text{if } -1 - \frac{1}{k} \leq t \leq 0, \\ \frac{k+1-kt}{k+1}, & \text{if } 0 \leq t \leq 1 + \frac{1}{k}, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Now taking $k = 3$ and $u_k = 1 = n$ for all $k \in \mathbb{N}$, we have that

$$\triangle_3^1 X_k(t) = \begin{cases} \frac{tk^2+2k^2+3kt+8k+3}{2k^2+3kt+8k+3}, & \text{if } -2 - \frac{1}{k} - \frac{1}{k+3} \leq t \leq 0, \\ \frac{-tk^2-3kt+2k^2+8k+3}{2k^2+3kt+8k+3}, & \text{if } 0 \leq t \leq 2 + \frac{1}{k} + \frac{1}{k+3}, \\ \bar{0}, & \text{otherwise,} \end{cases}$$

and $\lim_k \triangle_3^1 X_k(t) = X$, where

$$X(t) = \begin{cases} \frac{t+2}{2}, & \text{if } -2 \leq t \leq 0, \\ \frac{2-t}{2}, & \text{if } 0 \leq t \leq 2, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Thus, $(X_k) \in c^F(\triangle_3^1)$. Now consider

$$Y_k(t) = \begin{cases} \frac{t+k}{k}, & \text{if } \; -k \leq t \leq 0, \\ \frac{k-t}{k}, & \text{if } \; 0 \leq t \leq k, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

$$\triangle_3^1 Y_k(t) = \begin{cases} \frac{t+2k+3}{2k+3}, & \text{if } \; -2k-3 \leq t \leq 0, \\ \frac{2k+3-t}{2k+3}, & \text{if } \; 0 \leq t \leq 2k+3, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Clearly $(Y_k) \in c^F(\triangle_3^1)$ is not convergence free. □

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