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# KRASNOSELSKII-ZABREIKO FIXED POINT THEOREM FOR IMPLICIT $\psi$-CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS UNDER MIXED CONDITIONS

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**Abstract.** This paper delves into the exploration of the existence, uniqueness, and stability of solutions for $\psi$-Caputo fractional differential equations subjected to mixed boundary conditions. By utilizing the Krasnoselskii-Zabreiko fixed point theorem and the Banach contraction principle, we establish the uniqueness of solutions. Furthermore, we derive the conditions for generalized Ulam-Hyers stability. Additionally, an example is included to illustrate the practical application of our findings.

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## 1. Introduction

A mathematical model of coronavirus disease (COVID-19) serves as a valuable tool for understanding and predicting the virus's spread within a population. Researchers and public health experts have developed various mathematical models to study the disease's dynamics, evaluate the effectiveness of interventions, and guide informed decision-making. The SIR model, frequently utilized in the study of infectious diseases, partitions the population into three compartments: Susceptible (S), Infected (I), and Recovered (R). However, the basic SIR model does not encompass all the complexities of COVID-19. Therefore, more sophisticated models have been developed to incorporate additional factors. Here's an overview of one such complex mathematical model used for COVID-19:

1. Compartmental Model: This model divides the population into distinct compartments, including Susceptible (S), Exposed (E), Infected (I), Asymptomatic (A), Hospitalized (H), Recovered (R), and Deceased (D). Each compartment represents individuals with specific characteristics related to the disease.
2. Transmission Dynamics: This aspect of the model includes parameters such as the basic reproduction number (R0), indicating the average number of individuals infected by one person in a susceptible population. R0 aids in estimating the initial spread of the disease.
3. Differential Equations: The model employs a system of differential equations to depict the movement of individuals among various compartments over time. These equations quantify the rates of infection, recovery, hospitalization, and mortality.
4. Intervention Strategies: The model can incorporate the effects of diverse intervention strategies, including social distancing, mask-wearing, testing, contact tracing, vaccination, and healthcare capacity. These interventions can be simulated to assess their impact on disease transmission dynamics and to guide policy decisions.
5. Data Fitting and Calibration: Models are typically calibrated using real-world data to estimate the values of model parameters and assess how well the model aligns with observed trends. This helps validate the model and make it more accurate for predicting future scenarios.
6. Scenario Analysis: Once the model is calibrated, it can be used to simulate different scenarios and predict the future course of the disease under various conditions. This can help evaluate the effectiveness of different interventions and guide decision-making.

It's important to acknowledge that mathematical models are simplifications of reality and are reliant on assumptions and available data. The accuracy of the model's predictions hinges on the quality of input data, the validity of assumptions, and the timeliness of updates as new information emerges. Throughout the COVID-19 pandemic, researchers and public health authorities worldwide have extensively utilized mathematical models. These models have played crucial roles in guiding public health strategies, evaluating intervention effectiveness, and offering insights into disease spread and control.

Differential equations serve as powerful tools for constructing mathematical models of infectious disease spread, such as COVID-19. These models facilitate our comprehension of how the disease disseminates within a population and enable predictions about its future trajectory. Among the various types of differential equations utilized for this purpose, the compartmental model is the most prevalent, dividing the population into distinct compartments based on their disease status.

Let's revisit the SIR model. This model posits that individuals transition between compartments based on specific rates. Here, we denote the total population as $N$, and the variables $S(t)$, $I(t)$ and $R(t)$ respectively represent the number of individuals in each compartment at time $t$. The dynamics of the SIR model can be expressed through the following system of ordinary differential equations:

$$\begin{cases} \frac{dS}{dt} &= -\beta \cdot S \cdot \frac{I}{N} \\ \frac{dI}{dt} &= \beta \cdot S \cdot \frac{I}{N} - \gamma \cdot I \\ \frac{dR}{dt} &= \gamma \cdot I, \end{cases}$$

where, $\beta$ denotes the transmission rate, which governs how quickly susceptible individuals contract the infection upon contact with infectious individuals. $\gamma$ signifies the recovery rate, dictating the speed at which infected individuals recover and develop immunity. These equations depict the evolution of the number of individuals in each compartment over time. The first equation illustrates the rate at which susceptible individuals contract the infection, the second equation describes the rate at which infected individuals recover, and the third equation outlines the rate at which individuals transition from the infected to the recovered compartment To apply this model, you would need to define values for the parameters $\beta$ and $\gamma$, which can be estimated from data or inferred from known disease characteristics. The initial conditions $S(0)$, $I(0)$, and $R(0)$ represent the initial number of individuals in each compartment at the onset of the outbreak. Solving this system of equations numerically enables us to simulate the disease spread over time and forecast the epidemic's future trajectory. Various modifications and extensions to the basic SIR model exist to incorporate additional factors and make the model more realistic, such as population demographics, interventions (e.g., social distancing measures), and vaccination.

It's essential to recognize that mathematical models are simplifications of real-world phenomena and are built upon assumptions. The accuracy of a model's predictions hinges on the quality of input data, the appropriateness of the assumptions, and the precision of estimated parameters. Consequently, interpreting the results of any model requires careful consideration, alongside other sources of information and expert guidance.

The Caputo fractional differential equation generalizes ordinary differential equations by incorporating fractional derivatives. It is formulated using the Caputo fractional derivative, which extends the concept of the classical derivative.

The Caputo fractional differential equation with mixed conditions can be written as follows:

$$\begin{cases} {}^{C}\mathcal{D}^{q}_{a^+}u(t) & = f(t,u(t)),\ a < t < b, \\ u^{(k)}(a) & = g_k(a),\ 0 \leq k < m, \\ u^{(k)}(b) & = h_k(b),\ 0 \leq k < n, \end{cases}$$

where:

- ${}^{C}\mathcal{D}^{q}$ denotes the Caputo fractional derivative of order $q$, with $(0 < q \leq 1)$.
- $u(t)$ is the unknown function.
- $f(t, u(t))$ represents a given function.
- $a$ and $b$ are the boundaries of the interval.
- $u^{(k)}(a)$ and $u^{(k)}(b)$ stand for the initial and boundary conditions, respectively.
- $m$ and $n$ denote the order of the initial and boundary conditions, respectively.

The Caputo fractional derivative of order $q$ is defined as:

$${}^{C}\mathcal{D}^{q}_{a^+}u(t) = \frac{1}{\Gamma(1-q)}\int_a^t (t-s)^{-q}u'(s)ds,$$

where $\Gamma(\cdot)$ denotes the Gamma function. Solving Caputo fractional differential equations with mixed conditions can pose challenges, and a universal analytical method for all cases does not exist. However, numerical methods, including finite difference methods, finite element methods, and spectral methods, offer viable approaches to approximate the solutions. These methods entail discretizing the differential equation and solving the resulting system of algebraic equations. Various numerical software packages, such as MATLAB and Python libraries like SciPy, provide tools for solving fractional differential equations. It's important to note that the choice of solution technique depends on factors such as the nature of the given function $f(t, u(t))$, the initial and boundary conditions, and the characteristics of the Caputo fractional differential equation. For further details, refer to [6, 7, 10, 11, 14, 15, 16, 17, 19, 21].

In the past, COVID-19 has become an epidemic that has spread greatly around the world. Scientists and mathematicians have SIR model how disease spreads to figure out how to prevent it and find the best outcomes. The $\psi$-Caputo differential equation is one option that helps support expanded in general form and explains the SIR model more comprehensively.

In 2019, Tate and Dinde [23] investigated the existence and uniqueness of solutions for the boundary value problem concerning nonlinear implicit fractional differential equations. Their study focused on equations of the form:

$$\begin{cases} \mathcal{D}^{q}_{0^+}u(t) = \mathcal{F}(t, u(t), \mathcal{D}^{q}_{0^+}u(t)),\ t \in [0,1],\ 2 < q \leq 3, \\ \mathcal{D}^{q-1}_{0^+}u(0) = 0,\ \mathcal{D}^{q-1}_{0^+}u(1) = 0\ u(1) = 0, \end{cases}$$

where $\mathcal{D}^{q}_{0^+}$ denotes RiemannLiouville fractional derivative, and $\mathcal{F} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In another studied conducted in 2019, Borisut et al. [9] explored the $\psi$-Hilfer fractional differential equation with nonlocal multi point condition, which takes the form:

$$\begin{cases} \mathcal{D}^{q,p;\psi}_{a^+}u(t) = f(t, u(t), \mathcal{D}^{q,p;\psi}_{a^+}u(t)),\ t \in [a,b], \\ \mathcal{I}^{1-r;\psi}_{a^+}u(a) = \displaystyle\sum_{i=1}^{m} \beta_i u(\eta_i),\ q \leq r = q + p - qp < 1,\ \eta_i \in [a,b], \end{cases}$$

where $0 < q < 1,\ 0 \leq p \leq 1,\ m \in \mathbb{N},\ \beta_i \in \mathbb{R},\ i = 1, 2 \dots, m,\ -\infty < a < b < \infty$, $\mathcal{D}^{q,p;\psi}_{a^+}$ denotes the $\psi$-Hilfer fractional derivative, $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\mathcal{I}^{1-r;\psi}_{a^+}$ represents the $\psi$-Riemann-Liouville fractional integral of order $1 - r$.

Afterwards, Borisut and Bantaojai [8] delved into the study and exploration of the following implicit Caputo fractional derivative and nonlocal fractional integral conditions:

$$\begin{cases} {}^{C}\mathcal{D}^{q}_{0^+}u(t) = f(t, u(t), {}^{C}\mathcal{D}^{q}_{0^+}u(t)),\ t \in [0,T], \\ u(0) = \eta,\ u(T) = {}_{RL}\mathcal{I}^{p}_{0^+}u(\kappa),\ \kappa \in (0,T), \end{cases}$$

where $1 < q \leq 2,\ 0 < p \leq 1,\ \eta \in \mathbb{R},\ {}^{C}\mathcal{D}^{q}_{0^+}u(t)$ denotes the Caputo fractional derivative of order $q$, ${}_{RL}I^{p}_{0^+}$ represents the Riemann-Liouville fractional integral of order $p$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function. They utilized Krasnoselskii's fixed point theorem and Boyd-Wong non-linear contraction for their analysis.

Building upon the research outlined in [2, 5, 9, 11, 18, 20, 23, 24], this paper aims to establish the existence, uniqueness, and stability of solutions for $\psi$-Caputo fractional differential equations with mixed conditions:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu(t) = f(t,\mu(t),{}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu(t)), \ t \in [0,T], \\ \mu(0) = 0, \ {}^{C}\mathcal{D}_{0^+}^{\rho;\psi}\mu(0) = 0, \ \mu(T) = \sum_{i=1}^{m} \beta_i \mu(\kappa_i), \\ {}^{C}\mathcal{D}_{0^+}^{\rho;\psi}\mu(T) = \sum_{i=1}^{m} \beta_i \mu(\kappa_i), \end{cases} \tag{1.1}$$

where, $3 < q \leq 4, \ 1 < \rho \leq 2, \ \kappa \in (0,T), \ \beta \in \mathbb{R}, \ T > 0, \ {}^{C}\mathcal{D}_{0^+}^{q;\psi}$ denotes the $\psi$-Caputo fractional derivative, and $f : [0,T] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ represents a continuous function.

The paper is structured as follows: Section 2 provides an overview of fundamental concepts related to fractional derivatives. In Section 3, we present our main result, employing the Krasnoselskii-Zabreiko fixed point theorem, Banach contraction principle and generalized Ulam-Hyers stability, respectively. Additionally, we provide an illustrative example to support the main findings. Finally, we conclude our obtained results in Section 4.

## 2. Preliminaries

In this section, we provide essential notations, definitions, lemmas, and theorems that will be used to prove the main result.

The concept of fractional calculus extends the traditional notions of integration and differentiation to non-integer orders, providing a powerful tool for describing various complex phenomena. In this context, the Riemann-Liouville and Caputo fractional derivatives, along with their generalizations, are fundamental in deriving subsequent results.

**Definition 2.1.** ([25]) The Riemann-Liouville fractional integral of order $q > 0$ for a function $f : (0,\infty) \to \mathbb{R}$, with $\Gamma(\cdot)$ denoting the Gamma function defined as $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$, is defined by:

$$_{RL}\mathcal{I}_{0^+}^{q} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds.$$

**Definition 2.2.** ([13]) The $\psi$-Riemann-Liouville fractional integral of order $q, \ (n-1 < q < n)$ for an integrable function $f : [a,b] \to \mathbb{R}$ is defined as:

$$\mathcal{I}_{a^+}^{q;\psi} f(t) = \frac{1}{\Gamma(q)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{q-1} f(s) ds.$$

This integral is taken with respect to another function $\psi : [a,b] \to \mathbb{R}$ that is an increasing differential function such that $\psi'(t) \neq 0$ for all $t \in [a,b]$.

**Definition 2.3.** ([22]) The $\psi$-Hilfer fractional derivative of function $f \in C^n([a,b],\mathbb{R})$ of order $q$ and type $0 \le p \le 1$ is determined as:

$$\mathcal{D}^{q,p;\psi}_{a^+} f(t) = \mathcal{I}^{p(n-q);\psi}_{a^+} \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n \mathcal{I}^{(1-p)(n-q);\psi}_{a^+} f(t),\ t > a,$$

where $n-1 < q < n,\ n \in \mathbb{N}$ with $[a,b]$ is the interval such that $-\infty \le a < b \le +\infty$ and $\psi \in C^n([a,b],\mathbb{R})$ such that $\psi(t)$ is increasing and $\psi'(t) \neq 0$ for all $t \in [a,b]$. On the order hand, we have

$$\mathcal{D}^{q,\rho;\psi}_{a^+} f(t) = \mathcal{I}^{\rho(n-q);\psi}_{a^+} \mathcal{D}^{r;\psi}_{a^+} f(t),\ t > a,$$

where $\mathcal{D}^{r;\psi}_{a^+} f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n \mathcal{I}^{(1-\rho)(n-q);\psi}_{a^+} f(t)$ and $r = q + \rho - q\rho$.

**Definition 2.4.** ([22]) The $\psi$-Caputo fractional derivative of function $f \in C^n[a,b],\ n-1 < q < n$ with respect to another function $\psi$ is defined as:

$$\begin{aligned} {}^C\mathcal{D}^{q;\psi}_{a^+} f(t) &= \mathcal{I}^{n-q;\psi}_{a^+} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t) \\ &= \frac{1}{\Gamma(n-q)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-q-1} f^{[n]}_{\psi}(s)ds, \end{aligned}$$

where $n$ is the smallest integer greater than or equal to $q$,

$$f^{[n]}_{\psi}(s) = \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n f(s),$$

and $\psi$ is as defined in Definition 2.3.

**Lemma 2.5.** ([12]) *If $q > 0,\ 0 \le r < 1$ and $f \in L^1(a,b)$, then*

$$\mathcal{I}^{q;\psi}_{a^+}\mathcal{I}^{p;\psi}_{a^+} f(t) = \mathcal{I}^{q+p;\psi}_{a^+} f(t),\ t \in [a,b].$$

*In particular, if $f \in C[a,b]$, then $\mathcal{I}^{q;\psi}_{a^+}\mathcal{I}^{p;\psi}_{a^+} f(t) = \mathcal{I}^{q+p;\psi}_{a^+} f(t)$, $t \in [a,b]$.*

**Theorem 2.6.** ([4]) *Suppose that $f \in C^n[0,T],\ n-1 < q < n$ and $n \in \mathbb{N}$. Then*

$$\mathcal{I}^{q;\psi}_{0^+}\left({}^C\mathcal{D}^{q;\psi}_{0^+} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{[\psi(t)-\psi(0)]^k}{k!} f^{[n]}_{\psi}(0)$$

*for all $t \in (0,T]$. Moreover for $3 < q \le 4$ yields*

$$\mathcal{I}^{q;\psi}_{0^+}\left({}^C\mathcal{D}^{q;\psi}_{0^+} f(t)\right) = f(t) + c_0 + c_1(\psi(t)-\psi(0)) + c_2(\psi(t)-\psi(0))^2 + c_3(\psi(t)-\psi(0))^3,$$

*where $c_0, c_1, c_2, c_3$ are arbitrary constants on $\mathbb{R}$.*

**Lemma 2.7.** ([4]) *If* $f(t) = (\psi(t) - \psi(0))^k$, $k > \rho$, $n - 1 < \rho \leq n$ *and* $k, n \in \mathbb{N}$, *then*

$$ {}^C\mathcal{D}_{0^+}^{\rho;\psi} f(t) = \begin{cases} \frac{k!}{\Gamma(k+1-\rho)}(\psi(t) - \psi(0))^{k-\rho} & \text{if} \quad n \leq k, \\ 0 & \text{if} \quad n > k. \end{cases} $$

The following results are essential for proving our main theorems.

**Lemma 2.8.** ([25]) **(Arzela-Ascoli theorem)** *Let* $M \subseteq C[a,b]$. $M$ *is relatively compact in* $C[a,b]$ *if and only if* $M$ *is*

(1) *uniformly bounded (meaning that it is a bounded set in* $C[a,b]$*),*
(2) *equicontinuous on* $[a,b]$, *for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $|t_2 - t_1| < \delta$ *implies* $|f(t_2) - f(t_1)| < \epsilon$ *for any* $f \in M$.

**Theorem 2.9.** ([1]) **(Krasnoselskii-Zabreiko's Fixed Point Theorem)** *Let* $\mathcal{A} : E \to E$ *be a completely continuous operator, where* $(E, \|\cdot\|)$ *is a Banach space. Suppose* $\mathcal{L} : E \to E$ *is a bounded linear operator such that* 1 *is not an eigenvalue of* $\mathcal{L}$ *and*

$$ \lim_{\|\mu\| \to \infty} \frac{\|\mathcal{A}\mu - \mathcal{L}\mu\|}{\|\mu\|} = 0. $$

*Then* $\mathcal{A}$ *has a fixed point in* $E$.

**Theorem 2.10.** ([1]) **(Banach Contraction Principle)** *Let* $E$ *be a Banach space,* $D \subset E$ *be closed, and* $\mathcal{A} : D \to D$ *be a contraction mapping, meaning* $\|\mathcal{A}u - \mathcal{A}v\| \leq k\|u - v\|$ *for some* $k \in (0,1)$ *and for all* $u, v \in D$. *Then,* $\mathcal{A}$ *has a unique fixed point.*

Let us conclude this section by introducing the concepts of Ulam-Hyers stability and generalized Ulam-Hyers stability.

**Definition 2.11.** ([3]) Problem (1.1) is said to be *Ulam-Hyers stable* if there exists a positive real number $\mathcal{C}_{UH}$ with the following property: For every $\epsilon > 0$ and $u \in C[0,T]$, if

$$ \left| {}^C\mathcal{D}_{0^+}^{q;\psi} \mu(t) - f(t, \mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi} \mu(t)) \right| \leq \epsilon, \tag{2.1} $$

then there exists some $\mu^* \in C[0,T]$ satisfying

$$ \begin{cases} {}^C\mathcal{D}_{0^+}^{q;\psi} \mu^*(t) = f(t, \mu^*(t), {}^C\mathcal{D}_{0^+}^{q;\psi} \mu^*(t)), \ t \in [0,T], \\ \mu^*(0) = 0, \ {}^C\mathcal{D}_{0^+}^{\rho;\psi} \mu^*(0) = 0, \ \mu^*(T) = \displaystyle\sum_{i=1}^{m} \beta_i \mu^*(\kappa_i), \\ {}^C\mathcal{D}_{0^+}^{\rho;\psi} \mu^*(T) = \displaystyle\sum_{i=1}^{m} \beta_i \mu^*(\kappa_i), \end{cases} \tag{2.2} $$

where $q \in (3,4],\ \rho \in (1,2],\ \kappa_i \in (0,T),\ \beta_i \in \mathbb{R},\ m \in \mathbb{N}$ such that

$$|\mu(t) - \mu^*(t)| \leq \mathcal{C}_{UH}\epsilon,\ \forall t \in [0,T].$$

**Definition 2.12.** ([3]) Problem (1.1) is said to be *generalized Ulam-Hyers stable* if there exists a function $\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\phi(0) = 0$ such that for any solution $\mu^* \in C[0,T]$ of inequality (2.1), there exists a unique solution $\mu \in C[0,T]$ of problem (1.1) satisfying $|\mu(t) - \mu^*(t)| \leq \phi(\epsilon)$ for all $t \in [0,T]$.

## 3. Main results

In this section, we investigate the existence of solutions for the implicit $\psi$-Caputo fractional differential equation with mixed conditions represented by equation (1.1).

The subsequent results depend on the preceding ones to be proven.

**Lemma 3.1.** *Let* $3 < q \leq 4,\ 1 < \rho \leq 2$ *and* $f \in C([a,b])$, *then*

$$^C\mathcal{D}_{0^+}^{\rho;\psi}\mathcal{I}_{0^+}^{q;\psi}f(t) = \mathcal{I}_{0^+}^{q-\rho;\psi}f(t).$$

*Proof.* Consider

$$\begin{aligned}\left(\frac{1}{\psi'(t)} \cdot \frac{d}{dt}\right)^2 \mathcal{I}_{0^+}^{q;\psi}f(t) &= \frac{1}{(\psi'(t))^2}\frac{d^2}{dt^2}\left(\frac{1}{\Gamma(q)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{q-1}f(s)ds\right)\\ &= \frac{(q-1)(q-2)}{\Gamma(q)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{q-3}f(s)ds\\ &= \mathcal{I}_{0^+}^{q-2;\psi}f(t).\end{aligned}$$

From Definition 2.4, where $1 < \rho \leq 2$, we have

$$\begin{aligned}^C\mathcal{D}_{0^+}^{\rho;\psi}\mathcal{I}_{0^+}^{q;\psi}f(t) &= \mathcal{I}_{0^+}^{2-\rho;\psi}\left(\frac{1}{\psi'(t)} \cdot \frac{d}{dt}\right)^2 \mathcal{I}_{0^+}^{q;\psi}f(t)\\ &= \mathcal{I}_{0^+}^{2-\rho;\psi}\mathcal{I}_{0^+}^{q-2;\psi}f(t)\\ &= \mathcal{I}_{0^+}^{q-\rho;\psi}f(t).\end{aligned}$$

Hence, $^C\mathcal{D}_{0^+}^{\rho;\psi}\mathcal{I}_{0^+}^{q;\psi}f(t) = \mathcal{I}_{0^+}^{q-\rho;\psi}f(t),\ t \in (0,T)$. □

**Lemma 3.2.** *If $\mu$ belongs to $C([0,T])$, it satisfies the $\psi$-Caputo fractional differential equation with mixed conditions represented by problem* (1.1) *if and only if $\mu$ satisfies the integral equation:*

$$
\begin{cases}
\mu(t) = \mathcal{I}_{0^+}^{q;\psi} f(t,\mu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)) \\
+\frac{(\psi(t)-\psi(0))^2}{A_1B_2-A_2B_1}\Bigg\{(B_2-A_2)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right] \\
+\Big[A_2\mathcal{I}_{0^+}^{q-\rho;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) - B_2\mathcal{I}_{0^+}^{q;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\Big]\Bigg\} \\
+\frac{(\psi(t)-\psi(0))^3}{A_2B_1-A_1B_2}\Bigg\{(B_1-A_1)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right] \\
+\Big[A_1\mathcal{I}_{0^+}^{q-\rho;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) - B_1\mathcal{I}_{0^+}^{q;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\Big]\Bigg\},
\end{cases}
\tag{3.1}
$$

*where*

$$A_1 = (\psi(T)-\psi(0))^2 - \sum_{i=1}^{m}\beta_i(\psi(\kappa_i)-\psi(0))^2,$$

$$A_2 = (\psi(T)-\psi(0))^3 - \sum_{i=1}^{m}\beta_i(\psi(\kappa_i)-\psi(0))^3,$$

$$B_1 = \frac{2}{\Gamma(3-\rho)}(\psi(T)-\psi(0))^{2-\rho} - \sum_{i=1}^{m}\beta_i(\psi(\kappa_i)-\psi(0))^2,$$

$$B_2 = \frac{6}{\Gamma(4-\rho)}(\psi(T)-\psi(0))^{3-\rho} - \sum_{i=1}^{m}\beta_i(\psi(\kappa_i)-\psi(0))^3.$$

*Proof.* Applying $\mathcal{I}_{0^+}^{q;\psi}$ to both sides in equation (1.1) and using Lemma 2.6, we have

$$
\begin{aligned}
\mathcal{I}_{0^+}^{q;\psi}\left({}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)\right) &= \mathcal{I}_{0^+}^{q;\psi} f(t,\mu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)) \\
&\quad + c_0 + c_1(\psi(t)-\psi(0)) + c_2(\psi(t)-\psi(0))^2 \\
&\quad + c_3(\psi(t)-\psi(0))^3.
\end{aligned}
$$

From condition $\mu(0) = 0$, we get $c_0 = 0$.

$$
\begin{aligned}
\mu(t) \;=\; & \mathcal{I}_{0^+}^{q;\psi} f(t,\mu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)) \\
& + c_1(\psi(t)-\psi(0)) + c_2(\psi(t)-\psi(0))^2 + c_3(\psi(t)-\psi(0))^3.
\end{aligned}
\tag{3.2}
$$

By equation (3.2), Lemma 2.7 and $1 < \rho < 2$ which corresponds to condition ${}^C\mathcal{D}_{0^+}^{q;\psi}\mu(0) = 0$, this implies that $c_1 = 0$. So,

$$
\begin{aligned}
{}^C\mathcal{D}_{0^+}^{\rho;\psi}\mu(t) = \mathcal{I}_{0^+}^{q-\rho;\psi} f(t,\mu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)) + \frac{2c_2}{\Gamma(3-\rho)}(\psi(t)-\psi(0))^{2-\rho} \\
+ \frac{6c_3}{\Gamma(4-\rho)}(\psi(t)-\psi(0))^{3-\rho}.
\end{aligned}
$$

According to the conditions

$$\begin{aligned}\mu(T) &= \mathcal{I}_{0^+}^{q;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) + c_2(\psi(T)-\psi(0))^2 \\ &\quad + c_3(\psi(T)-\psi(0))^3. \end{aligned} \tag{3.3}$$

$$\begin{aligned}{}^C\mathcal{D}_{0^+}^{\rho;\psi}\mu(T) &= \mathcal{I}_{0^+}^{q-\rho;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) + \frac{2c_2}{\Gamma(3-\rho)}(\psi(T)-\psi(0))^{2-\rho} \\ &\quad + \frac{6c_3}{\Gamma(4-\rho)}(\psi(T)-\psi(0))^{3-\rho}. \end{aligned} \tag{3.4}$$

$$\begin{aligned}\mu(T) = {}^C\mathcal{D}_{0^+}^{\rho;\psi}\mu(T) &= \sum_{i=1}^{m}\beta_i\mu(\kappa_i) \\ &= \sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)) + c_2\sum_{i=1}^{m}\beta_i(\psi(\kappa_i)-\psi(0))^2 \\ &\quad + c_3\sum_{i=1}^{m}\beta_i(\psi(\kappa_i)-\psi(0))^3. \end{aligned} \tag{3.5}$$

Solve the equation (3.3), (3.4) and (3.5), we get

$$\begin{aligned} c_2 &= \frac{1}{A_1B_2-A_2B_2}\Bigg\{(B_2-A_1)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right] \\ &\quad + \left[A_2\mathcal{I}_{0^+}^{q-\rho} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) + B_2\mathcal{I}_{0^+}^{q} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\right]\Bigg\} \\ c_3 &= \frac{1}{A_2B_1-A_1B_2}\Bigg\{(B_1-A_1)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right] \\ &\quad + \left[A_1\mathcal{I}_{0^+}^{q-\rho} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) + B_1\mathcal{I}_{0^+}^{q} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\right]\Bigg\}. \end{aligned}$$

Hence, the result is given by

$$\begin{aligned}\mu(t) &= \mathcal{I}_{0^+}^{q;\psi} f(t,\mu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)) \\ &\quad + \frac{(\psi(t)-\psi(0))^2}{A_1B_2-A_2B_1}\Bigg\{(B_2-A_2)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right] \\ &\quad + \left[A_2\mathcal{I}_{0^+}^{q-\rho;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) - B_2\mathcal{I}_{0^+}^{q;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\right]\Bigg\} \\ &\quad + \frac{(\psi(t)-\psi(0))^3}{A_2B_1-A_1B_2}\Bigg\{(B_1-A_1)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right] \\ &\quad + \left[A_1\mathcal{I}_{0^+}^{q-\rho;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)) - B_1\mathcal{I}_{0^+}^{q;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\right]\Bigg\}. \end{aligned}$$

□

Now, let's delve into the space of continuous functions. Consider the space $E = C([0,T],\mathbb{R})$, rendering $E$ a Banach space equipped with the norm $\|\mu\|= \max_{t\in[0,T]}|\mu(t)|$. Define the operator $\mathcal{F} : E \to E$ as follows:

$$
\begin{aligned}
(\mathcal{F}\mu)(t) &= \mathcal{I}_{0^+}^{q;\psi} f(t,\mu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)) \\
&\quad + \frac{(\psi(t)-\psi(0))^2}{A_1B_2-A_2B_1}\left\{(B_2-A_2)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right]\right. \\
&\quad \left.+\left[A_2\mathcal{I}_{0^+}^{q-\rho;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))-B_2\mathcal{I}_{0^+}^{q;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\right]\right\} \\
&\quad + \frac{(\psi(t)-\psi(0))^3}{A_2B_1-A_1B_2}\left\{(B_1-A_1)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i))\right]\right. \\
&\quad \left.+\left[A_1\mathcal{I}_{0^+}^{q-\rho;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))-B_1\mathcal{I}_{0^+}^{q;\psi} f(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T))\right]\right\},
\end{aligned}
$$

then the problem (1.1) has solution if and only if the operator $\mathcal{F}$ has solution.

Subsequently, we demonstrate the existence and uniqueness of solutions for the $\psi$-Caputo fractional differential equation with mixed conditions employing the Krasnoselskii-Zabreiko fixed-point theorem and the Banach contraction principle. Additionally, we explore results concerning Ulam-Hyers stability and generalized Ulam-Hyers stability.

3.1. **Existence result via Krasnoselskii-Zabreiko fixed point theorem.** The existence of solutions to the problem (1.1) is the subject of our next investigation, which is guided by Krasnoselskii-Zabreiko fixed point theorem.

**Theorem 3.3.** *Assuming $f : [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous. Then the problem (1.1) has at least one solution on $[0,T]$ under the following conditions:*

(H1) *There exist positive constant $M > 0$ and $0 < N < 1$ such that*

$$|f(t,\mu,\nu)-f(t,\mu^*,\nu^*)|\le M|\mu-\mu^*|+N|\nu-\nu^*|$$

*for any $\mu,\nu,\mu^*,\nu^*\in\mathbb{R}$ and $t\in[0,T]$.*

(H2) *The function $f$ satisfies*

$$|f(t,\mu,\nu)|\le l_1(t)+l_2(t)|\mu|+l_3(t)|\nu|,$$

*where $l_1^* = \sup_{t\in[0,T]} l_1(t) < 1$, $l_2^* = \sup_{t\in[0,T]} l_2(t) < 1$, $l_3^* = \sup_{t\in[0,T]} l_3(t) < 1$.*

(H3) $\lim_{\|\mu\|\to\infty}\dfrac{f(t,\mu,\nu)}{\mu} = l(t)$ *and*

$$l^* = \max_{t\in[0,T]}|l(t)|$$

$$= l_{max} < 1/\left(\frac{[\psi(T)]^q}{\Gamma(q+1)} + \frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|}\left\{\frac{|B_2-A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}\beta_i + \frac{|A_2|[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{|B_2|[\psi(T)]^q}{\Gamma(q+1)}\right\} + \frac{(\psi(T)-\psi(0))^3}{|A_2B_1-A_1B_2|}\left\{\frac{|B_1-A_1|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}\beta_i + \frac{|A_1|[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{|B_1|[\psi(T)]^q}{\Gamma(q+1)}\right\}\right).$$

*Proof.* The proof will proceed in five steps. We define the operator $\mathcal{F}: E \to E$ by

$$\begin{aligned}
&(\mathcal{F}\mu)(t)\\
&= \mathcal{I}_{0^+}^{q;\psi} f(t,\mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t))\\
&\quad + \frac{(\psi(t)-\psi(0))^2}{A_1B_2-A_2B_1}\left\{(B_2-A_2)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f\left(\kappa_i,\mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right]\right.\\
&\quad \left. + \left[A_2\mathcal{I}_{0^+}^{q-\rho;\psi} f\left(T,\mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - B_2\mathcal{I}_{0^+}^{q;\psi} f\left(T,\mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right]\right\}\\
&\quad + \frac{(\psi(t)-\psi(0))^3}{A_2B_1-A_1B_2}\left\{(B_1-A_1)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f\left(\kappa_i,\mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right]\right.\\
&\quad \left. + \left[A_1\mathcal{I}_{0^+}^{q-\rho;\psi} f\left(T,\mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - B_1\mathcal{I}_{0^+}^{q;\psi} f\left(T,\mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right]\right\}.
\end{aligned} \tag{3.6}$$

**Step 1:** $\mathcal{F}$ is continuous. Let $\{\mu_n\}$ be sequence such that $\mu_n \to \mu$ in $E$ as $n \to \infty$. For each $t \in [0,T]$, we have

$$\begin{aligned}
&\left|\mathcal{F}\mu_n(t) - \mathcal{F}\mu(t)\right|\\
&\le \mathcal{I}_{0^+}^{q;\psi}\left|f\left(t,\mu_n(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu_n(t)\right) - f\left(t,\mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)\right)\right| + \frac{(\psi(t)-\psi(0))^2}{|A_1B_2-A_2B_1|}\\
&\quad \times\left\{|B_2-A_2|\left[\sum_{i=1}^{m}|\beta_i|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(\kappa_i,\mu_n(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu_n(\kappa_i)\right)\right.\right.\right.\\
&\qquad \left.\left. - f\left(\kappa_i,\mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right|\right]\\
&\quad + \left[|A_2|\,\mathcal{I}_{0^+}^{q-\rho;\psi}\left|f\left(T,\mu_n(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu_n(T)\right) - f\left(T,\mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right|\right.\\
&\quad \left.\left. + |B_2|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(T,\mu_n(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu_n(T)\right) - f\left(T,\mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right|\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\psi(t)-\psi(0))^3}{|A_2B_1-A_1B_2|}\times\left\{|B_1-A_1|\left[\sum_{i=1}^{m}|\beta_i|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(\kappa_i,\mu_n(\kappa_i),{}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu_n(\kappa_i)\right)\right.\right.\right.\\
&-f\left(\kappa_i,\mu(\kappa_i),{}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\\
&+\left[|A_1|\mathcal{I}_{0^+}^{q-\rho;\psi}\left|f\left(T,\mu_n(T),{}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu_n(T)\right)-f\left(T,\mu(T),{}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right|\right.\\
&\left.\left.\left.+|B_1|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(T,\mu_n(T),{}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu_n(T)\right)-f\left(T,\mu(T),{}^{C}\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right|\right]\right]\right\}\\
\le\ &\frac{M|\mu_n-\mu|[\psi(T)]^q}{(1-N)\Gamma(q+1)}+\frac{M|\mu_n-\mu|(\psi(T)-\psi(0))^2}{(1-N)(A_1B_2-A_2B_1)}\left\{\frac{(B_2-A_2)[\psi(T)]^q}{\Gamma(\Gamma(q+1))}\sum_{i=1}^{m}|\beta_i|\right.\\
&\left.+\frac{A_2[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}+\frac{B_2[\psi(T)]^q}{\Gamma(q+1)}\right\}+\frac{M|\mu_n-\mu|(\psi(t)-\psi(0))^3}{(1-N)(A_2B_1-A_1B_2)}\\
&\times\left\{\frac{(B_1-A_1)[\psi(T)]^q}{(\Gamma(q+1))}\sum_{i=1}^{m}|\beta_i|+\frac{A_1[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}+\frac{B_1[\psi(T)]^q}{\Gamma(q+1)}\right\}.
\end{aligned}
$$

Since $\|\mathcal{F}\mu_n-\mathcal{F}\mu\|\to 0$ as $n\to\infty$. This implies that the operator $\mathcal{F}$ is continuous.

**Step 2:** We will show $\mathcal{F}(\Omega_r)\subset\Omega_r$. Define $\Omega_r=\{\mu\in C[0,T]:\|\mu\|_{C[0,T]}\le r\}$, where

$$
\begin{aligned}
r\ge\ &l_1^*\left(\frac{[\psi(T)]^q}{\Gamma(q+1)}+\frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|}\left\{\frac{|B_2-A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|+\frac{|A_2|[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}\right.\right.\\
&\left.+\frac{|B_2|[\psi(T)]^q}{\Gamma(q+1)}\right\}+\frac{(\psi(T)-\psi(0))^3}{|A_2B_1-A_1B_2|}\left\{\frac{|B_1-A_1|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|+\frac{|A_1|[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}\right.\\
&\left.\left.+\frac{|B_1|[\psi(T)]^q}{\Gamma(q+1)}\right\}\right)/(1-l_3^*)\\
&-l_2^*\left(\frac{[\psi(T)]^q}{\Gamma(q+1)}+\frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|}\left\{\frac{|B_2-A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|+\frac{|A_2|[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}\right.\right.\\
&\left.+\frac{|B_2|[\psi(T)]^q}{\Gamma(q+1)}\right\}+\frac{(\psi(T)-\psi(0))^3}{|A_2B_1-A_1B_2|}\left\{\frac{|B_1-A_1|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|\right.\\
&\left.\left.+\frac{|A_1|[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}+\frac{|B_1|[\psi(T)]^q}{\Gamma(q+1)}\right\}\right).
\end{aligned}
$$

Let $\mu$ belong to $\Omega_r$. In order to prove that $\mathcal{F}\mu \in \Omega_r$ for $t \in [0,T]$, we get

$$\begin{aligned}
(\mathcal{F}\mu)(t) \leq\ & \mathcal{I}_{0^+}^{q;\psi} \left| f(t, \mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi} \mu(t)) \right| + \frac{(\psi(t)-\psi(0))^2}{|A_1B_2 - A_2B_1|} \\
& \times \left\{ |B_2 - A_2| \left[ \sum_{i=1}^{m} |\beta_i| \mathcal{I}_{0^+}^{q;\psi} \left| f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right) \right| \right] \right. \\
& + \left[ |A_2| \mathcal{I}_{0^+}^{q-\rho;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) \right| \right. \\
& \left. \left. + |B_2| \mathcal{I}_{0^+}^{q;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) \right| \right] \right\} + \frac{(\psi(t)-\psi(0))^3}{|A_2B_1 - A_1B_2|} \\
& \times \left\{ |B_1 - A_1| \left[ \sum_{i=1}^{m} |\beta_i| \mathcal{I}_{0^+}^{q;\psi} \left| f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right) \right| \right] \right. \\
& + \left[ |A_1| \mathcal{I}_{0^+}^{q-\rho;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) \right| \right. \\
& \left. \left. + |B_1| \mathcal{I}_{0^+}^{q;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) \right| \right] \right\} \\
\leq\ & \frac{(l_1^*+l_2^*r)[\psi(T)]^q}{(1-l_3^*)\Gamma(q+1)} + \frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|} \left\{ |B_2-A_2| \frac{(l_1^*+l_2^*r)[\psi(T)]^q}{(1-l_3^*)\Gamma(q+1)} \sum_{i=1}^{m} |\beta_i| \right. \\
& \left. + |A_2| \frac{(l_1^*+l_2^*r)[\psi(T)]^{q-\rho}}{(1-l_3^*)\Gamma(q-\rho+1)} + |B_2| \frac{(l_1^*+l_2^*r)[\psi(T)]^q}{(1-l_3^*)\Gamma(q+1)} \right\} \\
& + \frac{(\psi(T)-\psi(0))^3}{|A_2B_1 - A_1B_2|} \left\{ |B_1 - A_1| \frac{(l_1^*+l_2^*r)[\psi(T)]^q}{(1-l_3^*)\Gamma(q+1)} \sum_{i=1}^{m} |\beta_i| \right. \\
& \left. + |A_1| \frac{(l_1^*+l_2^*r)[\psi(T)]^{q-\rho}}{(1-l_3^*)\Gamma(q-\rho+1)} + |B_1| \frac{(l_1^*+l_2^*r)[\psi(T)]^q}{(1-l_3^*)\Gamma(q+1)} \right\} \\
\leq\ & r.
\end{aligned}$$

Thus, we have $\mathcal{F}(\Omega_r) \subset \Omega_r$.

**Step 3:** We will show that $\mathcal{F}(\Omega_r)$ is uniformly bounded and equicontinuous. From Step 2, we obtain $\mathcal{F}(\Omega_r) = \{\mathcal{F}\mu : \mu \in \Omega_r\}$. Therefore, for each $\mu \in \Omega_r$, we have $\|\mathcal{F}\mu\| \leq r$, implying that $\mathcal{F}(\Omega_r)$ is uniformly bounded.

Now, let $\eta_1, \eta_2 \in [0,T]$ with $\eta_1 < \eta_2$, and choose $\mu \in \Omega_r$. Define the operator $\mathcal{F}$. Then, we have

$$\begin{aligned}
|\mathcal{F}\mu(\tau_2) - \mathcal{F}\mu(\tau_1)]| \leq\ & \frac{l_1^* + l_2^* r}{(1-l_3^*)\Gamma(q+1)} \left[ (\psi(\eta_2)-\psi(0))^q - (\psi(\eta_1)-\psi(0))^q \right] \\
& + \frac{(\psi(\eta_2)-\psi(0))^2 - (\psi(\eta_1)-\psi(0))^2}{|A_1B_2 - A_2B_1|}
\end{aligned}$$

$$\times \Bigg\{ |B_2 - A_2| \Big[ \sum_{i=1}^{m} |\beta_i| \mathcal{I}_{0^+}^{q;\psi} \Big| f(\kappa_i, \mu(\kappa_i), {}^{C}\mathcal{D}_{0^+}^{q;\psi} \mu(\kappa_i)) \Big| \Big]$$

$$+ \Big[ |A_2| \mathcal{I}_{0^+}^{q-\rho;\psi} \Big| f(T, \mu(T), {}^{C}\mathcal{D}_{0^+}^{q;\psi} \mu(T))$$

$$+ |B_2| \mathcal{I}_{0^+}^{q;\psi} \Big| f(T, \mu(T), {}^{C}\mathcal{D}_{0^+}^{q;\psi} \mu(T)) \Big| \Big] \Bigg\}$$

$$+ \frac{(\psi(\eta_2) - \psi(0))^3 - (\psi(\eta_1) - \psi(0))^3}{|A_2 B_1 - A_1 B_2|}$$

$$\times \Bigg\{ |B_1 - A_1| \Big[ \sum_{i=1}^{m} |\beta_i| \mathcal{I}_{0^+}^{q;\psi} \Big| f(\kappa_i, \mu(\kappa_i), {}^{C}\mathcal{D}_{0^+}^{q;\psi} \mu(\kappa_i)) \Big| \Big]$$

$$+ \Big[ |A_1| \mathcal{I}_{0^+}^{q-\rho;\psi} \Big| f(T, \mu(T), {}^{C}\mathcal{D}_{0^+}^{q;\psi} \mu(T))$$

$$+ |B_1| \mathcal{I}_{0^+}^{q;\psi} \Big| f(T, \mu(T), {}^{C}\mathcal{D}_{0^+}^{q;\psi} \mu(T)) \Big| \Big] \Bigg\}.$$

The right-hand side of the above inequality tends to zero as $\eta_1 \to \eta_2$. Thus, $\mathcal{F}(\Omega_r)$ is both equicontinuous and uniformly bounded. Consequently, by the Arzel-Ascoli theorem, we can assert that $\mathcal{F} : \Omega_r \to \Omega_r$ is completely continuous.

Next, we approach problem (1.1) as a linear problem by defining

$$f(t, \mu(t), {}^{C}\mathcal{D}_{0^+}^{q;\psi} \mu(t))) = l(t)\mu(t).$$

Utilizing equation (3.1), we introduce the operator $\mathcal{L}$ as follows:

$$(\mathcal{L}\mu)(t) = \mathcal{I}_{0^+}^{q;\psi} l(t)\mu(t) + \frac{(\psi(t) - \psi(0))^2}{A_1 B_2 - A_2 B_1} \Bigg\{ (B_2 - A_2) \left[ \sum_{i=1}^{m} \beta_i \mathcal{I}_{0^+}^{q;\psi} l(\kappa_i)\mu(\kappa_i) \right]$$

$$+ \Big[ A_2 \mathcal{I}_{0^+}^{q-\rho;\psi} l(T)\mu(T) - B_2 \mathcal{I}_{0^+}^{q;\psi} l(T)\mu(T) \Big] \Bigg\}$$

$$+ \frac{(\psi(t) - \psi(0))^3}{A_2 B_1 - A_1 B_2} \Bigg\{ (B_1 - A_1) \left[ \sum_{i=1}^{m} \beta_i \mathcal{I}_{0^+}^{q;\psi} l(\kappa_i)\mu(\kappa_i) \right]$$

$$+ \Big[ A_1 \mathcal{I}_{0^+}^{q-\rho;\psi} l(T)\mu(T) - B_1 \mathcal{I}_{0^+}^{q;\psi} l(T)\mu(T) \Big] \Bigg\}.$$

**Step 4:** We assert that 1 cannot be an eigenvalue of the operator $\mathcal{L}$. Suppose 1 is an eigenvalue of $\mathcal{L}$. Then, we would have:

$$
\begin{aligned}
\|\mu\| &= \sup_{t\in[0,T]} |\mathcal{L}\mu(t)| \\
&\le l_{\max}\|\mu\| \frac{[\psi(T)]^q}{\Gamma(q+1)} + \frac{(\psi(T)-\psi(0))^2}{|A_1B_2 - A_2B_1|} \left\{ \frac{|B_2 - A_2| l_{\max}[\psi(T)]^q}{\Gamma(q+1)} \right. \\
&\quad \left. \times \sum_{i=1}^m |\beta_i| \|\mu\| + |A_2| l_{\max}\|\mu\| \frac{[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + |B_2| l_{\max}\|\mu\| \frac{[\psi(T)]^q}{\Gamma(q+1)} \right\} \\
&\quad + \frac{(\psi(T)-\psi(0))^3}{|A_2B_1 - A_1B_2|} \left\{ \frac{|B_1 - A_1| l_{\max}[\psi(T)]^q}{\Gamma(q+1)} \right. \\
&\quad \left. \times \sum_{i=1}^m |\beta_i| \|\mu\| + |A_1| l_{\max}\|\mu\| \frac{[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + |B_1| l_{\max}\|\mu\| \frac{[\psi(T)]^q}{\Gamma(q+1)} \right\} \\
&\le \left( \frac{[\psi(T)]^q}{\Gamma(q+1)} + \frac{(\psi(T)-\psi(0))^2}{|A_1B_2 - A_2B_1|} \left\{ \frac{|B_2 - A_2| [\psi(T)]^q}{\Gamma(q+1)} \sum_{i=1}^m \beta_i \right. \right. \\
&\quad \left. + \frac{|A_2| [\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{|B_2| [\psi(T)]^q}{\Gamma(q+1)} \right\} + \frac{(\psi(T)-\psi(0))^3}{|A_2B_1 - A_1B_2|} \\
&\quad \left. \times \left\{ \frac{|B_1 - A_1| [\psi(T)]^q}{\Gamma(q+1)} \sum_{i=1}^m \beta_i + \frac{|A_1| [\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{|B_1| [\psi(T)]^q}{\Gamma(q+1)} \right\} \right) l_{\max}\|\mu\| \\
&< \|\mu\|.
\end{aligned}
$$

This leads to a contradiction. Therefore, 1 cannot be an eigenvalue of the operator $\mathcal{L}$.

**Step 5:** We will show that $\frac{\|\mathcal{F}\mu - \mathcal{L}\mu\|}{\|\mu\|} \to 0$ as $\|\mu\| \to \infty$. For $t \in [0,T]$, we obtain

$$
\begin{aligned}
&|\mathcal{F}\mu(t) - \mathcal{L}\mu(t)| \\
&\le I_{0^+}^{q;\psi} \left| f\left(t, \mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)\right) - l(t)\mu(t) \right| + \frac{(\psi(t)-\psi(0))^2}{|A_1B_2 - A_2B_1|} \\
&\quad \times \left\{ |B_2 - A_2| \sum_{i=1}^m |\beta_i| I_{0^+}^{q;\psi} \left| f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right) - l(\kappa_i)\mu(\kappa_i) \right| \right. \\
&\quad + |A_2| I_{0^+}^{q-\rho;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T) \right| \\
&\quad \left. + |B_2| I_{0^+}^{q;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T) \right| \right\} \\
&\quad + \frac{(\psi(t)-\psi(0))^3}{|A_2B_1 - A_1B_2|} \left\{ |B_1 - A_1| \sum_{i=1}^m |\beta_i| I_{0^+}^{q;\psi} \left| f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right) \right. \right. \\
&\quad - l(\kappa_i)\mu(\kappa_i)| + |A_1| I_{0^+}^{q-\rho;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T) \right|
\end{aligned}
$$

$$+ |B_1| I_{0^+}^{q;\psi} \left| f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T) \right| \Bigg\}.$$

So, we have

$$\begin{aligned}
\frac{\|\mathcal{F}\mu - \mathcal{L}\mu\|}{\|\mu\|} \leq{}& I_{0^+}^{q;\psi} \left| \frac{f(t, \mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t))}{\mu(t)} - l(t) \right| + \frac{(\psi(t)-\psi(0))^2}{|A_1B_2 - A_2B_1|} \\
& \times \Bigg\{ |B_2 - A_2| \sum_{i=1}^{m} |\beta_i| I_{0^+}^{q;\psi} \left| \frac{f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right) - l(\kappa_i)\mu(\kappa_i)}{\mu(t)} \right| \\
& + |A_2| I_{0^+}^{q-\rho;\psi} \left| \frac{f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T)}{\mu(t)} \right| \\
& + |B_2| I_{0^+}^{q;\psi} \left| \frac{f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T)}{\mu(t)} \right| \Bigg\} \\
& + \frac{(\psi(t)-\psi(0))^3}{|A_2B_1 - A_1B_2|} \Bigg\{ |B_1 - A_1| \sum_{i=1}^{m} |\beta_i| I_{0^+}^{q;\psi} \\
& \times \left| \frac{f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)}{\mu(t)} - \frac{l(\kappa_i)\mu(\kappa_i)}{\mu(t)} \right| \\
& + |A_1| I_{0^+}^{q-\rho;\psi} \left| \frac{f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T)}{\mu(t)} \right| \\
& + |B_1| I_{0^+}^{q;\psi} \left| \frac{f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - l(T)\mu(T)}{\mu(t)} \right| \Bigg\}.
\end{aligned}$$

As $\|\mu\| \to \infty$, it follows that $\left| \frac{f(t,\mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t))}{\mu(t)} - l(t) \right| \to 0$. Therefore, we have

$$\lim_{\|\mu\| \to \infty} \frac{\|\mathcal{F}\mu - \mathcal{L}\mu\|}{\|\mu\|} = 0.$$

Consequently, by the Krasnoselskii-Zabreiko fixed point theorem, problem (1.1) possesses at least one nontrivial solution on $[0, T]$. □

3.2. **Existence and uniqueness result via Banach contraction principle.** The existence and uniqueness of the solution to the problem (1.1) is the subject of our next investigation, which is guided by Banachs fixed point theorem.

**Theorem 3.4.** *Suppose that* $(H1)$ *holds. If*

$$\begin{cases} \frac{M}{1-N}\left(\frac{[\psi(T)]^q}{\Gamma(q+1)}+\frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|}\left\{\frac{|B_2-A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|+\frac{A_2[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}\right.\right. \\ \left.+\frac{B_2[\psi(T)]^q}{\Gamma(q+1)}\right\}+\frac{(\psi(T)-\psi(0))^3}{|A_2B_1-A_1B_2|}\left\{\frac{|B_1-A_1|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|+\frac{A_1[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}\right. \\ \left.\left.+\frac{B_1[\psi(T)]^q}{\Gamma(q+1)}\right\}\right)<1, \end{cases} \tag{3.7}$$

*then problem* (1.1) *has a unique solution.*

*Proof.* Consider the operator $\mathcal{F}:C[0,T]\to C[0,T]$, defined by (3.6). Let $\mu,\nu\in C[0,T]$. Then we have:

$$\begin{aligned}
&|\mathcal{F}\mu(t)-\mathcal{F}\nu(t)|\\
&\leq \mathcal{I}_{0^+}^{q;\psi}\left|f\left(t,\mu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)\right)-f\left(t,\nu(t),{}^C\mathcal{D}_{0^+}^{q;\psi}\nu(t)\right)\right|+\frac{(\psi(t)-\psi(0))^2}{|A_1B_2-A_2B_1|}\\
&\quad\times\left\{|B_2-A_2|\left[\sum_{i=1}^{m}|\beta_i|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right.\right.\right.\\
&\quad-\left.\left.f\left(\kappa_i,\nu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\nu(\kappa_i)\right)\right|\right]+\left[|A_2|\mathcal{I}_{0^+}^{q-\rho;\psi}\left|f\left(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right.\right.\\
&\quad-\left.f\left(T,\nu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\nu(T)\right)\right|+|B_2|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right.\\
&\quad-\left.\left.\left.f\left(T,\nu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\nu(T)\right)\right|\right]\right\}+\frac{(\psi(t)-\psi(0))^3}{|A_2B_1-A_1B_2|}\\
&\quad\times\left\{|B_1-A_1|\left[\sum_{i=1}^{m}|\beta_i|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(\kappa_i,\mu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right.\right.\right.\\
&\quad-\left.\left.f\left(\kappa_i,\nu(\kappa_i),{}^C\mathcal{D}_{0^+}^{q;\psi}\nu(\kappa_i)\right)\right|\right]+\left[|A_1|\mathcal{I}_{0^+}^{q-\rho;\psi}\left|f\left(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right.\right.\\
&\quad-\left.f\left(T,\nu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\nu(T)\right)\right|+|B_1|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(T,\mu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right.\\
&\quad-\left.\left.\left.f\left(T,\nu(T),{}^C\mathcal{D}_{0^+}^{q;\psi}\nu(T)\right)\right|\right]\right\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|\mathcal{F}\mu(t)-\mathcal{F}\nu(t)\|\\
&\leq\frac{M}{1-N}\left(\frac{[\psi(T)]^q}{\Gamma(q+1)}+\frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|}\left\{\frac{|B_2-A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|\right.\right.
\end{aligned}$$

$$+ \frac{A_2[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{B_2[\psi(T)]^q}{\Gamma(q+1)}\Bigg\} + \frac{(\psi(T)-\psi(0))^3}{|A_2B_1 - A_1B_2|}\Bigg\{\frac{|B_1 - A_1|[\psi(T)]^q}{\Gamma(q+1)}$$

$$\times \sum_{i=1}^{m}|\beta_i| + \frac{A_1[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{B_1[\psi(T)]^q}{\Gamma(q+1)}\Bigg\}\Bigg) \|\mu - \nu\|.$$

Therefore, $\mathcal{F}$ is contraction. Consequently, by the Banach Contraction Principle, the operator $\mathcal{F}$ possesses a unique fixed point, which corresponds to the unique solution of problem (1.1). □

3.3. **Stability results.** This subsection of the paper is dedicated to presenting stability results for the proposed problem.

**Theorem 3.5.** *Consider a function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f \in C[0,T]$ for all $\mu \in C[0,T]$, and there exist positive constants $\mathcal{C}_{UH} > 0$ for each, satisfying the following inequalities:*

$$\begin{cases} \Big|\mu(t) - \mathcal{I}_{0^+}^{q;\psi} f\left(t, \mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)\right) \\ -\frac{(\psi(t)-\psi(0))^2}{A_1B_2 - A_2B_1}\Bigg\{(B_2 - A_2)\left[\sum_{i=1}^{m}\beta_i \mathcal{I}_{0^+}^{q;\psi} f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right] \\ -\left[A_2\mathcal{I}_{0^+}^{q-\rho;\psi} f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - B_2\mathcal{I}_{0^+}^{q;\psi} f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right]\Bigg\} \\ -\frac{(\psi(t)-\psi(0))^3}{A_2B_1 - A_1B_2}\Bigg\{(B_1 - A_1)\left[\sum_{i=1}^{m}\beta_i \mathcal{I}_{0^+}^{q;\psi} f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right] \\ -\left[A_1\mathcal{I}_{0^+}^{q-\rho;\psi} f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right) - B_1\mathcal{I}_{0^+}^{q;\psi} f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right]\Bigg\}\Big| \\ < \epsilon^*. \end{cases}$$

*Moreover, assume that inequality (3.7) holds, where $\epsilon = \epsilon^*$. If there exists a solution $\mu^* \in C[0,T]$ of problem (1.1) such that*

$$|\mu(t) - \mu^*(t)| \leq \mathcal{C}_{UH}\epsilon$$

*for $t \in [0,T]$, then the problem (1.1) is Ulam-Hyers stable.*

*Proof.* In light of Theorem 3.4, let $\mu \in C[0,T]$ be the unique solution of (1.1), given by equation (3.1), and let $\mu^* \in C[0,T]$ satisfy equation (3.7), which is

$$\mu^*(t) = \mathcal{I}_{0^+}^{q;\psi} f\left(t, \mu^*(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(t)\right)$$
$$+ \frac{(\psi(t)-\psi(0))^2}{A_1B_2 - A_2B_1}\Bigg\{(B_2 - A_2)\left[\sum_{i=1}^{m}\beta_i \mathcal{I}_{0^+}^{q;\psi} f\left(\kappa_i, \mu^*(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(\kappa_i)\right)\right]$$
$$+ \Big[A_2\mathcal{I}_{0^+}^{q-\rho;\psi} f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right)$$

$$
\begin{aligned}
&- B_2\mathcal{I}_{0^+}^{q;\psi} f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right) \\
&+ \frac{(\psi(t)-\psi(0))^3}{A_2B_1 - A_1B_2}\left\{(B_1 - A_1)\left[\sum_{i=1}^{m}\beta_i\mathcal{I}_{0^+}^{q;\psi} f\left(\kappa_i, \mu^*(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(\kappa_i)\right)\right]\right. \\
&+ \left[A_1\mathcal{I}_{0^+}^{q-\rho;\psi} f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right)\right. \\
&\left.\left. -B_1\mathcal{I}_{0^+}^{q;\psi} f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right)\right]\right\}.
\end{aligned}
$$

By the above equation, we have

$$
\begin{aligned}
&|\mu(t) - \mu^*(t)| \\
&\leq \epsilon^* + \mathcal{I}_{0^+}^{q;\psi}\left|f\left(t, \mu(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(t)\right) - f\left(t, \mu^*(t), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(t)\right)\right| \\
&\quad + \frac{(\psi(t)-\psi(0))^2}{|A_1B_2 - A_2B_1|}\left\{|B_2 - A_2|\left[\sum_{i=1}^{m}|\beta_i|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right.\right.\right. \\
&\quad - \left.\left. f\left(\kappa_i, \mu^*(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(\kappa_i)\right)\right|\right] + \left[|A_2|\mathcal{I}_{0^+}^{q-\rho;\psi}\left|f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right.\right. \\
&\quad - \left. f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right)\right| + |B_2|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right. \\
&\quad - \left.\left.\left. f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right)\right|\right]\right\} + \frac{(\psi(t)-\psi(0))^3}{|A_2B_1 - A_1B_2|} \\
&\quad \times \left\{|B_1 - A_1|\left[\sum_{i=1}^{m}|\beta_i|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(\kappa_i, \mu(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(\kappa_i)\right)\right.\right.\right. \\
&\quad - \left.\left. f\left(\kappa_i, \mu^*(\kappa_i), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(\kappa_i)\right)\right|\right] + \left[|A_1|\mathcal{I}_{0^+}^{q-\rho;\psi}\left|f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right.\right. \\
&\quad - \left. f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right)\right| + |B_1|\mathcal{I}_{0^+}^{q;\psi}\left|f\left(T, \mu(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu(T)\right)\right. \\
&\quad - \left.\left.\left. f\left(T, \mu^*(T), {}^C\mathcal{D}_{0^+}^{q;\psi}\mu^*(T)\right)\right|\right]\right\} \\
&\leq \epsilon^* + \frac{M}{1-N}\left(\frac{[\psi(T)]^q}{\Gamma(q+1)} + \frac{(\psi(T)-\psi(0))^2}{|A_1B_2 - A_2B_1|}\left\{\frac{|B_2 - A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|\right.\right. \\
&\quad + \left.\frac{A_2[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{B_2[\psi(T)]^q}{\Gamma(q+1)}\right\} + \frac{(\psi(T)-\psi(0))^3}{|A_2B_1 - A_1B_2|}\left\{\frac{|B_1 - A_1|[\psi(T)]^q}{\Gamma(q+1)}\right. \\
&\quad \times \left.\left.\sum_{i=1}^{m}|\beta_i| + \frac{A_1[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{B_1[\psi(T)]^q}{\Gamma(q+1)}\right\}\right)\|\mu - \mu^*\|.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&\|\mu-\mu^*\| \\
&\leq \epsilon^*/\Bigg(1-\frac{M}{1-N}\Bigg(\frac{[\psi(T)]^q}{\Gamma(q+1)}+\frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|}\Bigg\{\frac{|B_2-A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^m|\beta_i| \\
&\quad+\frac{A_2[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}+\frac{B_2[\psi(T)]^q}{\Gamma(q+1)}\Bigg\}+\frac{(\psi(T)-\psi(0))^3}{|A_2B_1-A_1B_2|}\Bigg\{\frac{|B_1-A_1|[\psi(T)]^q}{\Gamma(q+1)} \\
&\quad\times\sum_{i=1}^m|\beta_i|+\frac{A_1[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}+\frac{B_1[\psi(T)]^q}{\Gamma(q+1)}\Bigg\}\Bigg)\Bigg).
\end{aligned}
\tag{3.8}
$$

Upon simplification (3.8) yields

$$\|\mu-\mu^*\|\leq \mathcal{C}_{UH}\epsilon,$$

where

$$
\begin{aligned}
\mathcal{C}_{UH}&=1/\Bigg(1-\frac{M}{1-N}\Bigg(\frac{[\psi(T)]^q}{\Gamma(q+1)}+\frac{(\psi(T)-\psi(0))^2}{|A_1B_2-A_2B_1|}\Bigg\{\frac{|B_2-A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^m|\beta_i| \\
&\quad+\frac{A_2[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}+\frac{B_2[\psi(T)]^q}{\Gamma(q+1)}\Bigg\}+\frac{(\psi(T)-\psi(0))^3}{|A_2B_1-A_1B_2|}\Bigg\{\frac{|B_1-A_1|[\psi(T)]^q}{\Gamma(q+1)} \\
&\quad\times\sum_{i=1}^m|\beta_i|+\frac{A_1[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}+\frac{B_1[\psi(T)]^q}{\Gamma(q+1)}\Bigg\}\Bigg)\Bigg).
\end{aligned}
$$

Hence, the solution to problem (1.1) exhibits Ulam-Hyers stability. Furthermore, if there exists a function $\phi:[0,T]\to(0,\infty)$ such that $\phi(\epsilon)=\mathcal{C}_{UH}\epsilon$ with $\phi(0)=0$, then from (3.8), we obtain

$$\|\mu-\mu^*\|\leq\phi(\epsilon)\quad\text{for all}\quad t\in[0,T].$$

Thus, problem (1.1) demonstrates generalized Ulam-Hyers stability. □

Next, we give an example to support our results.

**Example 3.6.** Consider the following $\psi$-Caputo fractional differential equation and mixed conditions

$$
\begin{cases}
{}^C\mathcal{D}_{0^+}^{\frac{7}{2};t^3}\mu(t)=\dfrac{\sin^3(t)}{(999+e^t)}\frac{|\mu(t)|}{1+|\mu(t)|}+\frac{\cos^3(t)}{(9+e^t)}\dfrac{\left|{}^C\mathcal{D}_{0^+}^{\frac{7}{2};t^3}\mu(t)\right|}{1+\left|{}^C\mathcal{D}_{0^+}^{\frac{7}{2};t^3}\mu(t)\right|}+\frac{\sqrt{3}}{2},\ t\in[0,1],\\
\mu(0)=0,\ \mu(T)=\frac{1}{2}\mu\left(\frac{\pi}{6}\right)+\frac{1}{3}\mu\left(\frac{\pi}{4}\right)+\frac{1}{4}\mu\left(\frac{\pi}{3}\right),\\
{}^C\mathcal{D}_{0^+}^{\frac{3}{2};t^3}\mu(0)=0,{}^C\mathcal{D}_{0^+}^{\frac{7}{2};t^3}\mu(T)=\frac{1}{2}\mu\left(\frac{\pi}{6}\right)+\frac{1}{3}\mu\left(\frac{\pi}{4}\right)+\frac{1}{4}\mu\left(\frac{\pi}{3}\right).
\end{cases}
\tag{3.9}
$$

By comparing problem (1.1) and (3.9), we obtain: $q = 7/2,\ \rho = 3/2,\ \psi(t) = t^3,\ \beta_1 = 1/2,\ \beta_2 = 1/3,\ \beta_3 = 1/4,\ \kappa_1 = \pi/6,\ \kappa_2 = \pi/4,\ \kappa_3 = \pi/3,\ T = 1,$ and

$$f(t,\mu(t),\nu(t)) = \frac{\sin^3(t)}{(999+e^t)}\frac{|\mu(t)|}{1+|\mu(t)|} + \frac{\cos^3(t)}{(9+e^t)}\frac{|\nu(t)|}{1+|\nu(t)|} + \frac{\sqrt{3}}{2}$$

for any $\mu, \nu \in \mathbb{R}$. This implies that the function $f$ satisfies the hypotheses $(H1)$ of Theorem 3.3. Thus,

$$|f(t,\mu,\nu) - f(t,\mu^*,\nu^*)| \leq \frac{1}{1,000}|\mu-\mu^*| + \frac{1}{10}|\nu-\nu^*|$$

for any $\mu, \nu, \mu^*, \nu^* \in \mathbb{R},\ t \in [0,1]$ with $M = 1/1,000,\ N = 1/10$.

Hence, by simple calculation, we can observe that

$$\begin{aligned}\frac{M}{1-N}&\Bigg(\frac{[\psi(T)]^q}{\Gamma(q+1)} + \frac{(\psi(T)-\psi(0))^2}{|A_1B_2 - A_2B_1|}\left\{\frac{|B_2 - A_2|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i| + \frac{A_2[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)}\right.\\ &+ \left.\frac{B_2[\psi(T)]^q}{\Gamma(q+1)}\right\} + \frac{(\psi(T)-\psi(0))^3}{|A_2B_1 - A_1B_2|}\left\{\frac{|B_1 - A_1|[\psi(T)]^q}{\Gamma(q+1)}\sum_{i=1}^{m}|\beta_i|\right.\\ &+ \left.\frac{A_1[\psi(T)]^{q-\rho}}{\Gamma(q-\rho+1)} + \frac{B_1[\psi(T)]^q}{\Gamma(q+1)}\right\}\Bigg) \approx 0.9154.\end{aligned}$$

By Theorem 3.4, it follows that problem (1.1) has a unique solution. Moreover, as shown in Theorem 3.5, for every $\epsilon > 0$, if $\mu \in C[0,1]$ satisfies

$$\left| {}^C\mathcal{D}_{0^+}^{\frac{7}{2};t^3}\mu(t) - \frac{\sin^3(t)}{(999+e^t)}\frac{|\mu(t)|}{1+|\mu(t)|} - \frac{\cos^3(t)}{(9+e^t)}\frac{\left|{}^C\mathcal{D}_{0^+}^{\frac{7}{2};t^3}\mu(t)\right|}{1+\left|{}^C\mathcal{D}_{0^+}^{\frac{7}{2};t^3}\mu(t)\right|} - \frac{\sqrt{3}}{2}\right| \leq \epsilon,$$

then there exists a unique solution $\mu^* \in C[0,1]$ such that

$$\|\mu - \mu^*\| \leq \mathcal{C}_{CH}\epsilon, \tag{3.10}$$

where $\mathcal{C}_{CH} \approx 11.8203 > 0$. Thus, the solution is Ulam-Hyers stable. Therefore, the solution to problem (1.1) is Ulam-Hyers stable. Furthermore, let the function $\phi : [0,1] \to \mathbb{R}$ be defined $\phi(\epsilon) = \mathcal{C}_{UH}\epsilon$ with $\phi(0) = 0$. Then, from (3.10), we have

$$\|\mu - \mu^*\| \leq \phi(\epsilon)$$

for all $t \in [0,1]$. Hence, the problem (1.1) is also generalized Ulam-Hyers stable.

## 4. Conclusions

In this article, we have established the existence and uniqueness of solutions for implicit $\psi$-Caputo fractional differential equations with mixed conditions utilizing the fixed point theorems of Krasnoselskii-Zabreiko and the Banach contraction principle. Additionally, we have investigated the criteria for Ulam-Hyers stability and generalized Ulam-Hyers stability. To demonstrate the practical applicability of our results, we have included a comprehensive example.

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