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OPTIMIZED STRATEGIES FOR OPEN-LOOP NASH EQUILIBRIUM IN DIFFERENTIAL GAMES USING THE PICARD ITERATION METHOD

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Abstract. This study explores the application of the Picard iteration method to derive strategies for open-loop Nash equilibrium (OLNE) in differential games, particularly in a competitive market scenario involving three companies. By integrating the dynamics of differential equations with game theory, this study presents a methodological approach for approximating solutions to complex strategic interactions. Through comparative analysis and visual representations, the paper introduces a novel theorem to establish essential conditions for OLNE. This work provides valuable insights into competitive strategies, dynamic systems, and market analysis, thereby enhancing the understanding of differential games and demonstrating the effectiveness of the Picard iteration method in solving non-linear integro-differential equations within strategic frameworks.

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1. Introduction

Differential equations are fundamental in numerous disciplines, including physics and engineering [14], playing a crucial role in modeling various phenomena where change rates are pivotal. Such phenomena include population dynamics, chemical reactions, space exploration, disease spread, and climate variations. Within this realm, the differential game emerges as an application bridging differential equations and game theory. Game theory, a significant mathematical domain [6], finds its relevance across social sciences, logic, systems science, and computer science, notably impacting economics. A key figure in this field, John Forbes Nash, contributed significantly to game theory, especially in conceptualizing solutions for non-cooperative games involving multiple players. In these games, each participant is presumed to understand the equilibrium strategies of their counterparts, implying a mutual awareness of strategies among players. Differential games, as a subset of game theory, focus on conflicts or competitions represented through dynamic systems [1], where the evolution of state variables is governed by differential equations. These variables effectively encapsulate a systems characteristics, determining its future behavior in the absence of external influences.

In [2], Hegazy et al. explored the resolution of minmax zero-sum two-person continuous differential games with fuzzy controls and state trajectories. The study of differential games extends to various real-life applications [5, 9, 10, 11, 12, 13]. Their work delineated the necessary conditions for such games and provided a numerical example for illustration. Hemeda, in [3], introduced an integral iterative method (IIM), an enhancement over the Picard method (PM), to address nonlinear integro-differential equations and their systems. Another notable contribution is by Joseph in [4], who investigated a duopolistic market scenario where two firms engage in competitive selling over a defined period. Here, each firm's market share and advertising efforts constitute their strategic elements. Joseph applied the principle of maximum along with theorems on general inequality constraints and numerical techniques to develop solutions.

In our research, we take an alternate route, using a distinct theorem to determine the essential conditions for an open-loop Nash equilibrium in a differential game. Our approach involves using the Picard technique for approximating solutions, creating charts for a comparative study among three companies, and visually illustrating these comparisons.

This document is organized in the following manner: The second section introduces the dynamics of the system, the structure of payoff functionals, and the necessary conditions for an open-loop Nash equilibrium. The third section is dedicated to deriving a proximate solution through the Picard technique.

The fourth section offers a discussion on the results, contrasting them with earlier research, while the fifth and final section provides the conclusion of the study.

2. Problem formulation

In this segment, we delve into the dynamics governing the system, define the payoff functionals, and elaborate on the concept of OLNE.

2.1. **The Dynamical System.** We examine a competitive market scenario involving three entities: Enterprise A, Enterprise B, and Enterprise C. These entities are engaged in selling an identical product in a competitive marketplace [4]. Denote the market share of Enterprise A at time t as $x(t)$, that of Enterprise B as $y(t)$, and that of Enterprise C as $1 - x(t) - y(t)$. The control variables are characterized as follows: $u_1(t)$ represents the advertising effort exerted by Enterprise A at time t, $u_2(t)$ for Enterprise B, and $u_3(t)$ for Enterprise C. The growth in the customer base of Enterprise A is directly influenced by its advertising efforts. Conversely, the advertising initiatives of Enterprises B and C not only contribute to their growth but also potentially reduce the customer base of Enterprise A and each other. The dynamics of this system are thus formulated as:

$$\dot{x}(t) = u_1(t)(1 - x(t) - y(t)) - u_2(t)x(t) - u_3(t)x(t), \tag{2.1}$$

$$\dot{y}(t) = u_2(t)(1 - x(t) - y(t)) - u_1(t)y(t) - u_3(t)y(t) \tag{2.2}$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad t \in [0, T], \tag{2.3}$$

subject to the constraints

$$0 \leq x(t) + y(t) \leq 1. \tag{2.4}$$

Given the context of a differential game involving three participants, we introduce the following definition.

Definition 2.1. (Three-Player Differential Game) In a differential game encompassing three players over a time span $[t_0, t_f]$, the following elements are present:

(1) A set of players, designated as $N = \{1, 2, 3\}$.
(2) For each participant i within N, there is a group of control vectors $u_i(t)$ within $U_i \subseteq \mathbb{R}^{n_i}$, with U_i symbolizing the allowed control inputs for participant i.
(3) A duo of state variables $[x, y]$ within $X \subseteq \mathbb{R}^n$, where X denotes the set of allowable states.

(4) A specific strategy collection Ψ_i for each player, where the strategy ψ_i within Ψ_i is a rule for decision-making that establishes the control $u_i(t)$ from U_i, relying on the data accessible at time t.

2.2. **Payoff functionals and OLNE.** We now turn our attention to the state equation representing the games state and the associated payoff functionals. These are defined as

$$\dot{x} = f(x(t), y(t), u_1(t), u_2(t), u_3(t), t), \tag{2.5}$$

$$\dot{y} = g(x(t), y(t), u_1(t), u_2(t), u_3(t), t) \tag{2.6}$$

and for the payoff functionals,

$$J_i(u_1(t), u_2(t), u_3(t)) = \int_{t_0}^{t_f} I_i(x(t), y(t), u_1(t), u_2(t), u_3(t), t)\, dt, \;\; i = 1, 2, 3. \tag{2.7}$$

In this scenario, as the information structure is open-loop, each player's equilibrium strategy is denoted as $u_i^*(t)$ over the interval $t \in [t_0, t_f]$ for $i = 1, 2, 3$.

To derive these strategies, we introduce the Hamiltonian function H_i for each player i, given by

$$\begin{aligned} H_i(\lambda_i, u_1, u_2, u_3, t) = {} & I_i(x(t), y(t), u_1(t), u_2(t), u_3(t), t) \\ & + \lambda_i^T \left(f(x, y, u_1, u_2, u_3, t) + g(x, y, u_1, u_2, u_3, t) \right), \end{aligned} \tag{2.8}$$

$i = 1, 2, 3$, where λ_i is the co-state vector for player i.

Definition 2.2. In the context of the three-player differential game outlined in Definition 2.1, occurring over the interval $[t_0, t_f]$, the informational framework for a player i is considered open-loop when, at any moment t, the only information player i can access is the games initial state x_0. Consequently, the strategy set for player i is expressed as $\Psi_i(t) = x_0$ for $t \in [t_0, t_f]$.

Definition 2.3. Consider the cost functions:

$$J_1(u_1(t), u_2(t), u_3(t)), J_2(u_1(t), u_2(t), u_3(t)) \;\text{ and }\; J_3(u_1(t), u_2(t), u_3(t))$$

for players 1, 2, and 3, respectively. A set of control strategies (u_1^*, u_2^*, u_3^*) constitutes a Nash equilibrium strategy if, for each player i, where $i = 1, 2, 3$, the condition $J_i(u_1^*, u_2^*, u_3^*) \leq J_i(u_1, u_2^*, u_3^*)$ holds true.

In simpler terms, the Nash equilibrium implies that no player can improve their optimization criterion by unilaterally changing their strategy.

We can now define the payoff functionals for the three firms in our problem as follows:

$$J_1 = \int_0^T e^{-r_1 t}\left[\phi_1 x(t) - c_1 u_1(t)\right] dt, \tag{2.9}$$

$$J_2 = \int_0^T e^{-r_2 t}\left[\phi_2 y(t) - c_2 u_2(t)\right] dt, \tag{2.10}$$

$$J_3 = \int_0^T e^{-r_3 t}\left[\phi_3 (1 - x(t) - y(t)) - c_3 u_3(t)\right] dt, \tag{2.11}$$

where the interval $[t_0, t_f] = [0, T]$, r_i is the interest rate for Firm i, ϕ_i represents the fractional revenue potential for Firm i, and $c_i(s)$ is the advertising cost function for the three firms. We assume $c_i(s) = \frac{k_i s^2}{2}$, where k_i is a positive constant. The functions f and I_i are defined as

$$\begin{aligned} f(x(t), y(t), u_1(t), u_2(t), u_3(t), t) &= u_1(t)(1 - x(t) - y(t)) \\ &\quad - u_2(t)x(t) - u_3(t)x(t), \end{aligned} \tag{2.12}$$

$$\begin{aligned} g(x(t), y(t), u_1(t), u_2(t), u_3(t), t) &= u_2(t)(1 - x(t) - y(t)) \\ &\quad - u_1(t)y(t) - u_3(t)y(t), \end{aligned} \tag{2.13}$$

$$I_1(x(t), u_1(t), u_2(t), u_3(t), t) = \phi_1 x(t) - c_1(u_1(t)), \tag{2.14}$$

$$I_2(x(t), u_1(t), u_2(t), u_3(t), t) = \phi_2 y(t) - c_2(u_2(t)), \tag{2.15}$$

$$I_3(x(t), u_1(t), u_2(t), u_3(t), t) = \phi_3(1 - x(t) - y(t)) - c_3(u_3(t)). \tag{2.16}$$

The Hamiltonian functions for the players are then given by

$$H_1(\lambda_1, x, y, u_1, u_2, u_3, t) = \phi_1 x(t) - \frac{k_1 u_1^2}{2} + \lambda_1[f + g], \tag{2.17}$$

$$H_2(\lambda_2, x, y, u_1, u_2, u_3, t) = \phi_2 y(t) - \frac{k_2 u_2^2}{2} + \lambda_2[f + g], \tag{2.18}$$

$$H_3(\lambda_3, x, y, u_1, u_2, u_3, t) = \phi_3(1 - x(t) - y(t)) - \frac{k_3 u_3^2}{2} + \lambda_3[f + g]. \tag{2.19}$$

The following theorem captures the complex dynamics and strategic interactions among three competing firms in a market, each employing strategic control actions to optimize its respective payoff.

Theorem 2.4. (Open-Loop Nash Equilibrium Conditions) *Given that* $f(x(t), y(t), u_1(t), u_2(t), u_3(t), t)$ *and* $g(x(t), y(t), u_1(t), u_2(t), u_3(t), t)$ *describe the market share dynamics of firms* 1 *and* 2*, respectively, and assuming*

$$I_i(x(t), y(t), u_1(t), u_2(t), u_3(t), t),$$

for $i = 1, 2, 3$ *are continuously differentiable over* $\mathbb{R}^n$*, that is,* $f, g : \mathbb{R}^n \times \mathbb{R}^s \times [0, T] \to \mathbb{R}$ *with* $f, g \in C^1$*,* $s = \sum_{j=1}^3 s_j$*, and* $I_i \in C^1$ *for* $i = 1, 2, 3$*. If*

$u_i^*(t)$, $t_0 \leq t \leq t_f$ are open-loop Nash equilibrium strategies and $x^*(t)$, $y^*(t)$, $t_0 \leq t \leq t_f$ are the corresponding state trajectories, then there exist three costate vectors $\lambda_i : [t_0, t_f] \to \mathbb{R}^n$, and three Hamiltonian functions:

$$\begin{aligned} H_i(\lambda_i, x, y, u_1, u_2, u_3, t) = I_i(x(t), y(t), u_1(t), u_2(t), u_3(t), t) \\ + \lambda_i^T f(x, y, u_1, u_2, u_3, t) \\ + \lambda_i^T g(x, y, u_1, u_2, u_3, t) \end{aligned} \tag{2.20}$$

such that the following conditions are met:

(1)

$$\dot{x}^* = f(x^*(t), y^*(t), u_1^*(t), u_2^*(t), u_3^*(t), t),$$

$$\dot{y}^* = g(x^*(t), y^*(t), u_1^*(t), u_2^*(t), u_3^*(t), t),$$

$$x^*(0) = x_0, \quad y^*(0) = y_0.$$

(2) *The dynamics of the costate vectors are defined by:*

$$\begin{aligned} \dot{\lambda}_i(t) = &- \frac{\partial H_i(\lambda_i, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial x} \\ &- \frac{\partial H_i(\lambda_i, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial y} \end{aligned} \tag{2.21}$$

for $i = 1, 2, 3$.

(3) *The optimality conditions for the control strategies are:*

$$\frac{\partial H_1(\lambda_1, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial u_1} = 0, \tag{2.22}$$

$$\frac{\partial H_2(\lambda_2, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial u_2} = 0, \tag{2.23}$$

$$\frac{\partial H_3(\lambda_3, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial u_3} = 0 \tag{2.24}$$

with boundary conditions for the initial states $x^*(0) = x_0$, $y^*(0) = y_0$, and terminal conditions for the costate vectors $\lambda_i(t_f) = 0$ for $i = 1, 2, 3$.

Proof. Existence of State Trajectories: By the definition of the state trajectories, we have

$$\dot{x}^* = f(x^*(t), y^*(t), u_1^*(t), u_2^*(t), u_3^*(t), t), \tag{2.25}$$

$$\dot{y}^* = g(x^*(t), y^*(t), u_1^*(t), u_2^*(t), u_3^*(t), t) \tag{2.26}$$

with initial conditions $x^*(0) = x_0$, $y^*(0) = y_0$.

Costate Equations: For each player i, the Hamiltonian H_i is given by

$$H_i(\lambda_i, x, y, u_1, u_2, u_3, t) = I_i(x(t), y(t), u_1(t), u_2(t), u_3(t), t) + \lambda_i^T f(x, y, u_1, u_2, u_3, t) + \lambda_i^T g(x, y, u_1, u_2, u_3, t). \tag{2.27}$$

The dynamics of the costate vectors λ_i are

$$\dot{\lambda}_i(t) = -\frac{\partial H_i(\lambda_i, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial x} - \frac{\partial H_i(\lambda_i, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial y}. \tag{2.28}$$

Optimality Conditions: The optimality conditions require that

$$\frac{\partial H_1(\lambda_1, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial u_1} = 0, \tag{2.29}$$

$$\frac{\partial H_2(\lambda_2, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial u_2} = 0, \tag{2.30}$$

$$\frac{\partial H_3(\lambda_3, x^*, y^*, u_1^*, u_2^*, u_3^*, t)}{\partial u_3} = 0. \tag{2.31}$$

Boundary Conditions: The boundary conditions are given by

$$x^*(0) = x_0, \quad y^*(0) = y_0, \quad \lambda_i(t_f) = 0, \quad \text{for } i = 1, 2, 3. \tag{2.32}$$

These conditions together ensure that the strategies $u_i^*(t)$ for $i = 1, 2, 3$ form an open-loop Nash equilibrium for the differential game, as they satisfy the necessary conditions for optimality and state evolution. □

3. Implementing the Picard technique for approximate solutions

In this section, we investigate the solution's existence for the system outlined in equations (2.21) to (2.24) as per [7, 8]. We employ the Picard method to ascertain an approximate solution and examine its convergence. The system, simplified from the necessary conditions of an OLNE game, is represented as

$$\dot{x} = f_1(x, y, \lambda_1, \lambda_2, \lambda_3, t), \quad \dot{y} = f_2(x, y, \lambda_1, \lambda_2, \lambda_3, t), \tag{3.1}$$

$$\dot{\lambda}_1 = f_3(x, y, \lambda_1, \lambda_2, \lambda_3, t) - \phi_1, \tag{3.2}$$

$$\dot{\lambda}_2 = f_4(x, y, \lambda_1, \lambda_2, \lambda_3, t) - \phi_2, \tag{3.3}$$

$$\dot{\lambda}_3 = f_5(x, y, \lambda_1, \lambda_2, \lambda_3, t) + 2\phi_3 \tag{3.4}$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \tag{3.5}$$

$$\lambda_1(T) = 0, \quad \lambda_2(T) = 0, \quad \lambda_3(T) = 0. \tag{3.6}$$

Under these assumptions for our problem:

(1) Functions $f_i(x(t), y(t), \lambda_1(t), \lambda_2(t), \lambda_3(t), t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \to \mathbb{R}$ are continuous, with positive constants M_i such that $|f_i| \leq M_i$, for $i = 1, 2, 3, 4, 5$.

(2) These functions f_i fulfill the Lipschitz condition with constants L_i, where $0 < L_i < 1$, for $i = 1, 2, 3, 4, 5$ as follows:

$$|f_1(x, y, \lambda_1, \lambda_2, \lambda_3, t) - f_2(x', y, \lambda_1, \lambda_2, \lambda_3, t)| \leq L_1|x - x'|, \tag{3.7}$$

$$|f_2(x, y, \lambda_1, \lambda_2, \lambda_3, t) - f_1(x, y', \lambda_1, \lambda_2, \lambda_3, t)| \leq L_2|y - y'|, \tag{3.8}$$

$$|f_3(x, y, \lambda_1, \lambda_2, \lambda_3, t) - f_2(x, y, p_1, \lambda_2, \lambda_3, t)| \leq L_3|\lambda_1 - p_1|, \tag{3.9}$$

$$|f_4(x, y, \lambda_1, \lambda_2, \lambda_3, t) - f_3(x, y, \lambda_1, p_2, \lambda_3, t)| \leq L_4|\lambda_2 - p_2|, \tag{3.10}$$

$$|f_5(x, y, \lambda_1, \lambda_2, \lambda_3, t) - f_4(x, y, \lambda_1, \lambda_2, p_3, t)| \leq L_5|\lambda_3 - p_3|. \tag{3.11}$$

To demonstrate the existence of the solution for the system (3.1)-(3.6), we integrate equations (3.1), (3.2), (3.3), and (3.4) yielding:

$$x(t) = x_0 + \int_0^t f_1(x, y, \lambda_1, \lambda_2, \lambda_3, t)\, dt, \tag{3.12}$$

$$y(t) = y_0 + \int_0^t f_2(x, y, \lambda_1, \lambda_2, \lambda_3, t)\, dt, \tag{3.13}$$

$$\lambda_1(t) = -\phi_1(t - T) + \int_t^T f_3(x, \lambda_1, \lambda_2, \lambda_3, t)\, dt, \tag{3.14}$$

$$\lambda_2(t) = -\phi_2(t - T) + \int_t^T f_4(x, \lambda_1, \lambda_2, \lambda_3, t)\, dt, \tag{3.15}$$

$$\lambda_3(t) = 2\phi_3(t - T) + \int_t^T f_5(x, \lambda_1, \lambda_2, \lambda_3, t)\, dt. \tag{3.16}$$

Upon differentiating the integral forms in equations (3.12) to (3.16), we derive:

$$\dot{x} = f_1(x, y, \lambda_1, \lambda_2, \lambda_3, t), \tag{3.17}$$

$$\dot{y} = f_2(x, y, \lambda_1, \lambda_2, \lambda_3, t), \tag{3.18}$$

$$\dot{\lambda}_1 = f_3(x, y, \lambda_1, \lambda_2, \lambda_3, t) - \phi_1, \tag{3.19}$$

$$\dot{\lambda}_2 = f_4(x, y, \lambda_1, \lambda_2, \lambda_3, t) - \phi_2, \tag{3.20}$$

$$\dot{\lambda}_3 = f_5(x, y, \lambda_1, \lambda_2, \lambda_3, t) + 2\phi_3. \tag{3.21}$$

Inserting $t = 0$ in equations (3.12) and (3.13) and $t = T$ in equations (3.14), (3.15), and (3.16) yields:

$$x(0) = x_0 + \int_0^0 f_1(x, y, \lambda_1, \lambda_2, \lambda_3, t)\, dt = x_0, \tag{3.22}$$

$$y(0) = y_0 + \int_0^0 f_2(x, y, \lambda_1, \lambda_2, \lambda_3, t)\, dt = y_0, \tag{3.23}$$

$$\lambda_1(T) = -\phi_1(T - T) + \int_T^T f_3(x, y, \lambda_1, \lambda_2, \lambda_3, t)\, dt = 0, \tag{3.24}$$

$$\lambda_2(T) = -\phi_2(T - T) + \int_T^T f_4(x, y, \lambda_1, \lambda_2, \lambda_3, t)\, dt = 0, \tag{3.25}$$

$$\lambda_3(T) = 2\phi_3(T - T) + \int_T^T f_5(x, y, \lambda_1, \lambda_2, \lambda_3, t)\, dt = 0. \tag{3.26}$$

The equivalence of the system (3.1)-(3.6) with these integral equations (3.12)-(3.16) confirms the existence of a solution.

Applying the Picard method to the integral equations (3.12)-(3.16), we construct solutions as sequences:

$$x_n(t) = x_0 + \int_0^t f_1(x_{n-1}, y_{n-1}, \lambda_{1,n-1}, \lambda_{2,n-1}, \lambda_{3,n-1}, t)\, dt,\ x(0) = x_0, \tag{3.27}$$

$$y_n(t) = y_0 + \int_0^t f_2(x_{n-1}, y_{n-1}, \lambda_{1,n-1}, \lambda_{2,n-1}, \lambda_{3,n-1}, t)\, dt,,\ y(0) = y_0, \tag{3.28}$$

$$\lambda_{1,n}(t) = \phi_1(t - T) + \int_t^T f_3(x_{n-1}, y_{n-1}, \lambda_{1,n-1}, \lambda_{2,n-1}, \lambda_{3,n-1}, t)\, dt,\ \lambda_{1,0} = 0, \tag{3.29}$$

$$\lambda_{2,n}(t) = -\phi_2(t - T) + \int_t^T f_4(x_{n-1}, y_{n-1}, \lambda_{1,n-1}, \lambda_{2,n-1}, \lambda_{3,n-1}, t)\, dt,\ \lambda_{2,0} = 0, \tag{3.30}$$

$$\lambda_{3,n}(t) = \phi_3(t - T) + \int_t^T f_5(x_{n-1}, y_{n-1}, \lambda_{1,n-1}, \lambda_{2,n-1}, \lambda_{3,n-1}, t)\, dt,\ \lambda_{3,0} = 0 \tag{3.31}$$

for $n = 1, 2, 3, \ldots$.

If the sequences $\{x_n(t)\}, \{y_n(t)\}, \{\lambda_{1,n}(t)\}, \{\lambda_{2,n}(t)\}, \{\lambda_{3,n}(t)\}$ converge, then the infinite series for $(x_j - x_{j-1}), (y_j - y_{j-1}), (\lambda_{1,j} - \lambda_{1,j-1}), (\lambda_{2,j} - \lambda_{2,j-1})$, and $(\lambda_{3,j} - \lambda_{3,j-1})$ are convergent. Hence, the solutions will be $x, y, \lambda_1, \lambda_2$, and λ_3, where

$$x(t) = \lim_{n\to\infty} x_n(t), \tag{3.32}$$

$$y(t) = \lim_{n\to\infty} y_n(t), \tag{3.33}$$

$$\lambda_1(t) = \lim_{n\to\infty} \lambda_{1,n}(t), \tag{3.34}$$

$$\lambda_2(t) = \lim_{n\to\infty} \lambda_{2,n}(t), \tag{3.35}$$

$$\lambda_3(t) = \lim_{n\to\infty} \lambda_{3,n}(t). \tag{3.36}$$

To establish the uniform convergence of these sequences, we consider their associated series:

$$\sum_{n=1}^{\infty}(x_n - x_{n-1}), \quad \sum_{n=1}^{\infty}(y_n - y_{n-1}), \tag{3.37}$$

$$\sum_{n=1}^{\infty}(\lambda_{1,n} - \lambda_{1,n-1}), \quad \sum_{n=1}^{\infty}(\lambda_{2,n} - \lambda_{2,n-1}), \quad \sum_{n=1}^{\infty}(\lambda_{3,n} - \lambda_{3,n-1}). \tag{3.38}$$

For $n = 1$, we have

$$x_1 - x_0 = \int_0^t f_1(x_0, y_0, (\lambda_1)_0, (\lambda_2)_0, (\lambda_3)_0, t)\, dt, \tag{3.39}$$

$$|x_1 - x_0| \leq M_1 T, \tag{3.40}$$

$$y_1 - y_0 = \int_0^t f_2(x_0, y_0, (\lambda_1)_0, (\lambda_2)_0, (\lambda_3)_0, t)\, dt, \tag{3.41}$$

$$|y_1 - y_0| \leq M_2 T, \tag{3.42}$$

$$(\lambda_1)_1 - (\lambda_1)_0 = -\phi_1(t-T) + \int_t^T f_3(x_0, y_0, (\lambda_1)_0, (\lambda_2)_0, (\lambda_3)_0, t)\, dt, \tag{3.43}$$

$$|(\lambda_1)_1 - (\lambda_1)_0| \leq (\phi_1 + M_3)T, \tag{3.44}$$

$$(\lambda_2)_1 - (\lambda_2)_0 = -\phi_2(t-T) + \int_t^T f_4(x_0, y_0, (\lambda_1)_0, (\lambda_2)_0, (\lambda_3)_0, t)\, dt, \tag{3.45}$$

$$|(\lambda_2)_1 - (\lambda_2)_0| \leq (\phi_2 + M_4)T, \tag{3.46}$$

$$(\lambda_3)_1 - (\lambda_3)_0 = 2\phi_3(t - T) + \int_t^T f_5(x_0, y_0, (\lambda_1)_0, (\lambda_2)_0, (\lambda_3)_0, t)\, dt, \tag{3.47}$$

$$|(\lambda_3)_1 - (\lambda_3)_0| \leq (\phi_3 + M_5)T. \tag{3.48}$$

Subsequent estimation for $x_n - x_{n-1}, y_n - y_{n-1}, (\lambda_1)_n - (\lambda_1)_{n-1}, (\lambda_2)_n - (\lambda_2)_{n-1}$, and $(\lambda_3)_n - (\lambda_3)_{n-1}$ gives:

$$|x_n - x_{n-1}| \leq L_1 T |x_{n-1} - x_{n-2}|, \tag{3.49}$$

$$|y_n - y_{n-1}| \leq L_2 T |y_{n-1} - y_{n-2}|, \tag{3.50}$$

$$|(\lambda_1)_n - (\lambda_1)_{n-1}| \leq L_3 T |(\lambda_1)_{n-1} - (\lambda_1)_{n-2}|, \tag{3.51}$$

$$|(\lambda_2)_n - (\lambda_2)_{n-1}| \leq L_4 T |(\lambda_2)_{n-1} - (\lambda_2)_{n-2}|, \tag{3.52}$$

$$|(\lambda_3)_n - (\lambda_3)_{n-1}| \leq L_5 T |(\lambda_3)_{n-1} - (\lambda_3)_{n-2}|. \tag{3.53}$$

Applying estimation for $n = 2$, we obtain:

$$|x_2 - x_1| \leq L_1 T |x_1 - x_0|, \tag{3.54}$$

$$|x_2 - x_1| \leq L_1 M_1 T^2. \tag{3.55}$$

For $n = 3$, it follows that:

$$|x_3 - x_2| \leq L_1 T |x_2 - x_1|, \tag{3.56}$$

$$|x_3 - x_2| \leq L_1^2 M_1 T^3, \tag{3.57}$$

and in general,

$$|x_n - x_{n-1}| \leq L_1^{n-1} M_1 T^n. \tag{3.58}$$

Applying the same estimation for $y_n - y_{n-1}$, we get:

$$|y_2 - y_1| \leq L_2 T |y_1 - y_0|, \tag{3.59}$$

$$|y_2 - y_1| \leq L_2 M_2 T^2, \tag{3.60}$$

$$|y_3 - y_2| \leq L_2^2 M_2 T^3, \tag{3.61}$$

and in general,

$$|y_n - y_{n-1}| \leq L_2^{n-1} M_2 T^n. \tag{3.62}$$

Similarly, we estimate for $(\lambda_1)_n - (\lambda_1)_{n-1}$:

$$|(\lambda_1)_2 - (\lambda_1)_1| \leq L_3 T |(\lambda_1)_1 - (\lambda_1)_0|, \tag{3.63}$$

$$|(\lambda_1)_2 - (\lambda_1)_1| \leq L_3 (\phi_1 + M_3) T^2, \tag{3.64}$$

$$|(\lambda_1)_3 - (\lambda_1)_2| \leq L_3^2 (\phi_1 + M_3) T^3, \tag{3.65}$$

and in general,

$$|(\lambda_1)_n - (\lambda_1)_{n-1}| \leq L_3^{n-1} (\phi_1 + M_3) T^n. \tag{3.66}$$

We proceed similarly for $(\lambda_2)_n - (\lambda_2)_{n-1}$ and $(\lambda_3)_n - (\lambda_3)_{n-1}$:

$$|(\lambda_2)_2 - (\lambda_2)_1| \leq L_4 T |(\lambda_2)_1 - (\lambda_2)_0|, \tag{3.67}$$

$$|(\lambda_2)_2 - (\lambda_2)_1| \leq L_4 (\phi_2 + M_4) T^2, \tag{3.68}$$

$$|(\lambda_2)_3 - (\lambda_2)_2| \leq L_4^2 (\phi_2 + M_4) T^3, \tag{3.69}$$

$$|(\lambda_2)_n - (\lambda_2)_{n-1}| \leq L_4^{n-1} (\phi_2 + M_4) T^n, \tag{3.70}$$

and

$$|(\lambda_3)_2 - (\lambda_3)_1| \leq L_5 T |(\lambda_3)_1 - (\lambda_3)_0|, \tag{3.71}$$

$$|(\lambda_3)_2 - (\lambda_3)_1| \leq L_5 (\phi_3 + M_5) T^2, \tag{3.72}$$

$$|(\lambda_3)_3 - (\lambda_3)_2| \leq L_5^2 (\phi_3 + M_5) T^3, \tag{3.73}$$

$$|(\lambda_3)_n - (\lambda_3)_{n-1}| \leq L_5^{n-1} (\phi_3 + M_5) T^n. \tag{3.74}$$

Given that $L_i < 1$ for $i = 1,2,3,4,5$ and $T < 1$, the series $\sum_{n=1}^{\infty}(x_n - x_{n-1})$, $\sum_{n=1}^{\infty}(y_n - y_{n-1})$, $\sum_{n=1}^{\infty}((\lambda_1)_n - (\lambda_1)_{n-1})$, $\sum_{n=1}^{\infty}((\lambda_2)_n - (\lambda_2)_{n-1})$, and $\sum_{n=1}^{\infty}((\lambda_3)_n - (\lambda_3)_{n-1})$ are uniformly convergent. Therefore, the sequences $\{x_n(t)\}, \{y_n(t)\}, \{(\lambda_1)_n(t)\}, \{(\lambda_2)_n(t)\}$ and $\{(\lambda_3)_n(t)\}$ converge uniformly.

Applying the Picard method, we approximate a solution for our problem: Upon simplifying the equations for a three-player system, we derive the following expressions:

$$\dot{x} = \frac{\lambda_1}{k_1}(-x-2y+1)(1-x-y) - \frac{\lambda_2}{k_2}(2x+y-1)x + \frac{\lambda_3}{k_3}(x+y)x, \tag{3.75}$$

$$\dot{y} = \frac{\lambda_2}{k_2}(-2x-y+1)(1-x-y) - \frac{\lambda_1}{k_1}(-x-2y+1)y + \frac{\lambda_3}{k_3}(x+y)y, \tag{3.76}$$

$$\dot{\lambda}_1 = \lambda_1\left(\frac{3\lambda_1}{k_1}(-x-2y+1) + \frac{3\lambda_2}{k_2}(-2x-y+1) - \frac{2\lambda_3}{k_3}(x+y)\right), \tag{3.77}$$

$$\dot{\lambda}_2 = \lambda_2\left(\frac{3\lambda_1}{k_1}(-x-2y+1) + \frac{3\lambda_2}{k_2}(-2x-y+1) - \frac{2\lambda_3}{k_3}(x+y)\right) - \phi_1, \tag{3.78}$$

$$\dot{\lambda}_3 = \lambda_3\left(\frac{3\lambda_1}{k_1}(-x-2y+1) + \frac{3\lambda_2}{k_2}(-2x-y+1) - \frac{2\lambda_3}{k_3}(x+y)\right) + 2\phi_3, \tag{3.79}$$

with the initial and boundary conditions are given by

$$x(0) = x_0, \tag{3.80}$$

$$y(0) = y_0, \tag{3.81}$$

$$\lambda_1(T) = 0, \tag{3.82}$$

$$\lambda_2(T) = 0, \tag{3.83}$$

$$\lambda_3(T) = 0. \tag{3.84}$$

Integrating equations (3.75) through (3.79), the system can be represented as:

$$\begin{aligned} x(t) = x_0 + \int_0^t \Big(&\frac{\lambda_1}{k_1}(-x-2y+1)(1-x-y) \\ &- \frac{\lambda_2}{k_2}(2x+y-1)x + \frac{\lambda_3}{k_3}(x+y)x\Big)dt, \end{aligned} \tag{3.85}$$

$$\begin{aligned} y(t) = y_0 + \int_0^t \Big(&\frac{\lambda_2}{k_2}(-2x-y+1)(1-x-y) \\ &- \frac{\lambda_1}{k_1}(-x-2y+1)y + \frac{\lambda_3}{k_3}(x+y)y\Big)dt, \end{aligned} \tag{3.86}$$

$$\lambda_1(t) = \int_t^T \Big(\frac{3\lambda_1}{k_1}(-x-2y+1) + \frac{3\lambda_2}{k_2}(-2x-y+1) - \frac{2\lambda_3}{k_3}(x+y) - \phi_1\Big)dt, \tag{3.87}$$

$$\lambda_2(t) = \int_t^T \Big(\frac{3\lambda_1}{k_1}(-x-2y+1) + \frac{3\lambda_2}{k_2}(-2x-y+1) - \frac{2\lambda_3}{k_3}(x+y) - \phi_2\Big)dt, \tag{3.88}$$

$$\lambda_3(t) = \int_t^T \Big(\frac{3\lambda_1}{k_1}(-x-2y+1) + \frac{3\lambda_2}{k_2}(-2x-y+1) - \frac{2\lambda_3}{k_3}(x+y) + 2\phi_3\Big)dt. \tag{3.89}$$

Applying the Picard method to equations (3.85) to (3.89), we formulate:

$$x_n(t) = x_0 + \int_0^t \Big(\frac{(\lambda_1)_{n-1}}{k_1}(-x_{n-1}-2y_{n-1}+1)(1-x_{n-1}-y_{n-1}) - \frac{(\lambda_2)_{n-1}}{k_2}(2x_{n-1}+y_{n-1}-1)x_{n-1} + \frac{(\lambda_3)_{n-1}}{k_3}(x_{n-1}+y_{n-1})x_{n-1}\Big)dt, \tag{3.90}$$

$$y_n(t) = y_0 + \int_0^t \Big(\frac{(\lambda_2)_{n-1}}{k_2}(-2x_{n-1}-y_{n-1}+1)(1-x_{n-1}-y_{n-1}) - \frac{(\lambda_1)_{n-1}}{k_1}(-x_{n-1}-2y_{n-1}+1)y_{n-1} + \frac{(\lambda_3)_{n-1}}{k_3}(x_{n-1}+y_{n-1})y_{n-1}\Big)dt, \tag{3.91}$$

$$(\lambda_1)_n(t) = \int_t^T \Big(\frac{3(\lambda_1)_{n-1}}{k_1}(-x_{n-1}-2y_{n-1}+1) + \frac{3(\lambda_2)_{n-1}}{k_2}(-2x_{n-1}-y_{n-1}+1) - \frac{2(\lambda_3)_{n-1}}{k_3}(x_{n-1}+y_{n-1}) + \phi_1\Big)dt, \tag{3.92}$$

$$(\lambda_2)_n(t) = \int_t^T \Big(\frac{3(\lambda_1)_{n-1}}{k_1}(-x_{n-1}-2y_{n-1}+1) + \frac{3(\lambda_2)_{n-1}}{k_2}(-2x_{n-1}-y_{n-1}+1) - \frac{2(\lambda_3)_{n-1}}{k_3}(x_{n-1}+y_{n-1}) + \phi_2\Big)dt, \tag{3.93}$$

$$(\lambda_3)_n(t) = \int_t^T \Big(\frac{3(\lambda_1)_{n-1}}{k_1}(-x_{n-1} - 2y_{n-1} + 1) + \frac{3(\lambda_2)_{n-1}}{k_2}(-2x_{n-1} - y_{n-1} + 1) - \frac{2(\lambda_3)_{n-1}}{k_3}(x_{n-1} + y_{n-1}) + 2\phi_3\Big)dt \tag{3.94}$$

with initial conditions:

$$x(0) = x_0, \quad y(0) = y_0, \quad (\lambda_1)_0 = 0, \quad (\lambda_2)_0 = 0, \quad (\lambda_3)_0 = 0. \tag{3.95}$$

At the initial iteration $n = 1$, the following expressions are obtained:

$$x_1(t) = x_0 + \int_0^t \left(\frac{(\lambda_1)_0}{k_1}(-x_0 - 2y_0 + 1)(1 - x_0 - y_0) - \frac{(\lambda_2)_0}{k_2}(2x_0 + y_0 - 1)x_0 + \frac{(\lambda_3)_0}{k_3}(x_0 + y_0)x_0\right) dt, \tag{3.96}$$

$$y_1(t) = y_0 + \int_0^t \left(\frac{(\lambda_2)_0}{k_2}(-2x_0 - y_0 + 1)(1 - x_0 - y_0) - \frac{(\lambda_1)_0}{k_1}(-x_0 - 2y_0 + 1)y_0 + \frac{(\lambda_3)_0}{k_3}(x_0 + y_0)y_0\right) dt, \tag{3.97}$$

$$(\lambda_1)_1(t) = \int_t^T \left(\frac{3(\lambda_1)_0}{k_1}(-x_0 - 2y_0 + 1) + \frac{3(\lambda_2)_0}{k_2}(-2x_0 - y_0 + 1) - \frac{2(\lambda_3)_0}{k_3}(x_0 + y_0) + \phi_1\right) dt, \tag{3.98}$$

$$(\lambda_2)_1(t) = \int_t^T \left(\frac{3(\lambda_1)_0}{k_1}(-x_0 - 2y_0 + 1) + \frac{3(\lambda_2)_0}{k_2}(-2x_0 - y_0 + 1) - \frac{2(\lambda_3)_0}{k_3}(x_0 + y_0) + \phi_2\right) dt, \tag{3.99}$$

$$(\lambda_3)_1(t) = \int_t^T \left(\frac{3(\lambda_1)_0}{k_1}(-x_0 - 2y_0 + 1) + \frac{3(\lambda_2)_0}{k_2}(-2x_0 - y_0 + 1) - \frac{2(\lambda_3)_0}{k_3}(x_0 + y_0) - 2\phi_3\right) dt \tag{3.100}$$

with the initial conditions:

$$x(0) = x_0, \quad y(0) = y_0, \quad (\lambda_1)_0 = 0, \quad (\lambda_2)_0 = 0, \quad (\lambda_3)_0 = 0. \tag{3.101}$$

The first approximations for x, λ_1, λ_2, and λ_3 are thus given by:

$$x_1(t) = x_0, \quad y_1(t) = y_0, \tag{3.102}$$
$$(\lambda_1)_1(t) = \phi_1(t-T), \tag{3.103}$$
$$(\lambda_2)_1(t) = \phi_2(t-T), \tag{3.104}$$
$$(\lambda_3)_1(t) = -2\phi_3(t-T). \tag{3.105}$$

Consequently, the initial approximations for u_1, u_2, u_3, c_1, c_2, and c_3 are:

$$(u_1)_1 = \frac{\phi_1(t-T)(-x_0-2y_0+1)}{k_1}, \tag{3.106}$$
$$(u_2)_1 = \frac{\phi_2(t-T)(-2x_0-y_0+1)}{k_2}, \tag{3.107}$$
$$(u_3)_1 = \frac{-2\phi_3(t-T)(x_0+y_0)}{k_3}, \tag{3.108}$$
$$(c_1)_1 = \frac{k_1}{2}\left(\frac{\phi_1(t-T)(-x_0-2y_0+1)}{k_1}\right)^2, \tag{3.109}$$
$$(c_2)_1 = \frac{k_2}{2}\left(\frac{\phi_2(t-T)(-2x_0-y_0+1)}{k_2}\right)^2, \tag{3.110}$$
$$(c_3)_1 = \frac{k_3}{2}\left(\frac{-2\phi_3(t-T)(x_0+y_0)}{k_3}\right)^2. \tag{3.111}$$

At the second iteration $n=2$, the system evolves as follows:

$$x_2(t) = x_0 + \int_0^t \Big(\frac{(\lambda_1)_1}{k_1}(-x_1-2y_1+1)(1-x_1-y_1) - \frac{(\lambda_2)_1}{k_2}(2x_1+y_1-1)x_1 + \frac{(\lambda_3)_1}{k_3}(x_1+y_1)x_1\Big)dt, \tag{3.112}$$
$$y_2(t) = y_0 + \int_0^t \Big(\frac{(\lambda_2)_1}{k_2}(-2x_1-y_1+1)(1-x_1-y_1) - \frac{(\lambda_1)_1}{k_1}(-x_1-2y_1+1)y_1 + \frac{(\lambda_3)_1}{k_3}(x_1+y_1)y_1\Big)dt, \tag{3.113}$$
$$(\lambda_1)_2(t) = \int_t^T \Big(\frac{3(\lambda_1)_1}{k_1}(-x_1-2y_1+1) + \frac{3(\lambda_2)_1}{k_2}(-2x_1-y_1+1) - \frac{2(\lambda_3)_1}{k_3}(x_1+y_1) + \phi_1\Big)dt, \tag{3.114}$$

$$(\lambda_2)_2(t) = \int_t^T \Big(\frac{3(\lambda_1)_1}{k_1}(-x_1 - 2y_1 + 1) + \frac{3(\lambda_2)_1}{k_2}(-2x_1 - y_1 + 1) - \frac{2(\lambda_3)_1}{k_3}(x_1 + y_1) + \phi_2\Big)dt, \tag{3.115}$$

$$(\lambda_3)_2(t) = \int_t^T \Big(\frac{3(\lambda_1)_1}{k_1}(-x_1 - 2y_1 + 1) + \frac{3(\lambda_2)_1}{k_2}(-2x_1 - y_1 + 1) - \frac{2(\lambda_3)_1}{k_3}(x_1 + y_1) - 2\phi_3\Big)dt. \tag{3.116}$$

Thus, the second approximation for x, y, λ_1, λ_2, and λ_3 can be expressed as:

$$x_2(t) = x_0 + \int_0^t \Bigg(\frac{\phi_1(t-T)}{k_1}(-x_0 - 2y_0 + 1)(1 - x_0 - y_0) - \frac{\phi_2(t-T)}{k_2}(2x_0 + y_0 - 1)x_0 - \frac{2\phi_3(t-T)}{k_3}(x_0 + y_0)x_0\Bigg)\, dt, \tag{3.117}$$

$$y_2(t) = y_0 + \int_0^t \Big(\frac{\phi_2(t-T)}{k_2}(-2x_0 - y_0 + 1)(1 - x_0 - y_0) - \frac{\phi_1(t-T)}{k_1}(-x_0 - 2y_0 + 1)y_0 - \frac{2\phi_3(t-T)}{k_3}(x_0 + y_0)y_0\Big)dt, \tag{3.118}$$

$$(\lambda_1)_2(t) = \int_t^T \Big(\frac{3\phi_1(t-T)}{k_1}(-x_0 - 2y_0 + 1) + \frac{3\phi_2(t-T)}{k_2}(-2x_0 - y_0 + 1) - \frac{2\phi_3(t-T)}{k_3}(x_0 + y_0) + \phi_1\Big)dt, \tag{3.119}$$

$$(\lambda_2)_2(t) = \int_t^T \Big(\frac{3\phi_1(t-T)}{k_1}(-x_0 - 2y_0 + 1) + \frac{3\phi_2(t-T)}{k_2}(-2x_0 - y_0 + 1) - \frac{2\phi_3(t-T)}{k_3}(x_0 + y_0) + \phi_2\Big)dt, \tag{3.120}$$

$$(\lambda_3)_2(t) = \int_t^T \Big(\frac{3\phi_1(t-T)}{k_1}(-x_0 - 2y_0 + 1) + \frac{3\phi_2(t-T)}{k_2}(-2x_0 - y_0 + 1) - \frac{2\phi_3(t-T)}{k_3}(x_0 + y_0) - 2\phi_3\Big)dt. \tag{3.121}$$

After solving the integrals, the equations become:

$$\begin{aligned}
x_2(t) = x_0 + \frac{t^2}{2k_1k_2k_3}\Big(&- 2k_1k_2x_0^2\phi_3 - 2k_1k_2x_0y_0\phi_3 - 2k_1k_3x_0^2\phi_2 \\
&- k_1k_3x_0y_0\phi_2 + k_1k_3x_0\phi_2 + k_2k_3x_0^2\phi_1 + 3k_2k_3x_0y_0\phi_1 \\
&- 2k_2k_3x_0\phi_1 + 2k_2k_3y_0^2\phi_1 - 3k_2k_3y_0\phi_1 + k_2k_3\phi_1\Big) \\
&+ \frac{t^2}{Tk_1k_2k_3}\Big(k_1k_2x_0^2\phi_3 + k_1k_2x_0y_0\phi_3 + k_1k_3x_0^2\phi_2 \\
&+ \frac{1}{2}k_1k_3x_0y_0\phi_2 - k_2k_3x_0^2\phi_1 - \frac{3}{2}k_2k_3x_0y_0\phi_1 \\
&+ k_2k_3y_0^2\phi_1 - \frac{3}{2}k_2k_3y_0\phi_1 + \frac{1}{2}k_2k_3\phi_1\Big),
\end{aligned} \tag{3.122}$$

$$\begin{aligned}
y_2(t) = y_0 + \frac{t^2}{2k_1k_2k_3}\Big(&- 2k_1k_2x_0y_0\phi_3 - 2k_1k_2y_0^2\phi_3 + 2k_1k_3x_0^2\phi_2 \\
&+ 3k_1k_3x_0y_0\phi_2 - 3k_1k_3x_0\phi_2 + k_1k_3y_0^2\phi_2 - 2k_1k_3y_0\phi_2 \\
&+ k_1k_3\phi_2 + k_2k_3x_0y_0\phi_1 + 2k_2k_3y_0^2\phi_1 - k_2k_3y_0\phi_1\Big) \\
&+ \frac{t^2}{Tk_1k_2k_3}\Big(k_1k_2x_0y_0\phi_3 + k_1k_2y_0^2\phi_3 - k_1k_3x_0^2\phi_2 \\
&- \frac{3}{2}k_1k_3x_0y_0\phi_2 + k_2k_3x_0y_0\phi_1 + 2k_2k_3y_0^2\phi_1 - \frac{1}{2}k_2k_3y_0\phi_1\Big),
\end{aligned} \tag{3.123}$$

$$\begin{aligned}
(\lambda_1)_2(t) = -\frac{T^3}{3k_1k_2k_3}\Big(&8k_1k_2x_0\phi_1\phi_3 + 8k_1k_2y_0\phi_1\phi_3 \\
&+ 6k_1k_3x_0\phi_1\phi_2 + 3k_1k_3y_0\phi_1\phi_2 - 3k_1k_3\phi_1\phi_2 \\
&+ 3k_2k_3x_0\phi_1 + 6k_2k_3y_0\phi_1 - 3k_2k_3\phi_1\Big) \\
&+ \frac{t^3}{3k_1k_2k_3}\Big(8k_1k_2x_0\phi_1\phi_3 + 8k_1k_2y_0\phi_1\phi_3 \\
&+ 6k_1k_3x_0\phi_1\phi_2 + 3k_1k_3y_0\phi_1\phi_2 - 3k_1k_3\phi_1\phi_2 \\
&- 3k_2k_3x_0\phi_1 - 6k_2k_3y_0\phi_1 + 3k_2k_3\phi_1\Big),
\end{aligned} \tag{3.124}$$

$$\begin{aligned}
(\lambda_2)_2(t) = -\frac{T^3}{3k_1k_2k_3}\Big(&8k_1k_2x_0\phi_2\phi_3 + 8k_1k_2y_0\phi_2\phi_3 \\
&+ 6k_1k_3x_0\phi_2 + 3k_1k_3y_0\phi_2 - 3k_1k_3\phi_2 \\
&+ 3k_2k_3x_0\phi_1\phi_2 + 6k_2k_3y_0\phi_1\phi_2 - 3k_2k_3\phi_1\phi_2\Big) \\
&+ \frac{t^3}{3k_1k_2k_3}\Big(8k_1k_2x_0\phi_2\phi_3 + 8k_1k_2y_0\phi_2\phi_3 + 6k_1k_3x_0\phi_2 \\
&+ 3k_1k_3y_0\phi_2 - 3k_1k_3\phi_2 - 3k_2k_3x_0\phi_1\phi_2 \\
&- 6k_2k_3y_0\phi_1\phi_2 + 3k_2k_3\phi_1\phi_2\Big),
\end{aligned} \tag{3.125}$$

$$
\begin{aligned}
(\lambda_3)_2(t) = {} & \frac{T^3}{3k_1k_2k_3}\Big(16k_1k_2x_0\phi_3^2 + 16k_1k_2y_0\phi_3^2 + 12k_1k_3x_0\phi_2\phi_3 \\
& + 6k_1k_3y_0\phi_2\phi_3 - 6k_1k_3\phi_2\phi_3 + 6k_2k_3x_0\phi_1\phi_3 \\
& + 12k_2k_3y_0\phi_1\phi_3 - 6k_2k_3\phi_1\phi_3\Big) \\
& - \frac{t^3}{3k_1k_2k_3}\Big(16k_1k_2x_0\phi_3^2 + 16k_1k_2y_0\phi_3^2 + 12k_1k_3x_0\phi_2\phi_3 \\
& + 6k_1k_3y_0\phi_2\phi_3 - 6k_1k_3\phi_2\phi_3 + 6k_2k_3x_0\phi_1\phi_3 \\
& + 12k_2k_3y_0\phi_1\phi_3 - 6k_2k_3\phi_1\phi_3\Big). \qquad (3.126)
\end{aligned}
$$

The second-order approximations for the control and cost functions are given by:

$$(u_1)_2 = \frac{(\lambda_1)_2(t)(-x_2(t) - 2y_2(t) + 1)}{k_1}, \qquad (3.127)$$

$$(u_2)_2 = \frac{(\lambda_2)_2(t)(-2x_2(t) - y_2(t) + 1)}{k_2}, \qquad (3.128)$$

$$(u_3)_2 = \frac{-(\lambda_3)_2(t)(x_2(t) + y_2(t))}{k_3}, \qquad (3.129)$$

$$(c_1)_2 = \frac{k_1}{2}\left((u_1)_2\right)^2, \qquad (3.130)$$

$$(c_2)_2 = \frac{k_2}{2}\left((u_2)_2\right)^2, \qquad (3.131)$$

$$(c_3)_2 = \frac{k_3}{2}\left((u_3)_2\right)^2. \qquad (3.132)$$

With specified values $x_0 = 0, y_0 = 0, \phi_1 = 0.1, \phi_2 = 0.3, k_1 = 0.25, k_2 = 0.5, \phi_3 = 0.2, k_3 = 0.75$, and $T = 1$, a comparative analysis of the approximate solutions for the three firms in the model is presented in the ensuing figures.

Figure 1 illustrates the approximate optimal state solutions for the problem. In this scenario, x, y, and $1 - x - y$ denote the market shares of Firm 1, Firm 2, and Firm 3, respectively, at various time points t. The figure captures the evolution of market shares for the three firms over time.

Figure 2 displays the approximate optimal control solutions. The variables u_1, u_2, and u_3 signify the advertising efforts of Firm 1, Firm 2, and Firm 3.

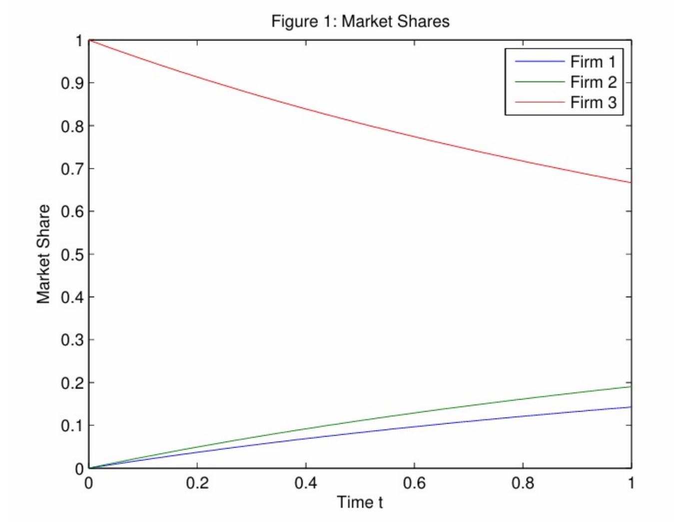

FIGURE 1. Market Share Dynamics of Firm 1, Firm 2, and Firm 3 Over Time.

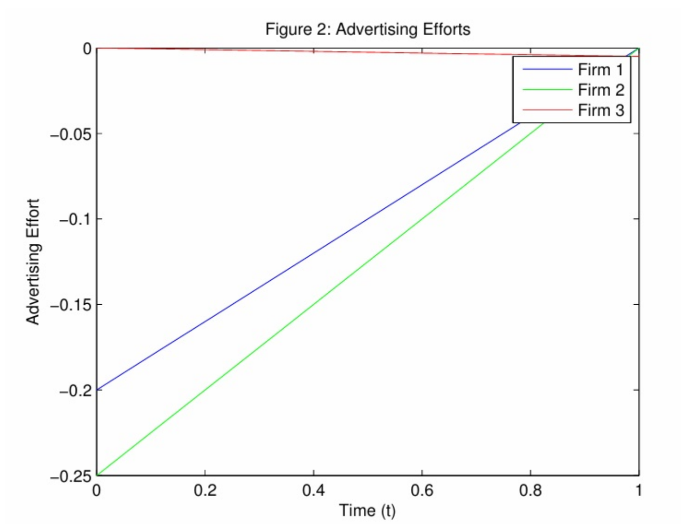

FIGURE 2. Advertising Strategies of Firm 1, Firm 2, and Firm 3 Over Time.

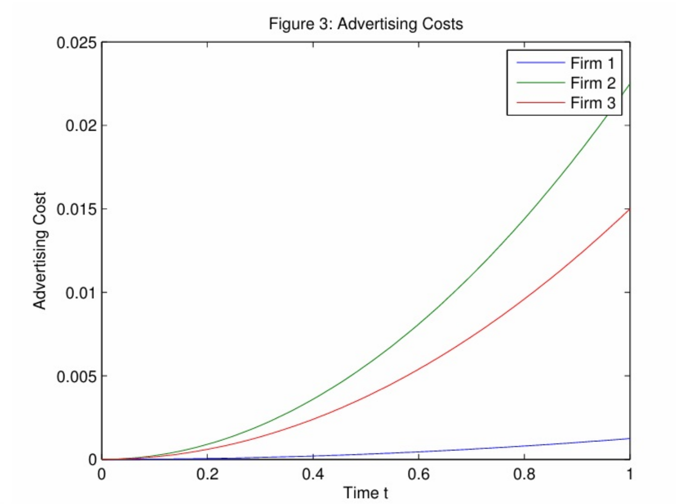

Figure 3. Advertising Costs of Firm 1, Firm 2, and Firm 3 Over Time.

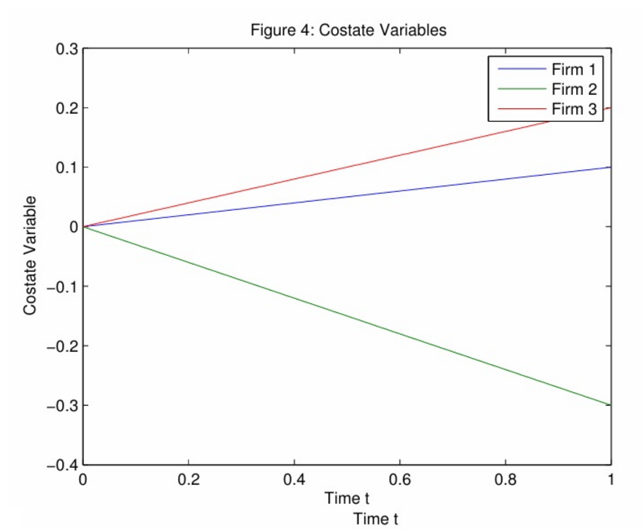

Figure 4. Costate Variable Trends for Firm 1, Firm 2, and Firm 3 Over Time.

4. Conclusion

The studys exploration into the application of the Picard method within the context of Nash equilibrium in differential games marks a significant advancement in the understanding of strategic dynamics in competitive markets. The results not only validate the methods applicability but also open avenues for its use in broader strategic and economic modeling. Future research could expand this approach to more complex scenarios, incorporating varying market conditions and player behaviors.

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