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NEW ITERATION APPROACH FOR APPROXIMATING FIXED POINTS

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Abstract. In this paper, we introduce and study a novel iterative approach for fixed point (FP) approximation under nonexpansive mapping of the Reich–Suzuki type(RSN). We further show that the suggested approach converges quicker than some existing iterative schemes for this kind of mapping. Furthermore, we provide some strong convergence findings as well as weak convergence results for our novel iterative approach for FPs of RSN mapping. Additionally, we conduct a numerical experiment to demonstrate the effectiveness of our innovative iterative approach.

1. Introduction

Functional analysis is a mathematical discipline that expands vector and space notions from finite to infinite dimensions. It has emerged as a fundamental basis for contemporary applied mathematics in recent decades. FP

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theory, on the other hand, is an important and rapidly emerging area of nonlinear functional analysis. It refers to outcomes that describe the presence of FPs. It ensures a meaningful solution to these problems. A large literature on FP theory is accessible in several publications [3, 4, 16, 17, 18, 23, 24, 28]. The fundamental concepts of FP theory can be divided into two primary categories. The first is to determine the required and sufficient criteria under which an operator allows FPs. Another option is to use schematic techniques to find such FPs. Another significant idea in FP theory is the study of the behaviors of FPs, like stability, data dependence, etc. However, a variety of iterative approaches have been developed to approximate the FPs of various mapping classes.

A novel iterative approach was presented by Mann [10] in 1953.

$$\eta_{s+1} = (1-\alpha_s)\,\eta_s + \alpha_s \Omega\ \eta_s,\ \forall s \geq 0, \tag{1.1}$$

where $\Omega:\ \mathcal{A} \to \mathcal{A}$ is a mapping such that $\mathcal{A}$ is a subset of Banach space $\mathbb{B}$, $as \in (0,1)$ and a sequence $\{\eta_s\}$ is generated by $\eta_0 \in \mathcal{A}$.

The Mann method is expanded from a single step to two steps by the Ishikawa [7] iterative method, which is as follows:

$$\begin{cases} \eta_{s+1} = (1-\alpha_s)\,\eta_s + \alpha_s \Omega\ \mu_s, \\ \mu_s = (1-\beta_s)\,\eta_s + \beta_s \Omega\ \eta_s, \end{cases} \tag{1.2}$$

where $\alpha_s,\ \beta_s \in (0,1)$.

The Mann and Ishikawa iterative methods are both extended by the Noor [11] iterative method, which is stated as follows:

$$\begin{cases} \eta_{s+1} = (1-\alpha_s)\,\eta_s + \alpha_s \Omega\ \mu_s, \\ \mu_s = (1-\beta_s)\,\eta_s + \beta_s \Omega\ \sigma_s, \\ \sigma_s = (1-\rho_s)\,\eta_s + \rho_s \Omega\ \eta_s. \end{cases} \tag{1.3}$$

The technique developed by Agarwal et al. [2] is a minor modification of the Ishikawa method, as described below:

$$\begin{cases} \eta_{s+1} = (1-\alpha_s)\,\Omega\eta_s + \alpha_s \Omega\ \mu_s, \\ \mu_s = (1-\beta_s)\,\eta_s + \beta_s \Omega\ \eta_s. \end{cases} \tag{1.4}$$

The three-step iterative method utilized by Abbas and Nazir [1] is as follows:

$$\begin{cases} \eta_{s+1} = (1-\alpha_s)\,\Omega\ \mu_s + \alpha_s \Omega\ \sigma_s, \\ \mu_s = (1-\beta_s)\,\Omega\eta_s + \beta_s \Omega\ \sigma_s, \\ \sigma_s = (1-\rho_s)\,\eta_s + \rho_s \Omega\ \eta_s. \end{cases} \tag{1.5}$$

In recent years, distinguished mathematicians have proposed numerous iterative schemes that expedite convergence to the fixed point [5, 9, 14, 15, 22, 27]. Regarding the estimation of the FPs of nonexpansive mapping of the RSN type, we provide a new iterative technique:

$$\begin{cases} \eta_s \in \mathcal{A}, s \in N, \\ \mu_s = \Omega((1-\alpha_s)\,\eta_s + \alpha_s \Omega\ \eta_s), \\ \xi_s = \Omega(\Omega\,(\mu_s)), \\ \eta_{s+1} = \Omega(\,(1-\beta_s)\,\Omega\mu_s + \beta_s \Omega \xi_s), \end{cases} \tag{1.6}$$

for all $s \geq 0$, where $\alpha_s,\ \beta_s \in (0,1)$.

This study is made especially for RSN mapping, the iterative approach (1.6) converges more quickly than the well-known iterative methods now in use. For nonexpansive mapping of the RSN type, we demonstrate both weak and strong convergence findings of the approach (1.6). We further demonstrate numerically that, in comparison to many other iterative methods currently in use, the iterative approach (1.6) converges more quickly. The results of this work improve and extend the corresponding results in the literature.

2. Preliminaries

In this section, the basic definitions and facts related to this work are recalled. Throughout this work, $\mathcal{F}(\Omega)$ denotes the collection of all FPs of Ω and $\mathcal{A}$ is a nonempty subset of a Banach space $\mathbb{B}$ (briefly BN-space). Let $\mathbb{B}^*$ be the dual of $\mathbb{B}$ and consider $< .,. >$ indicates the generalized duality pairing between $\mathbb{B}$ and $\mathbb{B}^*$. Then, for any $u \in \mathbb{B}$ the normalized duality map is the multi-valued map $\mathcal{J}$: $\mathbb{B} \to 2^{\mathbb{B}^*}$ which is described by:

$$\mathcal{J}\,(u) = \{q \ \in \mathbb{B}^* : < u, q >= \|u\|^2 = \|q\|^2\}.$$

Consider $\mathcal{D} = \{u \ \in \mathbb{B} : \ \|u\| = 1\}$. Then $\mathbb{B}$ is termed as smooth if the limit

$$\lim_{e \to 0} \frac{\|u + eq\| + \|u\|}{e}$$

exists for any $u,\ q \ \in \mathcal{D}$. Assume that the limit of the equation above exists and is uniformly obtained for $q \ \in \mathcal{D}$. In this situation, the norm of $\mathbb{B}$ is termed Fréchet differentiable.

Definition 2.1. A BN-space $\mathbb{B}$ is termed as uniformly convex (UC-Space) if, for every $\epsilon \in \ (0,\ 2]$, there exists $\delta > \ 0$ such that for
$\|u_1\| \leq 1$, $\|u_2\| \leq 1$, $\|u_1 - u_2\| > \epsilon$, then $\frac{\|u_1+u_2\|}{2} \leq \delta \ \ u_1, u_2 \ \in \ \mathbb{B}$.

Definition 2.2. ([21]) A mapping $\Omega : \ \mathcal{A} \to \mathcal{A}$ is termed as satisfying condition (C) if $\frac{1}{2}\|u_1 - \Omega u_1\| \leq \|u_1 - u_2\|$ implies $\|\Omega u_1 - \Omega u_2\| \leq \|u_1 - u_2\|$ for any $u_1, u_2 \ \in \ \mathcal{A}$.

Definition 2.3. ([6]) A map $\Omega:\ \mathcal{A}\to\mathbb{B}$ is demiclosed at $u\in\mathcal{A}$ if for every sequence $\{u_s\}$ in $\mathcal{A}$ and $q\in\ \mathbb{B}$, $u_s\to q$ and $\Omega u_s\to u$ indicates that $q\in\mathcal{A}$ and $\Omega q=u$.

Definition 2.4. ([13]) A mapping $\Omega:\ \mathcal{A}\to\mathcal{A}$ is termed as Reich–Suzuki-type nonexpansive mapping(RSN) if there is $\overline{\gamma}\in[0,\ 1)$ such that for

$$\frac{1}{2}\left\|u_1-\Omega u_1\right\|\le\left\|u_1-u_2\right\|,$$

then

$$\begin{aligned}\left\|\Omega u_1-\Omega u_2\right\|\le\overline{\gamma}\left\|u_1-\Omega u_1\right\|+\overline{\gamma}\left\|u_2-\Omega u_2\right\|\\ +(1-2\overline{\gamma})\left\|u_1-u_2\right\|\end{aligned}\tag{2.1}$$

for all $u_1,u_2\ \in\ \mathcal{A}$.

Lemma 2.5. ([25]) *Let* Ω: $\mathcal{A}\to\mathcal{A}$ *be a mapping. If* Ω *is RSN mapping with* $\mathcal{F}(\Omega)\neq\emptyset$*, then the followings are true:*

(i) *If* Ω *is RSN map, then* $\|\Omega u-\Omega u^*\|\le\|u-u^*\|$ *for any choice of* $u\in\mathcal{A}$ *and* $u^*\in\mathcal{F}(\Omega)$.

(ii) *If* Ω *fulfills condition (C), then* Ω *is RSN map.*

Lemma 2.6. ([19]) *Let* $\mathbb{B}$ *be a* $UC-$*space and* $\{\eta_s\}$ *be a real sequence with* $0<u_s\le\eta_s\le q_s<1$ *for each* $s\ge 1$*. Assume that* $\{u_s\}$, $\{q_s\}$ *are two sequences of* $\mathbb{B}$ *with* $\operatorname{limsup}_{s\to\infty}\|u_s\|\ \le\vartheta$, $\operatorname{limsup}_{s\to\infty}\|q_s\|\ \le\vartheta$ *and* $\operatorname{limsup}_{s\to\infty}\|\eta_s u_s+(1-\eta_s)q_s\|\ =\vartheta$ *hold for some* $\vartheta\ge 0$*. Then*

$$\operatorname*{limsup}_{s\to\infty}\|u_s-q_s\|=0\ .$$

Definition 2.7. ([12]) Let $\mathcal{A}\subseteq\mathbb{B}$, where $\mathcal{A}$ nonempty closed convex subset, and $\{u_s\}$ be a bounded sequence in $\mathbb{B}$. We put

$$r\left(u,\{u_s\}\right):=\operatorname*{limsup}_{s\to\infty}\|u-u_s\|$$

for $u\ \in\mathbb{B}$, then the asymptotic radius of $\{u_s\}$ concerning $\mathcal{A}$ is

$$r\left(\ \mathcal{A},\{u_s\}\right):=\inf\{r\left(u,\{u_s\}\right):u\ \in\mathbb{B}\}.$$

The asymptotic center of $\{u_s\}$ concerning $\mathcal{A}$ is

$$A\left(\ \mathcal{A},\{u_s\}\right):=\{u\ \in\mathbb{B}:\ r\left(u,\{u_s\}\right)=r\left(\ \mathcal{A},\{u_s\}\right)\}.$$

Lemma 2.8. ([26]) *Let* Ω: $\mathcal{A}\to\mathcal{A}$ *be a mapping. If* Ω *is RSN map, then for every* $u_1,u_2\in\mathcal{A}$*, the following inequality is true:*

$$\|u_1-\Omega u_2\|\le\left(\frac{3+\omega}{1-\omega}\right)\|u_1-\Omega u_1\|+\|u_1-u_2\|.\tag{2.2}$$

Definition 2.9. ([20]) A mapping Ω: $\mathcal{A}\to\mathcal{A}$ is said to have a fulfilled condition (I) if there exists a function $h : [0,\infty) \to [0,\infty)$ (nondecreasing) with $h(0) = 0$ and for $m > 0$, then $h(m) > 0$ with

$$\|u - \Omega u\| \geq h(d(u, \Omega u))$$

for every $u \in \mathcal{A}$, where $d(u, \Omega u) = \inf_{u^* \in \mathcal{F}(\Omega)} \|u - u^*\|$.

3. Main results

Some convergence findings of the iterative (1.6) for nonexpansive mapping of the RSN type are shown in this section.

Lemma 3.1. *Let Ω: $\mathcal{A}\to\mathcal{A}$ be RSN mapping where $\mathcal{A} \subseteq \mathbb{B}$ (nonempty, closed and convex) of BN-space $\mathbb{B}$ with $\mathcal{F}(\Omega)\neq\emptyset$. Assume $\{\eta_s\}$ be the sequence produced by the iterative* (1.6)*. Then*

$$\lim_{s\to\infty} \|\eta_s - \eta^*\|$$

exists for each $\eta^* \in \mathcal{F}(\Omega)$.

Proof. Consider $\eta^* \in \mathcal{F}(\Omega)$. Based on Lemma 2.5, we have

$$\begin{aligned} \|\mu_s - \eta^*\| &= \|\,\Omega\,(\,(1-\alpha_s)\,\eta_s + \alpha_s\Omega\,\eta_s) - \Omega\eta^*\| \\ &\leq (1-\alpha_s)\,\|\eta_s - \eta^*\| + \alpha_s\,\|\Omega\,\eta_s - \Omega\eta^*\| \\ &\leq (1-\alpha_s)\,\|\eta_s - \eta^*\| + \alpha_s\,\|\,\eta_s - \eta^*\| \\ &= \|\eta_s - \eta^*\|\,, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \|\xi_s - \eta^*| &= \|\Omega\,(\Omega\,(\mu_s)) - \eta^*\| \\ &\leq \|\Omega\,(\mu_s) - \eta^*\| \\ &\leq \|\mu_s - \eta^*\| \\ &\leq \|\eta_s - \eta^*\| \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|\eta_{s+1} - \eta^*\| &= \|\Omega(\,(1-\beta_s)\,\Omega\mu_s + \beta_s\Omega\xi_s) - \eta^*\| \\ &\leq \|(1-\beta_s)\,\Omega\mu_s + \beta_s\Omega\xi_s - \eta^*\| \\ &\leq (1-\beta_s)\,\|\Omega\mu_s - \eta^*\| + \beta_s\,\|\Omega\xi_s - \eta^*\| \\ &\leq (1-\beta_s)\,\|\mu_s - \eta^*\| + \beta_s\,\|\,\xi_s - \eta^*\| \\ &\leq (1-\beta_s)\,\|\eta_s - \eta^*\| + \beta_s\,\|\eta_s - \eta^*\| \\ &= \|\eta_s - \eta^*\|\,. \end{aligned} \tag{3.3}$$

Thus, $\lim_{s\to\infty} \|\eta_s - \eta^*\|$ exists. □

Lemma 3.2. *Let* Ω: $\mathcal{A}\to\mathcal{A}$ *be RSN mapping where* $\mathcal{A} \subseteq \mathbb{B}$ *(*$\mathcal{A}$ *a nonempty closed convex). Let* $\{\eta_s\}$ *produced by the iterative* (1.6). *Then* $\mathcal{F}(\Omega)\neq\emptyset$ *if and only if* $\{\eta_s\}$ *is bounded and* $\lim_{s\to\infty}\|\Omega\eta_s-\eta_s\| = 0$.

Proof. Assume $\mathcal{F}(\Omega)\neq\emptyset$ and $\eta^*\in\mathcal{F}(\Omega)$. From Lemma 3.1, $\lim_{s\to\infty}\|\eta_s-\eta^*\|$ exists and $\{\eta_s\}$ is bounded. Put

$$\lim_{s\to\infty}\|\eta_s-\eta^*\| = b. \tag{3.4}$$

Then, from (3.1), (3.2) and (3.4), we have

$$\limsup_{s\to\infty}\|\mu_s-\eta^*\| \le b \tag{3.5}$$

and

$$\limsup_{s\to\infty}\|\xi_s-\eta^*\| \le b. \tag{3.6}$$

Recalling Lemma 2.5, we have

$$\limsup_{s\to\infty}\|\Omega\eta_s-\eta^*\| \le \limsup_{s\to\infty}\|\eta_s-\eta^*\| = b. \tag{3.7}$$

Also, from (1.6) and Lemma 3.1, we obtain

$$\begin{aligned}\|\eta_{s+1}-\eta^*\| &= \|\Omega(\,(1-\beta_s)\,\Omega\mu_s+\beta_s\Omega\xi_s)-\eta^*\|\\ &\le \|(1-\beta_s)\,\Omega\mu_s+\beta_s\Omega\xi_s\;-\eta^*\|\\ &\le (1-\beta_s)\,\|\Omega\mu_s\;-\eta^*\|+\beta_s\,\|\Omega\xi_s\;-\eta^*\|\\ &\le (1-\beta_s)\,\|\mu_s\;-\eta^*\|+\beta_s\,\|\,\xi_s\;-\eta^*\|\\ &\le \|\mu_s\;-\eta^*\|-\beta_s\,\|\mu_s\;-\eta^*\|+\beta_s\,\|\,\xi_s\;-\eta^*\|\\ &\le \|\eta_s-\eta^*\|-\beta_s\,\|\eta_s-\eta^*\|+\beta_s\,\|\,\xi_s\;-\eta^*\|\,.\end{aligned}$$

This implies that

$$\begin{aligned}\|\eta_{s+1}-\eta^*\|-\|\eta_s-\eta^*\| &\le \frac{\|\eta_{s+1}-\eta^*\|-\|\eta_s-\eta^*\|}{\beta_s}\\ &\le \|\,\xi_s\;-\eta^*\|\;-\|\eta_s-\eta^*\|\,.\end{aligned} \tag{3.8}$$

Therefore, we have

$$b \le \liminf_{s\to\infty}\|\xi_s-\eta^*\|\,.$$

From (3.6) and (3.8), we obtain

$$b = \lim_{s\to\infty}\|\xi_s-\eta^*\|\,. \tag{3.9}$$

Using (1.6), we have

$$\begin{aligned} b &= \lim_{s\to\infty} \|\xi_s - \eta^*\| \\ &= \lim_{s\to\infty} \|\Omega\left(\Omega\left(\mu_s\right)\right) - \eta^*\| \\ &= \lim_{s\to\infty} \|\left(\Omega\left(\mu_s\right)\right) - \eta^*\| \\ &= \lim_{s\to\infty} \|\mu_s - \eta^*\| \\ &= \lim_{s\to\infty} \|[\Omega((1-\alpha_s)\,\eta_s + \alpha_s\Omega\ \eta_s)] - \eta^*\| \\ &= \lim_{s\to\infty} \|[(1-\alpha_s)\,\eta_s + \alpha_s\Omega\ \eta_s)] - \eta^*\| \\ &= \lim_{s\to\infty} \|(1-\alpha_s)\,(\eta_s - \eta^*) + \alpha_s(\Omega\ \eta_s - \eta^*)\|\,. \end{aligned}$$

Since $0\ < \alpha_s <\ 1$, then from Lemma 2.6, we have

$$\lim_{s\to\infty} \|\Omega\ \eta_s - \eta_s)\| = 0. \tag{3.10}$$

In contrast, assume that $\{\eta_s\}$ is bounded and

$$\lim_{s\to\infty} \|\Omega\ \eta_s - \eta_s)\| = 0.$$

Let $\eta^* \in A(\mathcal{A}, \{\eta_s\})$. Then, based on Lemma 2.8, we have

$$\begin{aligned} r\,(\ \Omega\eta^*, \{\eta_s\}) &= \lim_{s\to\infty} sup\, \|\eta_s - \Omega\eta^*)\| \\ &\le \left(\frac{3+\omega}{1-\omega}\right) \lim_{s\to\infty} sup\, \|\Omega\eta_s - \eta_s\| + \lim_{s\to\infty} sup\, \|\eta_s -\ \eta^*\| \\ &= \lim_{s\to\infty} sup\, \|\eta_s -\ \eta^*\| \\ &= r\,(\ \eta^*, \{\eta_s\})\,. \end{aligned}$$

This implies that $\Omega\eta^* \in\ A(\mathcal{A}, \{\eta_s\})$. Since $\mathbb{B}$ is UC-space, then $A(\mathcal{A}, \{\eta_s\})$ has only one element. Consequently, we acquire $\Omega\eta^* = \eta^*$. □

Lemma 3.3. *If Theorem 3.4's presumptions are all valid, then*

$$\lim_{s\to\infty} p_s(\mathcal{J}(p_1^*, p_2^*))$$

exists for any $p_1^*, p_2^* \in \mathcal{F}(\Omega)$; specifically,

$$\lim_{s\to\infty} (p - q, \mathcal{J}(p_1^*, p_2^*)) = 0$$

for $p, q \in \omega_w(p_s)$, where $\omega_w(p_s)$ represents the whole collection of weak points of $\{p_s\}$.

Proof. Lemma 2.3 in [8] provides the basis for the conclusion. □

Theorem 3.4. *Let $\mathbb{B}, \mathcal{A}, \Omega,$ and $\{\eta_s\}$ be as given in Lemma 3.2. Suppose either of the following hypotheses are valid:*

(a) $\mathbb{B}$ *meets Opials condition and* $I-\Omega$ *is demiclosed at zero;*
(b) $\mathbb{B}$ *possesses a Frchet differential norm.*

If $\mathcal{F}(\Omega) \neq \emptyset$*, then* $\{\eta_s\}$ *converges weakly to a point of* $\mathcal{A}$*.*

Proof. Based on Lemma 3.1, $\lim_{s\to\infty} \|\eta_s - \eta^*\|$ exists. It suffices to show that $\{\eta_s\}$ possesses a unique weak subsequential limit in $\mathcal{F}(\Omega)$.

Consider $\{\eta_{s_i}\}$ and $\{\eta_{s_j}\}$ are two subsequences of $\{\eta_s\}$, that converge weakly to b and y.

Consider case (a) is valid. Subsequently from Lemma 3.2, $\lim_{s\to\infty} \|\Omega(\eta_s) - \eta_s\| = 0$ and by the demiclosedness of $I-\Omega$, we have that $(I-\Omega)b = 0$. That is, $\Omega b = b$; equivalently, $y = \Omega y$.

Afterwards, we demonstrate uniqueness. Because $b, y \in \mathcal{F}(\Omega)$, then $\lim_{s\to\infty} \|\eta_s - b\|$ and $\lim_{s\to\infty} \|\eta_s - y\|$ exist. If $b \neq y$, so from Opial's condition, we have

$$\begin{aligned}
\lim_{s\to\infty} \|\eta_s - b\| &= \lim_{s_i\to\infty} \|\eta_{s_i} - b\| \\
&< \lim_{s_i\to\infty} \|\eta_{s_i} - y\| \\
&= \lim_{s\to\infty} \|\eta_s - y\| \\
&= \lim_{s_j\to\infty} \left|\eta_{s_j} - y\right| \\
&< \lim_{s_j\to\infty} \left|\eta_{s_j} - b\right| \\
&= \lim_{s\to\infty} \|\eta_s - b\| .
\end{aligned}$$

This leads to a contradiction, implying that $b = y$. Additionally, assuming that (b) is valid, we can use Lemma 3.3 we have $\langle \eta_s, \mathcal{J}(u_1, u_2)\rangle = 0$. Hence, $\|b-y\|^2 = \langle b-y, \mathcal{J}(b-y)\rangle$ indicates that $b = y$. □

Theorem 3.5. *Let* $\Omega, A, \mathbb{B}$ *be given as in Lemma* 3.2*. Then the sequence* $\{\eta_s\}$ *produced by* (1.6) *converges to an element of* $F(\Omega)$ *if and only if*

$$\liminf_{s\to\infty} d(\eta_s, F(\Omega)) = 0,$$

where $d(\eta_s, F(\Omega)) = \inf\{\|\eta_s - \eta^*\| : \eta^* \in F(\Omega)\}$*.*

Proof. Assume that $\liminf_{s\to\infty} d(\eta_s, F(\Omega)) = 0$ and consider the sequence $\eta^* \in F(\Omega)$. According to Lemma 3.1, $\lim_{s\to\infty} \|\eta_s - \eta^*\|$ exists. It is enough to prove $\{\eta_s\}$ is Cauchy in $\mathbb{A}$. Because

$$\liminf_{s\to\infty} d(\eta_s, F(\Omega)) = 0,$$

then given $\vartheta > 0$, there is $\omega_\circ \in N$ such that for each $s \geq \omega_\circ$,

$$d\left(\eta_s,\ \mathcal{F}\left(\Omega\right)\right) < \frac{\vartheta}{2},$$

that is,

$$\inf\{\|\eta_s - \eta^*\| \ : \eta^* \in \ \mathcal{F}(\Omega)\} < \frac{\vartheta}{2}.$$

In particular,

$$\inf\{\left\|\eta_{\omega_\circ} - \eta^*\right\| \ : \eta^* \in \ \mathcal{F}(\Omega)\} < \frac{\vartheta}{2}.$$

Thus, there exists $\eta^* \in \ \mathcal{F}\left(\Omega\right)$ such that

$$\left\|\eta_{\omega_\circ} - \eta^*\right\| < \frac{\vartheta}{2}.$$

Now for $s, \ \omega \geq \omega_\circ$, we have

$$\begin{aligned}\|\eta_{s+\omega} - \eta_s\| &\leq \|\eta_{s+\omega} - \eta^*\| + \|\eta_s - \eta^*\| \\ &\leq \left\|\eta_{\omega_\circ} - \eta^*\right\| + \left\|\eta_{\omega_\circ} - \eta^*\right\| \\ &= 2\left\|\eta_{\omega_\circ} - \eta^*\right\| \\ &< \vartheta.\end{aligned}$$

This means that $\{\eta_s\}$ is Cauchy in $\mathcal{A}$. Since $\mathcal{A}$ is closed, there is $q \in \mathcal{A}$ with $\lim_{s\to\infty} \eta_s = q$. Now,

$$\lim_{s\to\infty} d\left(\eta_s, \ \mathcal{F}\left(\Omega\right)\right) = 0$$

implying that

$$d\left(q, \ \ \mathcal{F}\left(\Omega\right)\right) = 0,$$

that is, $q \ \in \ \mathcal{F}\left(\Omega\right).$ □

4. Numrical result

To demonstrate the superior convergence rate of the new iterative approach compared to other iterative procedures, an illustrative example is displayed below.

Example 4.1. Let $(R, \|.\|)$ be a BN-space with the usual norm and $\mathcal{A} = [5, 7]$. Let $\Omega : \ \mathcal{A} \to \mathcal{A}$ be a mapping described by:

$$\Omega u = \begin{cases} \frac{u+20}{5}, & if \quad u < 7, \\ 4, & if \quad u = 7. \end{cases}$$

To demonstrate that Ω is RSN-mapping, the subsequent cases are considered.

Case 1: If $u, v < 7$, then

$$\begin{aligned}
\overline{\gamma}\left|u-\Omega u\right|+\overline{\gamma}\left|v-\Omega v\right|+(1-2\overline{\gamma})\left|u-v\right| &= \frac{1}{2}\left|u-(\frac{u+20}{5})\right|+\frac{1}{2}\left|v-(\frac{v+20}{5})\right| \\
&= \frac{1}{2}\left|\frac{4u-20}{5}\right|+\frac{1}{2}\left|\frac{4v-20}{5}\right| \\
&\geq \frac{1}{2}\left|\left(\frac{4u-20}{5}\right)-(\frac{4v-20}{5})\right| \\
&= \frac{1}{2}\left|\left(\frac{4u}{5}\right)-(\frac{4v}{5})\right| \\
&= \frac{2}{5}\left|u-v\right| \\
&\geq \frac{1}{5}\left|u-v\right| \\
&= \left|\Omega u-\Omega v\right|.
\end{aligned}$$

Case 2: If $u < 7$ and v =7, then

$$\begin{aligned}
\overline{\gamma}\left|u-\Omega u\right|+\overline{\gamma}\left|v-\Omega v\right|+(1-2\overline{\gamma})\left|u-v\right| &= \frac{1}{2}\left|\frac{4u-20}{5}\right|+\frac{1}{2}\left|7-4\right| \\
&= \frac{1}{2}\left|\frac{4u-20}{5}\right|+\frac{3}{2} \\
&\geq \left|\frac{u}{5}\right| \\
&= \left|\Omega u-\Omega v\right|.
\end{aligned}$$

Case 3: If u =7 and v <7, then

$$\begin{aligned}
\overline{\gamma}\left|u-\Omega u\right|+\overline{\gamma}\left|v-\Omega v\right|+(1-2\overline{\gamma})\left|u-v\right| &= \frac{1}{2}\left|7-4\right|+\frac{1}{2}\left|\frac{4v-20}{5}\right| \\
&= \frac{3}{2}+\frac{1}{2}\left|\frac{4v-20}{5}\right| \\
&\geq \left|\frac{v}{5}\right| \\
&= \left|\Omega u-\Omega v\right|.
\end{aligned}$$

Case 4: If u =7 and v =7, then

$$\begin{aligned}
\overline{\gamma}\left|u-\Omega u\right|+\overline{\gamma}\left|v-\Omega v\right|+(1-2\overline{\gamma})\left|u-v\right| \geq 0 &= \left|4-4\right| \\
&= \left|\Omega u-\Omega v\right|.
\end{aligned}$$

Therefore, Ω is RSN- mapping and has fixed point 5.

Table 1 and Figure 1 show how the new iterative approaches rate of convergence compares to other iterative approaches using Matlab.

Table 1. Comparison of convergence rates for various iteration approaches

Step	**New (1.6)**	**SP [12]**	**CR [13]**	**Mann [10]**	**Ishikawa [11]**
1	5.5	5.5	5.5	5.5	5.5
2	5.334	5.075	5.023	5.180	5.799
3	5.002	5.002	5.001	5.064	5.788
4	5.000	5.001	5.000	5.023	5.468
5	5.000	5.000	5.000	5.008	5.698
6	5	5.000	5.000	5.003	5.671
7		5.000	5.000	5.001	5.460
8		5.000	5.000	5.000	5.376
9		5.000	5.000	5.000	5.096
10		5.000	5.000	5.000	5.024
11		5.000	5.000	5.000	5.006
12		5.000	5.000	5.000	5.001
13		5.000	5	5.000	5.000
14		5		5.000	5.000
15				5.000	5.000
16				5.000	5.000
17				5.000	5.000
18				5.000	5.000
19				5.000	5.000
20				5.000	5.000
21				5.000	5.000
22				5.000	5.000
23				5.000	5.000
24				5.000	5.000
25				5.000	5.000
26				5.000	5.000
27				5.000	5
28				5.000	
29				5.000	
30				5.000	
31				5.000	
32				5.000	
33				5.000	
34				5.000	
35				5	

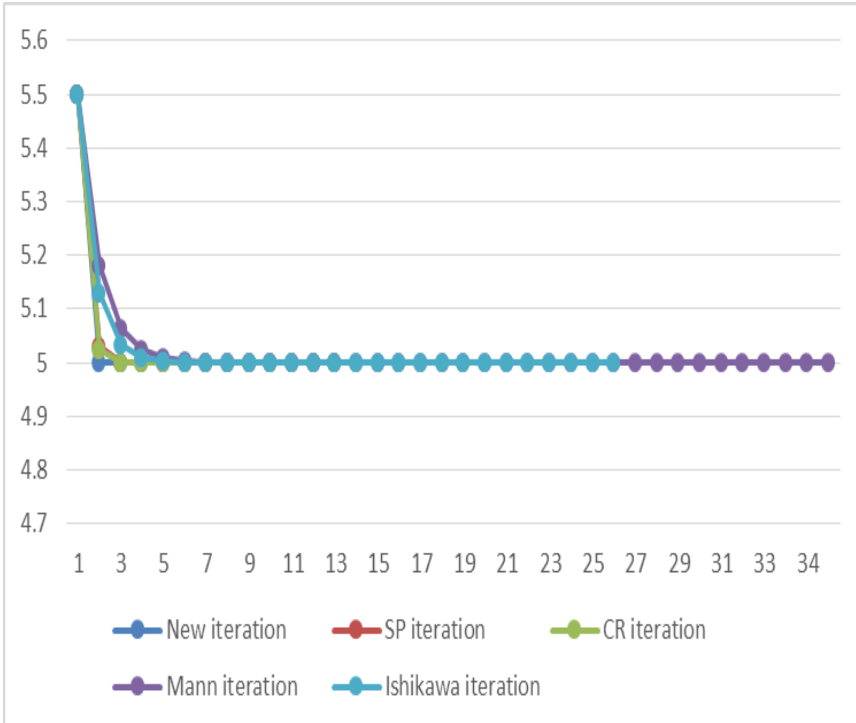

Figure 1. Graphic illustration of the convergence of iterative approaches

Table 2 indicates the number of iterations required for certain iterative approaches to reach the FP. The new iterative approach converges more quickly than the other methods, as shown in the data.

Table2. Number of iterations

Iterative approaches	Number of iterations
New iteration	6
SP iteration	14
CR iteration	13
Mann iteration	35
Ishikawa iteration	27

5. Conclusion

In this study, we introduce a novel iterative approach for estimating the FPs of RSN mapping. Our novel iterative method converges faster than previous well-known iterative algorithms, as shown by numerical evidence. In addition, we proved convergence results for RSN mapping in UC-space. A comparison of the convergence performance between the novel iterative approach (1.6) and some well-known iterative strategies is presented using an example of RSN mapping. In future work, it is also planned to study the non-expansion multi-valued mapping of the RSN type and establish some convergence theorems.

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