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SOME RESULTS ON FIXED POINTS IN b-METRIC SPACES THROUGH AN AUXILIARY FUNCTION

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Abstract. In this paper, we introduce a novel category of function that we employ to showcase a fresh set of fixed point outcomes in the context of b-metric spaces for P_φ-contractions. Furthermore, we provide several examples to elucidate our principal result. The outcomes we have obtained provide a broader scope of contraction mappings, such as the Kannan contraction, in a more general context.

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1. Introduction and Preliminary

Fixed point theory has captivated numerous researchers since 1922, primarily due to the renowned Banach contraction principle (BCP) [9]. The subject boasts an extensive body of literature and continues to thrive as a highly dynamic and vibrant area of research in the present era.

Fixed point theorems are well established principles that address the existence and properties of fixed points. For instance, Karapinar et al. [19] introduced the notion of Proinov-$\mathscr{C}b$-contraction mapping and explored its implications within b-metric spaces, which are recognized as a particularly fascinating abstract framework. In reference [5], the authors offered a detailed definition of cone metric spaces through the lens of neutrosophic theory, subsequently deriving various results associated with fixed points. The authors in [23] presented findings related to fixed point theory within the context of fuzzy b-metric spaces, along with several applications. In [2], the focus was on fixed point theory concerning modified ω-distance mappings in relation to quasi metric spaces. The studies in [25, 28] investigated the approximation of fixed points for specific mappings and provided applications in integral equations. Additionally, references [1, 3, 4, 20, 24, 26, 27] examined fixed point theory within the framework of G_b-metric and G-metric spaces. Also, in [30, 31] and references therein one can find a novel work on fixed point theory in various distance spaces.

These theorems hold significant value as they serve as crucial tools in establishing the existence and uniqueness of solutions for diverse mathematical models. These models encompass a wide range of phenomena encountered in various fields, including but not limited to steady-state temperature distribution, neutron transport theory, chemical equations, economic theories, and fluid flow. Theorems of this nature find application in differential equations, integral and partial differential equations, variational inequalities, numerical analysis, and real analysis, among others [6, 15, 17]. Indeed, it can be found in many applications formulated in terms of ordinary differential equations, partial differential equations, fractional differential equations, etc [12, 13, 14, 21, 22]. The concept of b-metric space has facilitated the adaptation and extension of Banach's principle in multiple directions, as evidenced by the works cited in [8, 10, 11, 16, 29] and the references included therein. In this manuscript, we commence by presenting the elegant class of P_ϕ-functions, which serves as the foundation for our formulation of novel contractions. Following this, we establish the existence and uniqueness of fixed point associated with these contractions. Subsequently, we derive a series of fixed point results that are grounded in our principal findings.

Kannan's Theorem [18], a well-known generalization of BCP, was famously demonstrated by Kannan to show that every contraction of Kannan-type has a distinct fixed point in a complete metric space. This theorem is particularly significant in the realm of analysis as it offers a valuable insight into the concept of metric completeness.

Theorem 1.1. (Kannan Theorem) *Suppose (X, d) is a complete metric space, and suppose $f : X \to X$ and f fulfills the following condition*

$$d(f\Omega,\ f\mu) \leq k\left(d(\Omega, f\Omega) + d(\mu, f\mu)\right),$$

where $0 \leq k < \frac{1}{2}$. Then f is characterized by having a unique fixed point.

The notion of b-metric spaces was proposed by Bakhtin [8] which has became well known by Czerwik [16].

Definition 1.2. A function $d_b : X \times X \to [0,\ \infty)$ is called a b-metric if there is $s \in [1,\ \infty)$ such that d_b satisfying:

(d_1) $d_b(\Omega,\ \mu) = 0$ if and only if $\Omega = \mu$,

(d_2) $d_b(\Omega,\ \mu) = d_b(\mu,\ \ \Omega),\ \forall\ \Omega, \mu \in X$,

(d_3) $d_b(\Omega,\ \mu) \leq s[d_b(\Omega,\ z) + d_b(z,\ \mu)],\ \ \forall\ \Omega, \mu, z \in X$.

The pair $(X,\ d_b,\ s)$ is called a b-metric space.

It should be noted that in the case where s equals 1, the triplet (X, d_b, s) forms a metric space. This implies that the properties of a metric space hold true when s is equal to 1. Henceforth, $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, $(X,\ d_b, s)$ means a b-metric space with base s. If $f : X \to X$, and $\Omega_0 \in X$, then the Picard sequence (Ω_r) generated by f within Ω_0 is denoted by $P_{seq}(\Omega_0, f)$; that is, the sequence (Ω_r) where $\Omega_r = f\Omega_{r-1},\ n \in \mathbb{N}$, also we refer by $\mathbf{Fix}(f)$ the set of all fixed points of f in X.

2. Main results

We commence by introducing the subsequent category of function that will be employed in the subsequent stages of this research.

Definition 2.1. Let P_φ denotes the set of all continuous functions $\varphi :\ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies the following condition:

$$\varphi(a,\ b) \leq 2\max\{a,\ b\} - \min\{a,\ b\}.$$

Subsequently, we present several examples pertaining to the class of P_φ functions.

Example 2.2. Let $\varphi_1, \varphi_2, \varphi_3 :\ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

(1) $\varphi_1(a, b) = \max\{a, b\}$,

(2) $\varphi_2(a,b) = |a-b|$,
(3) $\varphi_3(a,b) = \frac{a+b}{2}$.
Then $\varphi_1, \varphi_2, \varphi_3 \in P_\varphi$.

Following is the elucidation of P_φ-contraction, a concept of utmost significance in our result.

Definition 2.3. Suppose f is a self-mapping on $(X,\ d_b, s)$. Then f is said to be P_φ-contraction if there is P_φ-map φ such that for all $\Omega, \mu \in X$,

$$\Omega \neq \mu \implies d_b(f\Omega,\ f\mu) < k\left[d_b(\Omega,\mu) + \varphi\left(d_b(\Omega, f\Omega), d_b(\mu, f\mu)\right)\right], \qquad (2.1)$$

where $0 \leq k < \min\left\{\frac{1}{2s}, \frac{1}{s^2}\right\}$.

Lemma 2.4. *Let f be a self-map on (X, d_b, s) and let $\Omega_0 \in X$ such that f is P_φ-contraction. Then for the $P_{seq}(\Omega_0, f)$ if $\Omega_k \neq \Omega_{k+1}$ for each $k \in \mathbb{N}$, then*

$$\lim_{r\to\infty} d_b\left(\Omega_r, \Omega_{r+1}\right) = 0.$$

Proof. For each $r \in \mathbb{N}$, we have

$$\begin{aligned} d_b\left(\Omega_r, \Omega_{r+1}\right) &= d_b\left(f\Omega_{r-1}, f\Omega_r\right) \\ &< k\left[d_b\left(\Omega_{r-1}, \Omega_r\right) + \varphi\left(d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right)\right] \\ &\leq k\left[\ d_b\left(\Omega_{r-1}, \Omega_r\right) + 2\max\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\}\right. \\ &\qquad \left. -\min\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\}\ \right]. \end{aligned}$$

Case 1: If $\max\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\} = d_b\left(\Omega_{r-1}, \Omega_r\right)$, then

$$d_b\left(\Omega_r, \Omega_{r+1}\right) < \frac{3k}{1+k} d_b\left(\Omega_{r-1}, \Omega_r\right).$$

Case 2: If $\max\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\} = d_b\left(\Omega_r, \Omega_{r+1}\right)$, then

$$d_b\left(\Omega_r, \Omega_{r+1}\right) < 2k d_b\left(\Omega_r, \Omega_{r+1}\right) < d_b\left(\Omega_r, \Omega_{r+1}\right),$$

which is a contradiction. So, for each $r \in \mathbb{N}$, we have

$$d_b\left(\Omega_r, \Omega_{r+1}\right) < \frac{3k}{1+k} d_b\left(\Omega_{r-1}, \Omega_r\right) < \left(\frac{3k}{1+k}\right)^r d_b\left(\Omega_0, \Omega_1\right).$$

Hence,

$$\lim_{r\to\infty} d_b\left(\Omega_r, \Omega_{r+1}\right) = 0.$$

□

Lemma 2.5. *Let f be a self-map on (X, d_b, s) such that f is P_φ-contraction and let $\Omega_0 \in X$. Then $P_{seq}(\Omega_0, f)$ is a Cauchy sequence.*

Proof. Let $m > r$. Then

$$\begin{aligned} d_b(\Omega_r, \Omega_m) &\leq s[d_b(\Omega_r, \Omega_{r+1}) + d_b(\Omega_{r+1}, \Omega_m)] \\ &\leq s[d_b(\Omega_r, \Omega_{r+1}) + s(d_b(\Omega_{r+1}, \Omega_{m+1}) + d_b(\Omega_{m+1}, \Omega_m))] \\ &= sd_b(\Omega_r, \Omega_{r+1}) + s^2 d_b(\Omega_{r+1}, \Omega_{m+1}) + s^2 d_b(\Omega_{m+1}, \Omega_m) \\ &< sd_b(\Omega_r, \Omega_{r+1}) + s^2 k[d_b(\Omega_r, \Omega_m) \\ &\quad + \varphi(d_b(\Omega_r, \Omega_{r+1}), d_b(\Omega_m, \Omega_{m+1}))] + s^2 d_b(\Omega_{m+1}, \Omega_m) \\ &\leq sd_b(\Omega_r, \Omega_{r+1}) + s^2 k[d_b(\Omega_r, \Omega_m) + 2d_b(\Omega_r, \Omega_{r+1}) \\ &\quad - d_b(\Omega_m, \Omega_{m+1})] + s^2 d_b(\Omega_{m+1}, \Omega_m) \\ &= sd_b(\Omega_r, \Omega_{r+1}) + s^2 kd_b(\Omega_r, \Omega_m) + 2s^2 kd_b(\Omega_r, \Omega_{r+1}) \\ &\quad - s^2 kd_b(\Omega_m, \Omega_{m+1}) + s^2 d_b(\Omega_{m+1}, \Omega_m). \end{aligned}$$

So, we have

$$(1 - s^2 k)d_b(\Omega_r, \Omega_m) < (s + 2s^2 k)d_b(\Omega_r, \Omega_{r+1}) + (s^2 - s^2 k)d_b(\Omega_m, \Omega_{m+1}).$$

Hence by taking the limit as $m, r \to \infty$, we get

$$\lim_{m,n\to\infty} d_b(\Omega_r, \Omega_m) = 0,$$

and so, (Ω_r) is a Cauchy sequence. □

Theorem 2.6. *Suppose that (X, d_b, s) is complete and $f : X \to X$ is a P_φ-contraction. Then $\mathbf{Fix}(f)$ is characterized by having a unique element.*

Proof. Starting from $\Omega_0 \in X$, we construct $P_{seq}(\Omega_0, f)$. Hence, Lemma 2.5 ensures that $P_{seq}(\Omega_0, f)$ is Cauchy so, it is convergent in X. Say $\lim_{r\to\infty} (\Omega_r) = \varpi$. We claim that $f\varpi = \varpi$ as follows:

$$\begin{aligned} d_b(\Omega_{r+1}, f\varpi) &= d_b(f\Omega_r, f\varpi) \\ &< k[d_b(\Omega_r, \varpi) + \varphi(d_b(\varpi, f\varpi), d_b(\Omega_r, \Omega_{r+1}))] \\ &\leq k[d_b(\Omega_r, \varpi) + 2\max\{d_b(\varpi, f\varpi), d_b(\Omega_r, \Omega_{r+1})\} \\ &\quad -min\{d_b(\varpi, f\varpi), d_b(\Omega_r, \Omega_{r+1})\}]. \end{aligned}$$

So,

$$\limsup_{r\to\infty} d_b(\Omega_{r+1}, f\varpi) \leq 2kd_b(\varpi, f\varpi).$$

Now,

$$d_b(\varpi, f\varpi) \leq s[d_b(\varpi, \Omega_{r+1}) + d_b(\Omega_{r+1}, f\varpi)].$$

Taking lim sup to both sides whenever $r \to \infty$, we get

$$\begin{aligned} d_b(\varpi, f\varpi) &\leq s\left(0 + 2kd_b(\varpi, f\varpi)\right) \\ &= 2skd_b(\varpi, f\varpi). \end{aligned}$$

Hence, $(1-2sk)d_b(\varpi, f\varpi) \leq 0$, and therefore $d_b(\varpi, f\varpi) = 0$, that is, $\varpi = f\varpi$.
Now, to complete the proof, let $v \in X$ such that $fv = v$. Then

$$\begin{aligned} d_b(\varpi, v) &= d_b(f\varpi, fv) \\ &< k\left[d_b(\varpi, v) + \varphi\left(d_b(\varpi, f\varpi), d_b(v, fv)\right)\right] \\ &= kd_b(\varpi, v). \end{aligned}$$

Hence, $(1-k)\,d_b(\varpi, v) < 0$, which is a contradiction, and therefore, $\varpi = v$. □

According to Theorem 2.6 and the inherent nature of the class of P_φ functions, we are bestowed with a plethora of ensuing outcomes.

Corollary 2.7. *Suppose $(X,\ d_b, s)$ is complete and $f : X \to X$ fulfills the following condition:*

$$\Omega \neq \mu \implies d_b(f\Omega,\ f\mu) < k\left(d_b(\Omega, \mu) + |d_b(\Omega, f\Omega) - d_b(\mu, f\mu)|\right),$$

where $0 \leq k < \min\left\{\frac{1}{2s}, \frac{1}{s^2}\right\}$. Then $\mathbf{Fix}(f)$ is characterized by having a unique element.

Corollary 2.8. *Suppose $(X,\ d_b, s)$ is complete and $f : X \to X$ fulfills the following condition:*

$$\Omega \neq \mu \implies d_b(f\Omega,\ f\mu) < k\left(d_b(\Omega, \mu) + \max\left\{d_b(\Omega, f\Omega), d_b(\mu, f\mu)\right\}\right),$$

where $0 \leq k < \min\left\{\frac{1}{2s}, \frac{1}{s^2}\right\}$. Then $\mathbf{Fix}(f)$ is characterized by having a unique element.

Corollary 2.9. *Suppose $(X,\ d_b, s)$ is complete and $f : X \to X$ fulfills the following condition:*

$$\Omega \neq \mu \implies d_b(f\Omega,\ f\mu) < k\left(d_b(\Omega, \mu) + \frac{d_b(\Omega, f\Omega) + d_b(\mu, f\mu)}{2}\right),$$

where $0 \leq k < \min\left\{\frac{1}{2s}, \frac{1}{s^2}\right\}$. Then $\mathbf{Fix}(f)$ is characterized by having a unique element.

Corollary 2.10. *Suppose (X, d_b, s) is complete and $f : X \to X$ fulfills the following condition:*

$$\Omega \neq \mu \implies d_b(f\Omega,\ f\mu) < k\Big(d_b(\Omega,\mu) + \alpha \max\{d_b(\Omega, f\Omega), d_b(\mu, f\mu)\} - \min\{d_b(\Omega, f\Omega), d_b(\mu, f\mu)\}\Big),$$

where $1 \leq \alpha < 2$, $0 \leq k < \min\{\frac{1}{2s}, \frac{1}{s^2}\}$. Then $\mathbf{Fix}(f)$ *is characterized by having a unique.*

3. Examples

In this section, we present several examples to demonstrate the practicality and to elucidate our primary finding.

Example 3.1. The equation

$$\sqrt{6}\Omega - \sin\Omega - \sqrt{6} = 0 \tag{3.1}$$

has a unique solution in $[0, \frac{\pi}{2}]$.

In fact, it is clear that the solution of Equation (3.1) is the fixed point of the self-map f on $X = [0, \frac{\pi}{2}]$ which defined by $f\Omega = 1 + \frac{1}{\sqrt{6}} \sin\Omega$. Now, define $d_b : X \times X \to \infty$ by $d_b(\Omega, \mu) = (\Omega - \mu)^2$, also, define $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(a, b) = \frac{a+b}{2}$. Then, clearly $(X, d_b, 2)$ is a complete b-metric space, and also, $\varphi \in P_\varphi$. Moreover for $\Omega, \mu \in X$ with $\Omega \neq \mu$, we have

$$\begin{aligned} d_b(f\Omega, f\mu) &= \left(\tfrac{1}{\sqrt{6}} \sin\Omega - \tfrac{1}{\sqrt{6}} \sin\mu\right)^2 \\ &= \tfrac{1}{6}(\sin\Omega - \sin\mu)^2 \\ &\leq \tfrac{1}{6}(\Omega - \mu)^2 \\ &< \frac{1}{6}\left((\Omega - \mu)^2 + \frac{(\Omega - 1 - \frac{1}{10}\sin\Omega)^2 + (\mu - 1 - \frac{1}{10}\sin\mu)^2}{2}\right). \end{aligned}$$

Hence, f in a P_φ contraction, and so, Theorem 2.6 ensures that f has a unique fixed point.

Example 3.2. Define X as the set of all $n \times n$ matrices over the complex numbers, denoted as $M_n(\mathbb{C})$, and examine the spectral norm $\|.\| : X \to [0, \infty)$, also referred to as $\|S\| = s_1$, where s_1 is the greatest singular value of the matrix S.

It is evident that $(X,\ \|.\|)$ forms a Banach space due to the fact that X is a norm space with finite dimensionality.

Let $Q,\ A_i,\ B_i \in X$ for $i \in \{1,\ 2,\ \ldots, N\}$ be such that $\sum_{i=1}^{N} \|A_i\| \, \|B_i\| \leq 1$, define $d_b : X \times X \to [0,\ \infty)$ by $d_b(S,\ T) = \|S - T\|$, also, define $\varphi :\ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(a,\ b) = \max\{a,\ b\}$. Then the function $f : X \to X$ defined by $f(S) = Q + \frac{1}{4} \sum_{i=1}^{N} (A_i S B_i)$ has a unique fixed point in X.

In fact, it is Clear that $(X,\ d_b,\ 1)$ is a complete b-metric space, and also, $\varphi \in P_\varphi$. Moreover for $S,\ T \in X$ with $S \neq T$, we have

$$
\begin{aligned}
d_b\left(f(S),\ f(T)\right) &= \|f(S) - f(T)\| \\
&= \left\| Q + \frac{1}{4} \sum_{i=1}^{N} (A_i S B_i) - Q - \frac{1}{4} \sum_{i=1}^{N} (A_i T B_i) \right\| \\
&= \frac{1}{4} \left\| \sum_{i=1}^{N} (A_i S B_i) - \sum_{i=1}^{N} (A_i T B_i) \right\| \\
&= \frac{1}{4} \left\| \sum_{i=1}^{N} (A_i S B_i - A_i T B_i) \right\| \\
&= \frac{1}{4} \left\| \sum_{i=1}^{N} A_i (S - T) B_i \right\| \\
&\leq \frac{1}{4} \sum_{i=1}^{N} \|A_i (S - T) B_i\| \\
&\leq \frac{1}{4} \sum_{i=1}^{N} \|A_i\| \, \|S - T\| \, \|B_i\| \\
&= \frac{1}{4} \|S - T\| \sum_{i=1}^{N} \|A_i\| \, \|B_i\| \\
&\leq \frac{1}{4} \|S - T\| \\
&< \frac{1}{4} \Big[\|S - T\| + \max \Big\{ \left\| S - Q - \sum_{i=1}^{N} (A_i S B_i) \right\|, \\
&\qquad \left\| T - Q - \sum_{i=1}^{N} (A_i T B_i) \right\| \Big\} \Big].
\end{aligned}
$$

Hence, f is a P_φ-contraction, and so, Theorem 2.6 ensures that f has a unique fixed point.

4. Fixed point for P_φ-Kannan contractions

Definition 4.1. On $(X,\ d_b, s)$, a map $f : X \to X$ is said to be a P_φ-Kannan contraction if there is P_φ-map φ such that for all $x,\ y \in X$,

$$\Omega \neq \mu \implies d_b(f\Omega,\ f\mu) \leq k\left[d_b(\Omega, f\Omega) + \varphi\left(d_b(\Omega, f\Omega), d_b(\mu, f\mu)\right)\right],$$

where $0 \leq k < \frac{1}{2s}$.

Lemma 4.2. *Suppose f is a self-map on (X, d_b, s) and let $\Omega_0 \in X$ such that f is P_φ-Kannan contraction. Then for $P_{seq}(\Omega_0, f)$ if $\Omega_k \neq \Omega_{k+1}$ for each $k \in \mathbb{N}$, then $\lim_{r\to\infty} d_b\left(\Omega_r, \Omega_{r+1}\right) = 0$.*

Proof. For each $r \in \mathbb{N}$,

$$\begin{aligned} d_b\left(\Omega_r, \Omega_{r+1}\right) &= d_b\left(f\Omega_{r-1}, f\Omega_r\right) \\ &\leq k\left[d_b\left(\Omega_{r-1}, \Omega_r\right) + \varphi\left(d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right)\right] \\ &\leq k\left[\ d_b\left(\Omega_{r-1}, \Omega_r\right) + 2\max\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\}\right. \\ &\quad \left. - \min\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\}\ \right]. \end{aligned}$$

Case 1: If $\max\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\} = d_b\left(\Omega_{r-1}, \Omega_r\right)$, then

$$d_b\left(\Omega_r, \Omega_{r+1}\right) \leq \frac{3k}{1+k} d_b\left(\Omega_{r-1}, \Omega_r\right).$$

Case 2: If $\max\left\{d_b\left(\Omega_{r-1}, \Omega_r\right), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\} = d_b\left(\Omega_r, \Omega_{r+1}\right)$, then

$$d_b\left(\Omega_r, \Omega_{r+1}\right) \leq 2k d_b\left(\Omega_r, \Omega_{r+1}\right) < d_b\left(\Omega_r, \Omega_{r+1}\right),$$

which is a contradiction. So, for each $r \in \mathbb{N}$, we have

$$d_b\left(\Omega_r, \Omega_{r+1}\right) \leq \frac{3k}{1+k} d_b\left(\Omega_{r-1}, \Omega_r\right) \leq \left(\frac{3k}{1+k}\right)^r d_b\left(\Omega_0, \Omega_1\right).$$

Hence,

$$\lim_{r\to\infty} d_b\left(\Omega_r, \Omega_{r+1}\right) = 0.$$

□

Lemma 4.3. *Let f be a self-map on (X, d_b, s) such that f is P_φ-Kannan contraction and let $\Omega_0 \in X$. Then $P_{seq}(\Omega_0, f)$ is a Cauchy sequence.*

Proof. Let $m > r$. Then

$$\begin{aligned} d_b(\Omega_r, \Omega_m) &\leq s d_b(\Omega_r, \Omega_{r+1}) + s^2 d_b(\Omega_{r+1}, \Omega_{m+1}) + s^2 d_b(\Omega_{m+1}, \Omega_m) \\ &\leq s d_b(\Omega_r, \Omega_{r+1}) + s^2 k[d_b(\Omega_r, \Omega_{r+1}) + \varphi(d_b(\Omega_r, \Omega_{r+1}), \\ &\quad d_b(\Omega_m, \Omega_{m+1}))] + s^2 d_b(\Omega_{m+1}, \Omega_m) \\ &\leq s d_b(\Omega_r, \Omega_{r+1}) + s^2 k[d_b(\Omega_r, \Omega_{r+1}) + 2 d_b(\Omega_r, \Omega_{r+1}) \\ &\quad - d_b(\Omega_m, \Omega_{m+1})] + s^2 d_b(\Omega_{m+1}, \Omega_m). \end{aligned}$$

Hence by taking the limit as $m, r \to \infty$, we get $\lim_{m,n\to\infty} d_b(\Omega_r, \Omega_m) = 0$, and so, (Ω_r) is a Cauchy sequence. □

Theorem 4.4. *Suppose that $(X,\ d_b, s)$ is complete and f is a self-mapping on X such that f is a P_φ-Kannan contraction. Then* $\mathbf{Fix}(f)$ *is characterized by having a unique element.*

Proof. Starting from $\Omega_0 \in X$, we construct $P_{seq}(\Omega_0, f)$. Hence, Lemma 4.3 ensures that $P_{seq}(\Omega_0, f)$ is Cauchy so, it is convergent in X. Say $\lim_{r\to\infty} \Omega_r = \varpi$. We claim that $f\varpi = \varpi$ as follows:

$$\begin{aligned} d_b\left(\Omega_{r+1}, f\varpi\right) &= d_b\left(f\Omega_r, f\varpi\right) \\ &\leq k\left[d_b\left(\Omega_r, \Omega_{r+1}\right) + \varphi\left(d_b(\varpi, f\varpi), d_b\left(\Omega_r, \Omega_{r+1}\right)\right)\right] \\ &\leq k[d_b\left(\Omega_r, \Omega_{r+1}\right) + 2\max\left\{d_b(\varpi, f\varpi), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\} \\ &\quad - \min\left\{d_b(\varpi, f\varpi), d_b\left(\Omega_r, \Omega_{r+1}\right)\right\}]. \end{aligned}$$

So,

$$\limsup_{r\to\infty} d_b\left(\Omega_{r+1}, f\varpi\right) \leq 2k d_b(u, f\varpi).$$

Now,

$$d_b(\varpi, f\varpi) \leq s\left[d_b\left(\varpi, \Omega_{r+1}\right) + d_b\left(\Omega_{r+1}, f\varpi\right)\right].$$

Taking $\limsup$ to both sides whenever $r \to \infty$, we get

$$\begin{aligned} d_b(\varpi, f\varpi) &\leq s\left(0 + 2k d_b(\varpi, f\varpi)\right) \\ &= 2sk d_b(\varpi, f\varpi). \end{aligned}$$

Hence, $(1 - 2sk) d_b(\varpi, f\varpi) \leq 0$, and therefore $d_b(\varpi, f\varpi) = 0$ so, $\varpi = f\varpi$.

Now, let $v \in X$ such that $fv = v$, then if $\varpi \neq v$ we have

$$\begin{aligned} d_b(\varpi, v) &= d_b(f\varpi, fv) \\ &\leq k\left[d_b(\varpi, f\varpi) + \varphi\left(d_b(\varpi, f\varpi), d_b(v, fv)\right)\right] \\ &= 0. \end{aligned}$$

Therefore, $\varpi = v$. This completes the proof. □

By establishing the function $\varphi : R^+ \times R^+ \to R^+$ by $\varphi(a, b) = b$ and applying Theorem 4.4, we are able to obtain the subsequent result.

Remark 4.5. Theorem 1.1 is a consequence result of Theorem 4.4.

5. Conclusion

We have unveiled the P_φ function category, which we utilized to present a novel array of fixed point results within the realm of b-metric spaces specifically for contractions. In addition, we have offered a variety of examples to clarify our main findings. The results we have achieved expand the horizons of contraction mappings, including the Kannan contraction, within a more encompassing framework. This class of functions can be employed in alternative contexts of distance spaces to create diverse forms of contractions and to demonstrate novel fixed point theorems.

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