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CONTROL FUNCTION IN MENGER SPACE

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Abstract. The article presents a common fixed point result in Menger space in four self-mappings by using the control function [13] in the context of compatible mappings of type (P) and additionally provides some implications and applications as corollaries. This research generalizes the findings of Chaudhary et al. [9], and Pathak et al., [24] as well as expanding on some similar findings in the literature.

1. Introduction

In 1942, Menger [22] proposed the theory of probabilistic metric space. This probabilistic approach to metric space assigns a distribution function $M_{x,y}$ to any two points x and y. Schweizer and Sklar [25, 26] provided significant achievements in this area. Continuing this, Sehgal et al. [27] introduced the first fixed point theorem in Menger space in 1972. For additional information on this space, refer to [2, 14, 15, 17, 27, 29, 30, 33, 34, 37].

Many authors established fixed point theorems in single and multi-self-mapping in Menger space, a some of them refer to [3, 5, 18, 21, 32, 35, 36].

Sessa [28] coined the term "weakly commuting mapping" to improve commutativity. Jungck [19] soon expanded this notion to encompass compatible mappings in metric spaces. Mishra [23] established the idea of compatible mapping in Menger space. Singh and Jain [31] suggested the concept of weakly compatible mapping in Menger space, and many authors who

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worked on this space created various compatible mapping types see references [4, 6, 7, 8, 9, 10, 11, 12, 18, 20, 23, 24, 27].

Recently, Chaudhary et al. introduced compatible mappings of type (K) [10], type (P) [9] and weakly compatible mapping of type (P) [11] in Menger space.

This article uses the control function to create a common fixed point theorem in Menger space for four self-mappings. It generalizes the theorem of Chaudhary et al. [9], [24], and other related results in the literature.

2. Preliminaries

Definition 2.1. ([6]) A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is said to be distribution function if it is a non-decreasing function, left continuous with $\inf\{F(x) : x \in \mathbb{R}\} = 0$ and $\sup\{F(x) : x \in \mathbb{R}\} = 1$.

Here, we denote the set of all distribution functions by Ω while H denotes the specific distribution function defined by:

$$H(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ 1, & \text{if} \quad x > 0. \end{cases}$$

Definition 2.2. ([25]) A probabilistic metric space (PM-space) is an ordered pair (Y, M), where Y is any non-empty abstract set of elements and $M : Y \times Y \to \Omega$ is distribution function defined by $(p, q) \to M_{p,q}$, where $\Omega = \{M_{p,q} : p, q \in Y\}$ and the distribution function $M_{p,q}$ satisfy following conditions:

(P1) $M_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$ for every $p, q \in Y$;
(P2) $M_{p,q}(0) = 0$ for every $p, q \in Y$;
(P3) $M_{p,q}(x) = M_{q,p}(x)$ for every $p, q \in Y$, and
(P4) $M_{p,q}(x + y) = 1$ if and only if $M_{p,r}(x) = 1$ and $M_{r,q}(y) = 1$ for every $p, q, r \in Y$,

where, $M_{p,q}(x)$ represents the value of distribution function $M_{p,q}$ at $x \in \mathbb{R}$, and it is also denoted by $M(p, q, x)$.

Definition 2.3. ([17]) A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called triangular norm (shortly t-norm) if it satisfies the following conditions:

(T1) $t(0, 0) = 0$ and $t(a, 1) = a$ for all $a \in [0, 1]$;
(T2) $t(a, b) = t(b, a)$ for all $a, b \in [0, 1]$;
(T3) $t(a, b) \leq t(c, d)$ if $a \leq$ c and $b \leq d$ for every $a, b, c, d \in [0, 1]$; and
(T4) $t(a, t(b, c)) = t(t(a, b), c))$ for every $a, b, c \in [0, 1]$.

Definition 2.4. ([7]) A Menger space is a triplet (Y, M, t), where (Y, M) is PM-space and t is a triangular norm such that for all $p, q, r \in Y$ and $x, y \in \mathbb{R}\geq 0$:

(P5) $M_{p,q}(x + y) \geq t(M_{p,r}(x),\ M_{r,q}(y))$.

Remark 2.5. The following statement and results show how the metric space and probabilistic metric space are connected:

If (Y, d) is metric space then metric d induces a distribution function M defined by $M_{p,q}(t) = H(t - d(p, q))$. If f is contraction and $d(fp, fq)\leq kd(p, q)$ in metric space, then, in probabilistic metric space: $M_{fp,fq}(kt)\geq M_{p,q}(t)$, and when if $d(p, q) < t$ then $M_{p,q}(t) > 1 - t$. Also, $M_{fp,fq}(kt)\geq M_{p,q}(t)$, whenever $M_{p,q}(t) > 1 - t$.

Definition 2.6. ([7]) A mapping $A : Y \to Y$ in Menger space (Y, M, t) is said to be continuous at a point $p \in Y$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists $\epsilon_1 > 0$ and $\lambda_1 > 0$ such that if $M_{p,q}(\epsilon_1) > 1 - \lambda_1$, then $M_{Ap,Aq}(\epsilon) > 1 - \lambda$.

Definition 2.7. ([11]) Let (Y, M, t) be a Menger space and t be a continuous t-norm. Then,

(a) A sequence $\{y_n\}$ in Y is said to converge to a point y in Y if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = (N, \epsilon) > 0$ such that $M_{y_n,y}(\epsilon) > 1 - \lambda$ for all $n\geq N$. In this case, we write $\lim_{n\to\infty} y_n = y$.

(b) A sequence $\{y_n\}$ in Y is said to be Cauchy if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = (N, \epsilon) > 0$ such that $M_{y_n,y_m}(\epsilon) > 1 - \lambda$ for all $n, m\geq N$.

(c) A Menger space (Y, M, t) is said to be complete if every Cauchy sequence in Y converges to a point in Y.

Definition 2.8. ([23]) Two mappings $A, B : Y \to Y$ are said to be compatible in Menger space (Y, M, t) if

$$\lim_{n\to\infty} F_{ABx_n,BAx_n}(x) = 1 \quad \text{for all} \quad x > 0,$$

whenever $\{x_n\}$ is a sequence in Y such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = y$ for some y in Y.

Definition 2.9. ([9]) Two mappings $A, B : Y \to Y$ are said to be compatible of type (P) in Menger space (Y, M, t) if

$$\lim_{n\to\infty} M_{AAx_n,BBx_n}(x) = 1 \quad \text{for all} \quad x > 0,$$

whenever $\{x_n\}$ is a sequence in Y such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = y$ for some y in Y.

Example 2.10. Let (Y, d) be a metric space, where $Y = [0, \infty)$ with usual metric $d(x, y) = |x - y|$ and $t(a, b) = ab$ be t-norm. Defining distribution function as:

$$M_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

for all $x, y \in Y$. Then, (Y, M, t) is a Menger space.

If $A, B : Y \to Y$ are defined by

$$A(x) = \begin{cases} 5 & \text{for} \quad x \in [0, 1), \\ x & \text{for} \quad x \in [1, \infty) \end{cases}$$

and

$$B(x) = \begin{cases} 1 & \text{for} \quad x \in [0, 1), \\ 1/x & \text{for} \quad x \in [1, \infty). \end{cases}$$

Taking sequence $\{k_n\}$ where $k_n = 1 + \frac{1}{n}$, $n \in N$. Then, A, B are compatible with type P in Menger space but A, B are not compatible mapping.

Theorem 2.11. ([12]) *Let (Y, M, t) be a Menger space with the continuous $t-norm$ t and $A : Y \to Y$ be self-mapping. Then, A is continuous at a point $y \in Y$ if and only if for every sequence $\{y_n\}$ in Y converging to a point y, sequence $\{Ay_n\}$ converges to the point Ay, that is, if $y_n \to y$, then it implies $Ay_n \to Ay$.*

Proposition 2.12. ([12]) *In Menger space (Y, M, t), if $t(k, k) \geq k$ for all $k \in [0, 1]$, then $t(a, b) = \min(a, b)$ for all $a, b \in [0, 1]$.*

We need the following lemmas for the establishment of main results in the Menger space.

Lemma 2.13. ([31]) *Let (Y, M, t) be a Menger space. If there exists $k \in (0, 1)$ such that for all $p, q \in Y$, $M_{p,q}(kx) \geq M_{p,q}(x)$, then $p = q$.*

Lemma 2.14. ([32]) *Let $\{k_n\}$ be a sequence in Menger space (Y, M, t), where t is continuous $t-norm$ and $t(x, x) \geq x$ for all $x \in [0, 1]$. If there exists a constant $k \in [0, 1]$ such that $\lim_{n\to\infty} M_{k_n,k_n+1}(kx) \geq M_{k_n-1,k_n}(x)$ for all $x > 0$ and $n \in N$, then $\{k_n\}$ is a Cauchy sequence in Y.*

3. Main results

Theorem 3.1. *Let (Y, M, t) be a complete Menger space with $t(a, b) = \min(a, b)$ for all $a, b \in [0, 1]$ and $Q, S, R, T : Y \to Y$ be mappings such that*

(1) *$Q(Y) \subset T(Y)$ and $S(Y) \subset R(Y)$;*

(2) *the pairs* (Q, R) *and* (S, T) *are compatible mappings of type* (P)*;*
(3) R *and* T *are continuous, and*
(4) *there exists a constant* $k \in (0, 1)$ *such that*

$$M(Qx, Sy, kq) \geq \psi\{\min\{M(Rx, Ty, q), M(Sy, Ty, q), M(Qx, Ty, q)\}\}$$

for all $x, y \in Y$*, and* $q > 0$*, and where* $\psi : [0, 1] \to [0, 1]$ *satisfies*
 (i) ψ *is continuous and non-decreasing on* $[0, 1]$*;*
 (ii) $\psi(n) > n$ *for all* n *in* $[0, 1]$*.*
Noting that if $\psi \in \Psi$*, class of all mappings* $\psi : [0, 1] \to [0, 1]$*,*
then $\psi(0) = 0$*,* $\psi(1) = 1$*, and* $\psi(n) \geq n$ *for all in* $[0, 1]$*.*

Then Q, S, R, T *have a unique common fixed point in* Y*.*

Proof. Since we have $Q(Y) \subset T(Y)$ and $S(Y) \subset R(Y)$ for any $u_0 \in Y$, so there exists a point $u_1 \in Y$ such that $Qu_0 = Tu_1 = v_1$, and for u_1, we may choose $u_2 \in Y$ such that $Su_1 = Ru_2 = v_2$, and so on.

And inductively, we may construct sequence $\{u_n\}$ and $\{v_n\}$ in Y such that

$$Qu_{2n-2} = Tu_{2n-1} = v_{2n-1}$$

and

$$Su_{2n-1} = Ru_{2n} = v_{2n}$$

for $n = 1, 2, ...$ Putting $x = u_{2n}$ and $y = u_{2n+1}$ in condition (4), then we obtain

$$\begin{aligned} M(Qu_{2n}, Su_{2n+1}, kq) &= M(v_{2n+1}, v_{2n+2}, kq) \\ &\geq \psi\{\min\{M(Ru_{2n}, Tu_{2n+1}, q), M(Su_{2n+1}, Tu_{2n+1}, q), \\ &\quad M(Qu_{2n}, Tu_{2n+1}, q)\}\} \end{aligned}$$

or

$$\begin{aligned} M(v_{2n+1}, v_{2n+2}, kq) &\geq \psi\{\min\{M(v_{2n}, v_{2n+1}, q), M(v_{2n+2}, v_{2n+1}, q), \\ &\quad M(v_{2n+1}, v_{2n+1}, q)\}\} \\ &\geq \psi\{\min\{M(v_{2n}, v_{2n+1}, q), M(v_{2n+1}, v_{2n+2}, q)\}\}. \end{aligned}$$

Similarly, we obtain

$$M(v_{2n+2}, v_{2n+3}, kq) \geq \psi\{\min\{M(v_{2n+1}, v_{2n+2}, q), M(v_{2n+2}, v_{2n+3}, q)\}\}.$$

Therefore, for every $n \in N$,

$$M(v_n, v_{n+1}, kq) \geq \psi\{\min\{M(v_{n-1}, v_n, q), M(v_n, v_{n+1}, q)\}\}.$$

Consequently,

$$M(v_n, v_{n+1}, q) \geq \psi\{\min\{M(v_{n-1}, v_n, k^{-1}q), M(v_n, v_{n+1}, k^{-1}q)\}\} \tag{3.1}$$

and

$$M(v_n, v_{n+1}, k^{-1}q) \geq \psi\{\min\{M(v_{n-1}, v_n, k^{-2}q), M(v_n, v_{n+1}, k^{-2}q)\}\}. \tag{3.2}$$

After combining inequalities (3.1) and (3.2), we get

$$\begin{aligned} M(v_n, v_{n+1}, q) &\geq \psi\{\min\{M(v_{n-1}, v_n, k^{-1}q), \\ &\qquad \min\{M(v_{n-1}, v_n, k^{-2}q), M(v_n, v_{n+1}, k^{-2}q)\}\}\} \\ &= \psi\{\min\{\min\{M(v_{n-1}, v_n, k^{-1}q), M(v_{n-1}, v_n, k^{-2}q)\}, \\ &\qquad M(v_n, v_{n+1}, k^{-2}q)\}\}. \end{aligned} \tag{3.3}$$

Since $k \in (0,1)$ and M is non-decreasing, we obtain

$$\min\{M(v_{n-1}, v_n, k^{-1}q), M(v_{n-1}, v_n, k^{-2}q)\} = M(v_{n-1}, v_n, k^{-1}q).$$

Then inequality (3.3) gives

$$M(v_n, v_{n+1}, q) \geq \psi\{\min\{M(v_{n-1}, v_n, k^{-1}q), M(v_n, v_{n+1}, k^{-2}q)\}\}.$$

Continuing in this way, we get

$$M(v_n, v_{n+1}, q) \geq \psi\{\min\{M(v_{n-1}, v_n, k^{-1}q), M(v_n, v_{n+1}, k^{-m}q)\}\}.$$

Since $k \in (0,1)$ and M is non-decreasing and $\sup M = 1$ as $m \to \infty$,

$$M(v_n, v_{n+1}, k^{-m}q) \to 1.$$

So, it follows that

$$M(v_n, v_{n+1}, q) \geq \psi\{\min\{M(v_{n-1}, v_n, k^{-1}q)\}\}$$

or

$$M(v_n, v_{n+1}, kq) \geq \psi\{M(v_{n-1}, v_n, q)\}$$

for all $n \in N$ and $q > 0$.

Now, by property of ψ, we have

$$M(v_n, v_{n+1}, kq) \geq M(v_{n-1}, v_n, q)\}.$$

So, by Lemma 2.2, $\{v_n\}$ is Cauchy sequence in Y.

Since the Menger space (Y, M, t) is complete, so $\{v_n\}$ converges to a point z in Y and consequently the subsequences $\{Qu_{2n-2}\}, \{Tu_{2n-1}\}, \{Su_{2n-1}\}$, $\{Ru_{2n}\}$ of $\{v_n\}$ also converges to z. As $Qu_{2n}, Ru_{2n} \to z$ and (Q, R) is compatible mappings of type (P), then as $n \to \infty$

$$M(QQu_{2n}, RRu_{2n}, q/2) = 1. \tag{3.4}$$

Since $Ru_{2n} \to z$ and R is continuous, $RRu_{2n} \to Rz$.

$$M(RRu_{2n}, Rz, q/2) = 1 \tag{3.5}$$

as $n \to \infty$. Combining (3.4) and (3.5), we get

$$M(QQu_{2n}, Rz, q) = 1$$

as $n \to \infty$, that is,

$$QQu_{2n} \to Rz. \tag{3.6}$$

Similarly, we may prove when T is continuous as $n \to \infty$,

$$SSu_{2n-1} \to Tz. \tag{3.7}$$

Putting $x = Qu_{2n}$ and $y = Su_{2n-1}$ in relation (3.4), we get

$$\begin{aligned} M(QQu_{2n}, SSu_{2n+1}, kq) \geq \psi\{\min\{&M(RQu_{2n-1}, TSu_{2n-1}, q),\\ &M(SSu_{2n-1}, TSu_{2n-1}, q),\\ &M(QQu_{2n}, TSu_{2n-1}, q)\}\}. \end{aligned}$$

Taking the limit as $n \to \infty$ and using (3.6) and (3.7), we get

$$M(Rz, Tz, kq) \geq \psi\{\min\{M(Rz, Tz, q), M(Tz, Tz, q), M(Rz, Tz, q)\}\},$$

$$M(Rz, Tz, kq) \geq \psi\{\min\{M(Rz, Tz, q), M(Rz, Tz, q)\}\},$$

$$M(Rz, Tz, kq) \geq \psi\{M(Rz, Tz, q)\}$$

or

$$M(Rz, Tz, kq) \geq M(Rz, Tz, q),$$

by property of ψ. So, by Lemma 2.1, $Rz = Tz$. Taking $x = z$ and $y = Su_{2n-1}$ then from condition inequality (4)

$$\begin{aligned} M(Qz, SSu_{2n-1}, kq) \geq \psi\{\min\{&M(Rz, TSu_{2n-1}, q), M(SSu_{2n-1},\\ &TSu_{2n-1}, q), M(Qz, TSu_{2n-1}, q)\}\}. \end{aligned}$$

Taking the limit as $n \to \infty$, using (3.7) and $TSu_{2n-1} \to Tz$,

$$\begin{aligned} M(Qz, Tz, kq) &\geq \psi\{\min\{M(Rz, Tz, q), M(Tz, Tz, q), M(Qz, Tz, q)\}\}\\ &= \psi\{\min\{M(Tz, Tz, q), M(Tz, Tz, q), M(Qz, Tz, q)\}\}\\ &= \psi\{\min\{M(Qz, Tz, q)\}\}\\ &= \psi\{M(Rz, Tz, q)\}\\ &> M(Qz, Tz, q). \end{aligned}$$

So, by Lemma 2.1, $Qz = Tz$. Putting, $x = y = z$ in condition (4), we get,

$$\begin{aligned} M(Qz, Sz, kq) &\geq \psi\{\min\{M(Rz, Tz, q), M(Sz, Tz, q), M(Qz, Tz, q)\}\}\\ &= \psi\{\min\{M(Rz, Rz, q), M(Sz, Qz, q), M(Qz, Tz, q)\}\}\\ &= \psi\{\min\{M(Sz, Tz, q)\}\}\\ &= \psi\{\min\{M(Sz, Qz, q)\}\}\\ &= \psi\{M(Sz, Qz, q)\}\\ &> M(Sz, Qz, q). \end{aligned}$$

So, by Lemma 2.1, $Qz = Sz$. Thus, we get

$$Qz = Sz = Tz = Rz.$$

Lastly, from the condition (4),

$$M(Qu_{2n}, Sz, kq) \geq \psi\{\min\{M(Ru_{2n}, Tz, q), M(Sz, Tz, q), M(Qu_{2n}, Tz, q)\}\}.$$

Taking the limit as $n \to \infty$, using $Qu_{2n} \to z$ and $Ru_{2n} \to z$, we get

$$\begin{aligned} M(z, Sz, kq) &\geq \psi\{\min\{M(z, Sz, q), M(Sz, Sz, q), M(z, Sz, q)\}\} \\ &= \psi\{\min\{M(z, Sz, q)\}\} \\ &= \psi\{M(z, Sz, q)\} \\ &> M(z, Sz, q). \end{aligned}$$

So, by Lemma 2.1, $z = Sz$. Thus, we get $z = Qz = Sz = Tz = Rz$ and z is a common fixed point of Q, S, R, T.

For uniqueness, let w be another common fixed point of Q, S, R, T. Then,

$$\begin{aligned} M(z, w, kq) &= M(Qz, Sw, kq) \\ &\geq \psi\{\min\{M(Rz, Tw, q), M(Sw, Tw, q), M(Qz, Tw, q)\}\} \\ &= \psi\{\min\{M(z, w, q), M(Sw, Sw, q), M(z, w, q)\}\} \\ &= \psi\{M(z, w, q)\} \\ &> M(z, w, q). \end{aligned}$$

So, by Lemma 2.1, $z = w$. This completes the proof. □

If we take $R = T$, then we obtain the following result:

Corollary 3.2. *Let (Y, M, t) be a complete Menger space with $t(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $Q, S, R : Y \to Y$ be mappings such that*

(1) $Q(Y) \bigcup S(Y) \subset R(Y)$;
(2) *the pairs (Q, R) and (S, R) are compatible mappings of type (P);*
(3) *R be continuous, and*
(4) *there exists a constant $k \in (0, 1)$ such that*

$$M(Qx, Sy, kq) \geq \psi\{\min\{M(Rx, Ry, q), M(Sy, Ry, q), M(Qx, Ry, q)\}\}$$

for all $x, y \in Y$, and $q > 0$, and where $\psi : [0, 1] \to [0, 1]$ satisfies
(i) *ψ is continuous and non-decreasing on $[0, 1]$;*
(ii) *$\psi(n) > n$ for all n in $[0, 1]$.*

Then Q, S, R have a unique common fixed point in Y.

As the consequences of the above Theorem 3.1, we may establish the following results:

Corollary 3.3. *Let (Y, M, t) be a complete Menger space with $t(a, b) = min(a, b)$ for all $a, b \in [0, 1]$ and $Q, S, R, T : Y \to Y$ be mappings such that*

(1) *$Q(Y) \subset T(Y)$ and $S(Y) \subset R(Y)$;*
(2) *the pairs (Q, R) and (S, T) are compatible mappings of type (P);*

(3) *R and T be continuous, and*
(4) *there exists a constant $k \in (0,1)$ such that*

$$M(Qx, Sy, kq) \geq \psi\{\min\{M(Rx, Ty, q), M(Sy, Ty, q), M(Sy, Rx, 2q), M(Qx, Ty, q)\}\}$$

for all $x, y \in Y$, and $q > 0$, where $\psi : [0,1] \to [0,1]$ satisfies
(i) *ψ is continuous and non-decreasing on $[0,1]$;*
(ii) *$\psi(n) > n$ for all n in $[0,1]$.*

Then Q, S, R, T have a unique common fixed point in Y.

Proof. We have

$$M(Qx, Sy, kq) \geq \psi\{\min\{M(Rx, Ty, q), M(Sy, Ty, q), M(Sy, Rx, 2q), M(Qx, Ty, q)\}.$$

Since we have from definition of Menger space

$$M(Sy, Rx, 2q) \geq \min\{M(Sy, Ty, q), M(Ty, Rx, q)\}.$$

So, we obtain

$$\begin{aligned} M(Qx, Sy, kq) &\geq \psi\{\min\{M(Rx, Ty, q), M(Sy, Ty, q), M(Sy, Ty, q), M(Ty, Rx, q), M(Qx, Ty, q)\} \\ &= \psi\{\min\{M(Rx, Ty, q), M(Sy, Ty, q), M(Qx, Ty, q)\}. \end{aligned}$$

Hence, from Theorem (3.1), Q, S, R, T have a unique common fixed point in Y. □

Corollary 3.4. *Let (Y, M, t) be a complete Menger space with $t(a, b) = min(a, b)$ for all $a, b \in [0,1]$ and $Q, S, R, T : Y \to Y$ be mappings such that*

(1) *$Q(Y) \subset T(Y)$ and $S(Y) \subset R(Y)$;*
(2) *the pairs (Q, R) and (S, T) are compatible mappings of type (P);*
(3) *R, and T be continuous, and*
(4) *there exists a constant $k \in (0,1)$ such that*

$$M(Qx, Sy, kq) \geq \psi\{\min\{M(Rx, Ty, q), M(Qx, Ty, q)\}\}$$

for all $x, y \in Y$, and $q > 0$, where $\psi : [0,1] \to [0,1]$ satisfies
(i) *ψ is continuous and non-decreasing on $[0,1]$;*
(ii) *$\psi(n) > n$ for all n in $[0,1]$.*

Then Q, S, R, T have a unique common fixed point in Y.

Proof. We have

$$\begin{aligned} M(Qx,Sy,kq) &\geq \psi\{\min\{M(Rx,Ty,q),M(Qx,Ty,q)\}\} \\ &= \psi\{\min\{M(Rx,Ty,q),M(Qx,Ty,q),1\}\} \\ &= \psi\{\min\{M(Rx,Ty,q),M(Qx,Ty,q),M(Sy,Sy,4q)\}\} \\ &\geq \psi\{\min\{M(Rx,Ty,q),M(Qx,Ty,q), \\ &\qquad M(Sy,Rx,2q),M(Rx,Sy,2q)\}\} \\ &\geq \psi\{\min\{M(Rx,Ty,q),M(Qx,Ty,q),M(Sy,Rx,2q), \\ &\qquad M(Rx,Ty,q),M(Ty,Sy,q)\}\} \\ &\geq \psi\{\min\{M(Rx,Ty,q),M(Sy,Ty,q), \\ &\qquad M(Sy,Rx,2q),M(Qx,Ty,q)\}\}. \end{aligned}$$

Hence, from Corollary 3.3, Q,S,R,T have a unique common fixed point in Y. □

4. Conclusion

This research focuses on the Menger probabilistic metric space, with established results acquired by control functions. This remarkable work generalizes and extends the results of Chaudhary et al. [9] and Pathak et al. [24] by using control functions in Menger spaces and presents some extra consequences as an application of the basic Theorem 3.1.

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